Lifetime Ruin Minimization: Should Retirees Hedge Inflation or Just Worry About It?

Huaxiong Huang and Moshe A. Milevsky[1]

York University, Toronto, Canada

1 January 2009

[1]Dr. Huang is a Professor in the Department of Mathematics and Statistics. Dr. Milevsky (the contact author) is the Executive Director of The IFID Centre and Associate Professor at the Schulich School of Business. The contact author can be reached at Tel: (416) 736-2100 x 66014, Email: milevsky@yorku.ca. The authors acknowledge funding from the MITACS and NSERC program (Canada), and would like to thank Anna Abaimova for research assistance.

Abstract

Inflation for retirees is different from, and mostly higher than, the widely quoted macroeconomic inflation rate for the population. For example in the U.S. the Consumer Price Index (CPI) has a lesser known cousin called the CPI-E (for the elderly) in which the subcomponent weights are based on the consumption patterns of Americans above the age of 62. This suggests that Inflation Linked Bond Funds (ILBFs) – which are based on aggregate population inflation with differing maturities – might not be the best hedge for individual retirees' cost of living. Motivated by this question, the current paper extends lifetime ruin minimization (LRM) techniques to investigate the choice between an ILBF and a generic nominal investment fund, for a retiree facing an exogenous liability. Our model trades-off the benefits of an albeit imperfect insurance hedge against the cost of lower investment growth. And, to keep the financial economists happy, in the appendix we also solve the model by maximizing utility of lifetime consumption. Either way our numerical results suggest that ILBFs should be treated as just another asset class in the broad optimization problem, as opposed to a special or unique category.

1 Introduction and Motivation

As the first American baby boomer turned 62 on January 1st 2008 and among other things became eligible for Social Security – and with over 70 million to follow during the next 20 years – the financial services industry is shifting its attention from wealth accumulation to generating a sustainable retirement income. With the ongoing demise of Defined Benefit (DB) pensions, one of the main concerns in creating an appropriate income stream from a DC-style retirement account is the issue of inflation and price uncertainty over 25 to 35 potential years of retirement.

Against this backdrop, a number of financial commentators, including writers at the *Wall Street Journal* and the *New York Times*, have urged retirees to allocate a substantial portion of their retirement wealth to inflation-linked bonds such as TIPS and I-BONDS. These investments generate periodic income in real, as opposed to nominal units. In fact, a number of large financial services companies, such as TIAA-CREF, PIMCO, Vanguard and Fidelity actively promote their inflation-linked mutual funds as a hedge against an increasing cost of living, especially for retirees. In fact, the book *Risk Free Investing* by Bodie and Clowes (2005) makes the argument that a substantial portion of *any* investors wealth should be in inflation-adjusted products.

Oddly enough, there seems to be very little discussion as to whether the Urban Consumer Price Index (CPI-U) – which is the underlying adjustment mechanism for most inflation linked bonds – is an adequate measure of the inflation rate experienced by retirees. In fact, since the early 1980s the U.S. Bureau of Labor Statistics (BLS) has been calculating an experimental index called the CPI-E, for Americans above the age of 62. This index weights its expenditure sub-components based on the spending habits of the elderly, as opposed to the aggregate (urban dwelling or wage earning) population. See Amble and Stewart (1994) or Hobijn and Lagakos (2003) for more statistical and methodological details about this experimental index. The index was motivated by a group of legislators within the U.S. house of Representatives who wanted to introduce an alternative inflation index for retirees, one that could be used to better adjust Social Security payments. Currently, Social Security payments in the U.S. are adjusted each year based on the previous year's inflation rate as measured by the CPI-W, which is a sub-component of the CPI-U. (Of course, as of early

2009, the current fear in the market seems to be deflation as opposed to inflation, but that does not detract from the long-term nature of inflation concerns[1].)

Figure #1 (all tables and charts are placed in the appendix) exhibits the annualized gap between the CPI-E (elderly) versus the CPI-U (urban, used to adjust inflation linked bonds) during the last twenty five years. On average the gap is positive, by approximately 50 basis points per annum, which implies that the CPI-E increases at a greater rate, compared to the CPI-U. If both of these indices moved in lock-step, the gap would obviously be zero.

Figure #1 Placed Here

The reason for this outperformance – or excessive inflation rate – is that retirees have different consumption and spending habits compared to the general population. They spend more of their disposable income on medical care and housing, and of course these prices have increased more than other subcomponents during the last 25 years. Whether this will persist going forward, especially after the 2007/2008 decline in housing prices is debatable. But, once again, our agenda is not to forecast inflation. Table #1 displays the relative importance and weights of the major expenditure categories in the respective indices. Note that medical care is weighted 10.81% in the CPI-E but only 5.27% in the CPI-U, for example. Likewise, food is 12.87% in the CPI-E but is a higher 16.56% in the CPI-U. Needless to say, any financial instrument – such as Inflation-Linked Bond Funds (ILBF) – whose returns are linked to the CPI create an imperfect hedge for a typical retiree's personal inflation rate.

Table #1 Placed Here

Figure #2 illustrates the extent to which the investment returns from holding an ILBF might differ from changes (or shocks) to the inflation rate for the elderly. It displays the

[1]We obviously don't want to get caught-up in the minutia of inflation measurement and forecasting. Technically though, the CPI-W is for a representative household where more than one-half of the household's income must come from clerical or wage occupations and at least one of the household's earners must have been employed for at least 37 weeks during the previous 12 months. The CPI-W's population represents about 32 percent of the total U.S. population and is a subset or part of the CPI-U's (which stands for Urban) population. The CPI-U is used for computing the coupons and inflation-adjustments on Treasury Inflation Protection Securities (TIPS) as well as I-Bonds. From a practical perspective the CPI-W and the CPI-U are much closer to each other in composition, relative to the experimental CPI-E.

ongoing market value of $1 invested in a *hypothetical* ILBF in December 2003, as well as the total change in the CPI-E starting from December 2003. Two additional curves were traced in Figure #2, showing a constant rate of 3% and 4% applied to the initial $1 investment. This hypothetical fund was constructed (by the authors) by averaging the monthly total returns the four most popular inflation-linked bond funds available to retail investors in the U.S. (they are T. Rowe, Fidelity, TIAA CREF and Vanguard). In other words, at the end of the month the portfolio was rebalanced so that 25% was held in each of the four ILBFs. Notice how the value of this investment increases over time, somewhere between 3% and 4% per year, but is *more volatile* and often falls outside this range. In contrast, the CPI-E-adjusted value of the initial $1 stays within this 3% to 4% band.

Figure #2 Placed Here

Figure #3 takes this analysis one step further. It displays the results from regressing the monthly returns from the above-mentioned hypothetical portfolio of ILBFs against the changes in the unique CPI for the elderly. If the fit was perfect, the points would all fall on a 45 degree line. Clearly, this is not the case. The $R^2 = 0.06$, which is an estimate we will use later in our (continuous time) model. We do not have enough data points to conduct a statistically significant analysis for (the lower frequency) annual returns, although the R^2 is higher over longer return windows, as one would expect. On the other hand, we should note that when the analysis underlying Figure #2 and/or Figure #3 is conducted for any of the individual ILBFs, as opposed to the hypothetical average of the four – which diversifies the holder further against risks – the results are even more pronounced. More importantly, each individual retiree will has their own desired weightings for consumer expenditures which might further reduce the correlations between their cost of living (e.g. CPI-ME) and aggregate inflation. Thus, although any given ILBF (or inflation adjusted pension) is expected to increase over time as general prices increase, it is important not to confuse an investment that increases over time with an investment that provides perfect insurance against unexpected shocks.

Figure #3 Placed Here

Once again, we are not the first to point out that personal inflation rates are distinct from macro-economic inflation measures, especially as it pertains to retirees. Just as an example, a group of well-known economists, Cockburn, Douglas, Epstein and Grilliches (1997) used a unique data-set of prescription drug prices to focus on the challenges of measuring inflation for retirees. In fact, some researchers – such as Jennings (2005), or Aziz, Katz and Prisman (2001) – have argued that as a result of these imperfect measures, CPI-linked bonds (which are the building blocks of ILBFs) should be disaggregated into individual sub-components so that consumers can reconstruct the bonds using their preferred weights. In this way a suitably tailored bond fund would generate payments and provide income based on a rate that is linked to the medical care component of the CPI, for example. Retirees, and for that matter any consumer with a basket of consumption that differs from the general CPI, would be able to pick and choose their own ILBF that would hedge their own inflation rate.

And, while this idea is great in theory, consumers and current retirees, who are still waiting for access to this engineering innovation, must contend with a generic ILBF that is weakly correlated to their desired consumption's cost of living adjustments. The pragmatic and practical question is: *How much wealth should they allocate to an imperfect insurance policy? Are they better off taking their chances on a long-run equity risk premium that is expected to grow their portfolio over time?*

This is the impetus for our paper. In the real-world absence of a perfect hedge, we gauge whether retirees are better-off investing in an asset class that is expected to grow over time – such as a diversified equity mutual fund – versus investing in an asset class that may (or may not) hedge a retiree's cost of living. We develop a model which illustrates how the correlation, or lack thereof, between ILBF and a retiree's personal inflation rate impacts the demand and optimal allocation to this asset.

1.1 Our Position within the Literature

Most "lifecycle choice" models are built and calibrated assuming the consumer seeks to maximize their end-of-period utility of wealth or their ongoing utility of consumption under a variety of risk-aversion parameters and inter-temporal rates of substitutions. This approach to the dynamic asset allocation problem was initiated by Merton (1971) and has spawned hundreds of papers using these techniques. See Campbell and Viciera (2002) for an extensive literature review. A recent monograph by Ibbotson, et. al. (2007) takes the same utility-based approach as well. In related literature, Kothari and Shanken (2004) derive an optimal asset allocation with inflation-linked bonds, although not within the context of a retiree facing exogenous liabilities. We do use a number of their parameter estimates for our inputs, but differ in methodology and objective. (In fact, in the appendix to the paper we take this approach.) The "calibrating" asset allocation literature continues to grow, and due to the lack of space for a comprehensive literature review we refer the interested reader to recent papers by Horneff, Maurer and Stamos (2008) or Hoevenaars, Molenaar, Schotman and Steenkamp (2008) as well as Cairns, Blake and Dowd (2006).

Our primary model, however, is predicated on a retiree who seeks to minimize the probability of lifetime ruin, assuming their expenditures are stochastic (but exogenous) in nominal terms. The lifetime ruin minimization (LRM) framework has been adopted or discussed in a number of recent papers. Examples within this literature include the original paper by Browne (1995), Milevsky and Robinson (2000) in a static framework, Young (2006) in a dynamic framework, Moore and Young (2006), Bayraktar and Young (2007) as well as the recent working paper Robinson and Tahani (2007). More broadly, our paper is related to Freedman (2008), Gupta and Li (2007), Stamos (2007) and Haberman and Vigna (2002). In all of these articles, new normative models are proposed for individuals as they transition into retirement under a variety of objective functions. They do not necessarily maximize discounted utility of lifetime consumption, *a la* Merton (1971), but instead develop a more practically intuitive framework. The LRM approach resonates with some fraction of the wealth management community, which is one of the reasons we have adopted it as a framework for this analysis. More importantly, by stripping away time, risk and substitution preferences we are able to directly measure the impact of not having a perfect hedge on the probability of success and the affordability of a given projected liability.

¿From a methodological point of view *the contribution of this paper is to extend the existing literature on ruin-minimizing optimal control to the case where there are three – as opposed to just one – state variables, in an incomplete market.*

The additional level of complexity eliminates the ability to derive analytic solutions and expressions for the optimal control, since we are left with a collection of PDEs. Our paper provides numerical results that help shed light on the optimal demand for CPI-linked bonds, by retirees who are concerned with a simple objective function: minimizing the probability of running out of money before running out of life. Once again, in the appendix to this paper we derive the optimal asset allocation and optimal consumption policy under a more economically-driven objective function in which utility is derived from consumption itself, as opposed to it being (only) a constraint.

The remainder of this paper is organized as follows. Section #2 develops our underlying optimization model. Section #3 presents the derivation of the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the ruin probability and methodologies for solving the equation. Section #4 provides numerical examples and displays relevant results. Section #5 concludes the paper and Section #6 is an appendix in which the utility-maximizing model is derived.

2 The Underlying Model

Consistent with the LRM references cited above, our model retiree starts with initial (nest egg) wealth denoted by $W_0 = w$, from which he/she spends at a rate of \$1 per annum, initially. Practitioners refer to this as an *initial spending rate* of $1/w$. We assume that the retiree's exogenous liabilities or the annual rate of desired consumption – measured in nominal terms – evolve according to the diffusion process:

$$dL_t = \pi L_t dt + \xi L_t dB_t^{\pi}, \quad L_0 = 1 \tag{1}$$

where the parameter π is the expected cost of living adjustment (COLA) which is unique to the retiree, ξ is the volatility rate and B_t^{π} is the Brownian motion driving the uncertainty COLA uncertainty. For example, a retiree might expect his or her spending will increase by $\pi = 4\%$ each year, with a volatility of $\xi = 3\%$. Think of this as a personal inflation index/rate. Figure #2 – displayed earlier – represents one possible sample path of L_t over the Dec/03 to Dec/08 period.

In our simple model, the retiree's net-worth of $W_0 = w$ can be allocated between two types of investment funds. The first is a lower-risk ILBF that evolves according to the diffusion process:

$$dI_t = rI_t dt + \eta I_t dB_t^r, \quad I_0 = 1 \tag{2}$$

where r is the expected return – in nominal terms – and η is the volatility. In this case B_t^r denotes the Brownian motion driving the ILBF. It is important to emphasize that I_t does not represent the price of a particular CPI-linked bond, nor does it represent a coupon rate. Rather, one can best think of I_t as a unit value of a mutual fund[2] that invests in a collection of CPI-linked bonds across various maturities, where all coupons and maturity payments are re-invested in the fund. Figure #2 also presents a possible sample path for I_t, over the same Dec/03 to Dec/08 period. It was clearly more volatile than L_t, and had a low correlation, as per Figure #3. Typical parameter values could be an expected $r = 5\%$ nominal investment return, which would consist of 2% real-return and perhaps 2.5% inflation and a 0.5% inflation risk premium, although the exact decomposition is completely unnecessary for our analysis. The typical volatility η would be in the 8% range. These numbers are consistent with Ibbotson Associates estimates of the long-run volatility of a typical CPI-linked mutual fund, again in nominal terms.

The second investment alternative is an equity-based investment mutual fund that evolves according to the diffusion process:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t^y, \quad Y_0 = 1 \tag{3}$$

where μ is the expected nominal return and $\sigma > 0$ is the volatility of the risky fund. Typical parameter values are between 6% and 11% for the expected fund return μ and 10% to 30% for the fund volatility σ. Nothing new here.

The correlation structure between the Brownian motions driving the uncertainty is as follows. First, the correlation between B_t^π (driving the retiree's personal inflation rate or COLA) and B_t^r (driving the ILBF), is relatively high but not 100%. It is denoted by $\rho_{\pi r}$. Likewise, the correlation between B_t^π and B_t^y (the mutual fund) is obviously lower than $\rho_{\pi r}$, and will be denoted by $\rho_{\pi y}$. Finally, the correlation between B_t^y and B_t^r is denoted by ρ_{ry}. Visually, the correlation structure is as follows:

$\mathbf{L}_t$ (COLA)	$\mathbf{I}_t$ (ILBF)	$\mathbf{Y}_t$ (Stocks)	
1	$\rho_{\pi r}$	$\rho_{\pi y}$	$\mathbf{L}_t$
$\rho_{\pi r}$	1	ρ_{ry}	$\mathbf{I}_t$
$\rho_{\pi y}$	ρ_{ry}	1	$\mathbf{Y}_t$

(4)

[2]We obviously do not attempt to offer or develop a complete model that prices all CPI-linked bonds in the most advanced term-structure sense. For something along those lines, we refer interested readers to Jarrow and Yildirim (2003) for example. We are focused on a basic portfolio of CPI-linked bonds within a mutual fund structure.

which – to be invertible – imposes some natural conditions on the relationship between the three available correlations, which we will address later. In terms of calibration, Kothari and Shanken (2004) estimated the correlation between equity market returns and (hypothetical) CPI-linked funds returns are close to zero, so that $\rho_{ry} \approx 0$. For the most part, we will use this parameter estimate in our numerical examples. Likewise, the correlation $\rho_{\pi r}$ between the CPI-linked fund and the retiree's COLA is a matter of debate – and the core concern within our paper – so we will use two extreme cases; the first being a low correlation of 40% and the second being a high correlation of 95%. Finally, the correlation between the investment fund and the liabilities is partially constrained by the invertability of the correlation matrix (4), which we discuss later.

Putting this all together, the dynamics for the retiree's investment portfolio will obey:

$$dW_t = \alpha_t W_t \frac{dI_t}{I_t} + (1 - \alpha_t) W_t \frac{dY_t}{Y_t} - L_t dt, \quad W_0 = w \tag{5}$$

where α_t is the fraction of the portfolio that is allocated to the ILBF and $(1 - \alpha_t)$ is the fraction allocated to the investment or equity-based mutual fund. Again, L_t is defined as the consumption rate, i.e., consumption per unit of the time.

With the LRM approach, the retiree's problem is to minimize the probability of ruin over a retirement horizon $T = \infty$, but with a random time of death τ_d, under a given law of mortality. The infinite horizon might sound odd in the context of a human life, but obviously the force of mortality will eventually kill the retiree well before $T = \infty$. So in some sense it is just a technicality. Also, the uncertainty in the length of human life can be hedged using pension annuities and our model can always be framed in terms of the choice between CPI-linked payout annuities and market-linked variable annuities. In other words, whether the pose the problem as an optimal allocations to non-annuitized investments, or to income annuities, the underlying issue is the same.

Formally, the objective function can be stated as:

$$\psi(w, l, t) = \min_{0 \le \alpha_t \le 1} \Pr[\tau_0 < \tau_d | W_t = w, L_t = l, \tau_d > t], \tag{6}$$

where τ_0 is the time at which W_t first becomes zero. The additional constraint $\alpha_t \ge 0$ is imposed to preclude any short positions in the ILBF and $\alpha_t \le 1$ is imposed to preclude any leveraged position. Our technical objective is to locate the optimal control $\alpha_{t*}(w, t)$

as a function of the drift and diffusion parameters $\mu, \sigma, \pi, \xi, r, \eta$ as well as the correlation parameters $\rho_{\pi r}$, $\rho_{\pi y}$ and ρ_{ry}. As we have stated earlier, we are particularly interested in how $\alpha_{t*}(w, t)$ varies with respect to changes in the correlation $\rho_{\pi r}$ between the CPI and retiree's COLA or personal inflation rate.

Before we proceed to solve for the optimal control and the corresponding minimal ruin probability, we note the following qualitative observations. The qualitative interpretation of $\xi = 0$ is that the retiree's consumption liabilities will not fluctuate in nominal terms, although they can certainly fluctuate in CPI-adjusted terms. Likewise, the interpretation of $\eta = 0$ is that the ILBF grows deterministically over time at a rate or r, which means that there is no CPI risk in the system. Finally, when both $\xi = 0$ (i.e. no liability volatility) and $\eta = 0$ (no CPI risk), our problem boils down to minimizing the probability of ruin in a complete market, which most recently was examined by Moore and Young (2006). This paper extends those models to multiple state variables and incomplete markets.

Keep in mind that the absence of any risk-free asset in either nominal or personal-inflation adjusted terms, implies that the retiree can never reduce the ruin probability $\psi(w, t)$ to zero by immunizing (e.g. annuitizing) the consumption liability stream. The problem boils down to an economic trade-off between: 1) investing retirement wealth in an asset Y_t that is expected to earn $\mu > \pi$, and is greater than the projected increase in the retiree's cost of living, and 2) investing in the relatively safe (and highly correlated) asset I_t, that is not expected to earn as much as Y_t, since $r < \mu$. Hence the title of this paper; should retirees insure against their inflation rate by allocating a substantial portion of their wealth to an imperfect hedge, or should they just "worry" about it, and instead invest more aggressively.

3 Derivation and Solution of the HJB

In this section, we derive the Hamilton-Jacobi-Bellman (HJB) equation satisfied by $\psi(w, l, t)$ and discuss methodologies for solving the HJB.

3.1 Derivation of the HJB

The derivation is based on the method in Björk (1998, chapter 14). Assume an individual of age a at time zero. At any time t the individual aged $a + t$ continues the optimal asset allocation strategy within the range $0 \leq \alpha \leq 1$ if they survive until $t+h$. On the other hand, if the individual aged $a+t$ dies before time $t+h$, then the probability of ruin is zero. Let $\psi(w,l,t)$ denote the minimum ruin probability, i.e.,

$$\psi(w,l,t) = \min_{0\leqslant\alpha_t\leqslant 1} Pr[\tau_0 < \tau_d | W_t = w, L_t = l, \tau_d > t]. \tag{7}$$

Therefore,

$$\psi(w,l,t) \leq E[\psi(W_{t+h}, L_{t+h}, t+h)|W_t = w, L_t = l]_h p_{a+t} \tag{8}$$

$$+E[0|W_t = w, L_t = l]_h q_{a+t} \tag{9}$$

where the equality holds if and only if the optimal strategy is used. Here the symbols ${}_hp_{a+t}$ and ${}_hq_{a+t} := 1 - {}_hp_{a+t}$ are standard actuarial notation for the probabilities of surviving and dying between the time period h for an individual aged $a+t$, given by

$${}_hp_a = e^{-\int_0^h \lambda_{a+s} ds},$$

where λ_t is the instantaneous hazard rate of the force of mortality. Now, let $\mathcal{A}$ denote the operator defined as follows

$$\begin{aligned}\mathcal{A}f(w,l,t) &= f_t + [\alpha rw + (1-\alpha)\mu w - l]\, f_w \\ &+ \frac{1}{2}\left\{(\alpha\eta)^2 + [(1-\alpha)\sigma]^2 + 2\alpha(1-\alpha)\rho_{ry}\eta\sigma\right\} w^2 f_{ww} \\ &+ \pi l f_l + \frac{1}{2}(\xi l)^2 f_{ll} + [\alpha\rho_{\pi r}\eta + (1-\alpha)\rho_{\pi y}\sigma]\xi w l f_{wl},\end{aligned} \tag{10}$$

where f_t, f_w, f_l, etc. denote derivatives with respect to time, wealth and liability, respectively.

Using Ito's lemma on the function ψ, we have

$$\begin{aligned}\psi(W_{t+h}, L_{t+h}, t+h) &= \psi(w,l,t) + \int_t^{t+h} \mathcal{A}\psi(w_s, l_s, s) ds \\ &+ \int_t^{t+h} \psi_w(w_s, l_s, s)[\alpha\eta w_s dB_s^r + (1-\alpha)\sigma w_s dB_s^y] \\ &+ \int_t^{t+h} \psi_l(w_s, l_s, s)\xi l dB_s^\pi.\end{aligned} \tag{11}$$

Combining equation (8), (10) and (11), we obtain

$$\psi(w,l,t) \leq {}_hp_{a+t}\, E^{w,l,t}\left[\psi(w,l,t) + \int_t^{t+h} \mathcal{A}\psi(W_s, L_s, s) ds\right], \tag{12}$$

or

$$\psi(w,l,t)_h q_{a+t} \leq \ _h p_{a+t}\ E^{w,l,t}\left[\int_t^{t+h} \mathcal{A}\psi(W_s,L_s,s)ds\right]. \tag{13}$$

Taking the limit of $h \to 0$ and using $\lim_{h\to 0} h^{-1}{}_h q_{a+t} = \lambda_{a+t}$, we obtain

$$\psi(w,l,t)\lambda_{a+t} \leq \mathcal{A}\psi(w,l,t) \tag{14}$$

and the equality holds when the optimal strategy is used, which is either given by the first order condition or the boundary values $\alpha = 0$ or 1. Thus we obtain the HJB equation

$$\psi(w,l,t)\lambda_{a+t} = \min_{0\leq\alpha\leq 1}\{\mathcal{A}\psi(w,l,t)\}. \tag{15}$$

The boundary conditions are

$$\psi(\infty,l,t) = 0,\ \psi(0,l,t) = 1. \tag{16}$$

Assuming an infinite time horizon, the terminal condition is

$$\psi(w,l,\infty) = 0. \tag{17}$$

When the mortality rate λ_{a+t} is a constant, the infinite time horizon case leads to a time-independent version of (15) by dropping the term ψ_t. However, this is not the case (for realistic biological mortality) where the variable λ_{a+t} is time dependent and the solution ψ is therefore also time-dependent. In this case, time is equivalent to age, and we can rewrite the Hamilton-Jacobi-Bellman equation (15) using age a as the time variable

$$\lambda_a\psi = \psi_a + \min_{0\leq\alpha\leq 1}\mathcal{H}, \tag{18}$$

where

$$\begin{aligned}\mathcal{H} :=& \left[\alpha r w + (1-\alpha)\mu w - l\right]\psi_w + \pi l \psi_l \\ &+ \frac{1}{2}\left\{(\alpha\eta)^2 + [(1-\alpha)\sigma]^2 + 2\alpha(1-\alpha)\rho_{ry}\eta\sigma\right\} w^2\psi_{ww} \\ &+ \frac{1}{2}(\xi l)^2\psi_{ll} + [\alpha\rho_{\pi r}\eta + (1-\alpha)\rho_{\pi y}\sigma]\xi w l \psi_{wl}.\end{aligned} \tag{19}$$

The boundary conditions are still:

$$\psi(\infty,l,a) = 0,\ \psi(0,l,a) = 1,\ \psi(w,l,\infty) = 0. \tag{20}$$

3.2 Solution procedure

By introducing a similarity variable $x = w/l$, which is the ratio of wealth to the liabilities, it can be shown that the now two dimensional equation $\psi(x, a)$ satisfies

$$\lambda_a \psi = \psi_a + \mathcal{H}_*, \tag{21}$$

where

$$\begin{aligned}\mathcal{H}_* := \frac{1}{2}\left\{\left[\alpha_*^2\eta^2 + (1-\alpha_*)^2\sigma^2 + 2\alpha_*(1-\alpha_*)\rho_{ry}\eta\sigma\right] + \xi^2 - 2\left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi\right\} x^2\psi_{xx} \\ + \left\{\left[\alpha_* r x + (1-\alpha_*)\mu x - 1 - \pi x\right] + \xi^2 x - \left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi x\right\}\psi_x,\end{aligned} \tag{22}$$

and the value of α_* is either the boundary value (0 or 1) or given by the first order condition

$$\alpha_* = -\frac{(\rho_{\pi y}\sigma - \rho_{\pi r}\eta)\xi + r - \mu}{\eta^2 + \sigma^2 - 2\rho_{ry}\eta\sigma}\frac{\psi_x}{x\psi_{xx}} - \frac{\sigma(\rho_{ry}\eta - \sigma) + (\rho_{\pi y}\sigma - \rho_{\pi r}\eta)\xi}{\eta^2 + \sigma^2 - 2\rho_{ry}\eta\sigma}. \tag{23}$$

There are two ways one can use to actually derive equation (21). By treating $X_t = W_t/L_t$ as the main stochastic variable and applying the Ito's lemma

$$\begin{aligned}dX_t = \left\{\left[\alpha_* r + (1-\alpha_*)\mu X_t - 1 - \pi X_t\right] + \xi^2 X_t - \left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi X_t\right\} dt \\ - \xi X_t dB_t^\pi + \alpha\eta X_t dB_t^r + (1-\alpha_*)\sigma X_t dB_t^y,\end{aligned} \tag{24}$$

we can follow the same procedure outlined in Section 3.1, combined with the first order condition. That will lead to (21). Alternatively, one can simply apply the chain-rule to (19) and then use the first order condition (23). The derivations are straightforward in both cases and the details are omitted in this paper. The boundary conditions are

$$\psi(0, a) = 1,\ \psi(\infty, a) = 0,\ \psi(x, \infty) = 0. \tag{25}$$

To solve equation (21) numerically we need to truncate the domain in x as well as y and replace the boundary conditions in (25) by;

$$\psi(0, a) = 1,\ \psi(x_{max}, a) = 0,\ \psi(x, a_{max}) = 0 \tag{26}$$

where x_{max} and a_{max} are relatively large numbers. The computational results presented in this paper are obtained by choosing $x_{max} = 200$ and $a_{max} = 125$, based on the observation that further increasing the size of the domain produces almost identical solutions.

There are two possible ways to solve (21), one by discretising the time derivative ψ_a *explicitly* while the other by using an *implicitly* method. In both cases, we discretize a as $a_n = a_0 + n\delta a$ for $n = 0, 1, ..., N$, where $\delta a = (a_{max} - a_0)/N$, a_0 is the current age of the retiree.

3.2.1 Explicit method

Let $\psi^{(n)}$ denote ψ at time (age) a_n. The HJB (18) is approximated by

$$\psi^{(n+1)} = \psi^{(n)} + \delta a F(\psi^{(n)}), \tag{27}$$

$$\begin{aligned} F(\psi^{(n)}) = & -\lambda_a \psi^{(n)} \\ & + \frac{1}{2}\left\{ \left[\alpha_*^2\eta^2 + (1-\alpha_*)^2\sigma^2 + 2\alpha_*(1-\alpha_*)\rho_{ry}\eta\sigma\right] \right. \\ & \quad \left. + \xi^2 - 2\left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi \right\} x^2 \psi_{xx}^{(n)} \\ & + \left\{ \left[\alpha_* r x + (1-\alpha_*)\mu x - 1 - \pi x\right] + \xi^2 x \right. \\ & \quad \left. - \left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi x \right\} \psi_x^{(n)}. \end{aligned} \tag{28}$$

Let $\psi^{(n,*)}$(x) denotes the solution obtained using (28) and the first order condition (23), $\psi^{(n,0)}(x)$ and $\psi^{(n,1)}(x)$ the solution of (28) using $\alpha_* = 0$ or $\alpha_* = 1$. We choose the final solution as

$$\psi^{(n)}(x) = \min\left\{\psi^{(n,*)}(x), \psi^{(n,0)}(x), \psi^{(n,1)}(x)\right\}.$$

This method is conceptually simple and easy to implement. However, it is not efficient as small δa has to be used due to stability constraint associated with the explicit method.

3.2.2 Implicit method

Alternatively, we can use an implicit method by approximating the equation (18) as

$$\psi^{(n+1)} = \psi^{(n)} + \delta a F(\psi^{(n+1)}), \tag{29}$$

$$\begin{aligned} F(\psi^{(n+1)}) = & -\lambda_a \psi^{(n+1)} \\ & + \frac{1}{2}\left\{ \left[\alpha_*^2\eta^2 + (1-\alpha_*)^2\sigma^2 + 2\alpha_*(1-\alpha_*)\rho_{ry}\eta\sigma\right] \right. \\ & \quad \left. + \xi^2 - 2\left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi \right\} x^2 \psi_{xx}^{(n+1)} \\ & + \left\{ \left[\alpha_* r x + (1-\alpha_*)\mu x - 1 - \pi x\right] + \xi^2 x \right. \\ & \quad \left. - \left[\alpha_*\rho_{\pi r}\eta + (1-\alpha_*)\rho_{\pi y}\sigma\right]\xi x \right\} \psi_x^{(n+1)}. \end{aligned} \tag{30}$$

The simple approach used in the explicit method is not applicable here as we have to determine α_* before computing the probability $\psi^{(n+1)}$. In fact, we need to use the second order condition $\mathcal{H}_{\alpha\alpha} > 0$ at the minimum value. It can be verified that

$$\mathcal{H}_{\alpha\alpha} = (\eta^2 + \sigma^2 - 2\rho_{ry}\eta\sigma)^2 x^2 \psi_{xx}.$$

Note that ψ is not convex and $\mathcal{H}_{\alpha\alpha}$ can change signs. Therefore, we need both the first and second order conditions to determine the value of α_*.

- If $\mathcal{H}_{\alpha\alpha} > 0$ and α_* given by (23) satisfies $0 \leq \alpha_* \leq 1$, then that is the value for α_*;
- If $\mathcal{H}_{\alpha\alpha} > 0$ and α_* given by (23) satisfies $\alpha_* > 1$, then $\alpha_* = 1$;
- If $\mathcal{H}_{\alpha\alpha} > 0$ and α_* given by (23) satisfies $\alpha_* < 0$, then $\alpha_* = 0$;
- If $\mathcal{H}_{\alpha\alpha} \leq 0$, we need to compare $\mathcal{H}_\alpha$ at $\alpha = 0$ and $\alpha = 1$ and choose the α value which gives the smaller $\mathcal{H}_\alpha$.

By comparison, this implicit method is much more efficient than the explicit one presented earlier since a relatively large δa can be used for the computation.

4 Numerical Solutions and Examples

Before we present the results obtained by solving the HJB equation derived in the previous section we provide the rationale for the choice of relevant parameter values. The first risky asset Y_t, is a classical investment mutual fund that is expected to earn a nominal μ between 6% and 11% with a volatility of $\sigma = 20\%$. The second risky asset is the ILBF I_t, whose expected (nominal) return is $r = 5\%$ with a volatility of $\eta = 8\%$. In most of our numerical examples we assume that the two risky assets (ILBF and equity/stock fund) are uncorrelated (even though our methodology can readily handle any correlation between the two). In fact, one of the benefits of using two uncorrelated assets is that this deliberately biases our results away from investing in the equity-based fund Y_t as a hedge against the retiree's liabilities.

The retiree starts with a consumption liability rate of $L_0 = \$1$ per year at retirement but at a varying level of initial wealth W_0, allocated amongst Y_t and I_t. The retiree withdraws a stochastic $L_t, t > 0$ per year. As discussed within the derivation, we assume that $0 \leq \alpha_t \leq 1$, and thus prohibit leverage and short-sales. We assume that L_t increases by $\pi = 4\%$ per annum, under two separate cases: volatility of $\xi = 3\%$ (high) and 1% (low). We assume that the CPI-linked fund has a higher expected return r, relative to the liabilities and with greater volatility $\eta = 8\%$. Ultimately, we are interested in the minimal probability of ruin assuming the correlation between liabilities L_t and bond funds returns I_t is $\rho_{\pi r} = 95\%$ (high) and 40% (low). Moreover, we are interested in how the decreasing return of the equity-based asset will force a greater allocation to the bond-fund. We are also interested in the minimal ruin probability as a function of the correlation between the CPI-based bond fund and the consumption liability $\rho_{\pi r}$. The value of the remaining correlation coefficient is taken to be $\rho_{\pi y} = 0.25$ which is chosen so that the positivity of the correlation matrix is guaranteed.

All the computations are carried out with an initial retirement age of $a = 65$ and a time horizon of $T = 60$ (a maximum age of 125). We have experimented with a longer time horizon and the results are virtually identical, since obviously the retiree dies well before $T = 60$. We used the so-called Gompertz mortality rate in the numerical results, which implies that the actuarial force of mortality satisfies the following equation:

$$\lambda_a = \lambda_0 + \frac{1}{b}\exp\left(\frac{a-m}{b}\right) \tag{31}$$

The hazard/mortality rates becomes sufficiently large for a large T, so the survival rate is negligible after 125 years of life. For the result we displayed we used the set of parameter values: $m = 86.3$, $b = 9.5$ and $\lambda_0 = 0.003$. The source for these numbers is Milevsky (2006). Also, to carry out the computations, we have replaced the infinite domain $0 \le x < \infty$ by a finite one $0 \le x \le x_{max}$. We found that as long as x_{max} is sufficiently large, the effect on the solution is small. We have chosen $x_{max} = 200$. This is equivalent to assuming that the ruin probability is zero when the wealth to liability ratio is 200. A total of 1000 grid points are used for the state variable $x = w/l$ and 240 time steps in time. We also experimented with larger numbers of grid points in x and a and almost identical results were obtained. Finally, we have also compared the solutions obtained by explicit and implicit methods. Again, the results are virtually the same.

Table #2a Placed Here

In Table # 2a we present the computed values of the minimum ruin probability and the asset allocation percentage α_* for a high correlation $\rho_{\pi r} = 95\%$ between the COLA and the

CPI-linked fund, together with a high volatility for the consumption liability $\xi = 3\%$. We are interested in the interaction of the assumed equity return μ and the initial spending rate $1/w$.

Here is how to interpret the results. A 65 year-old retiree with initial investment wealth (nest egg) of \$1,000,000 wants to spend/consume \$40,000 per year adjusted for personal inflation. In the language of our model, this is a desired spending rate of 4% and is equivalent to an initial wealth to consumption ratio of $W_0 = 25$. The retiree estimates that this liability L_t will increase on average by $\pi = 4\%$ per year (using continuous compounding) with a volatility of $\xi = 3\%$. The retiree has a choice of investing wealth between a CPI-linked mutual fund that is expected to earn $r = 5\%$ per year with a volatility of $\eta = 8\%$, and a riskier equity based fund. The fund has a volatility of $\sigma = 20\%$, and the expected return is denoted by μ. Tables #2a and 2b display the optimal allocation to the CPI-linked fund depending on the retiree's subjective estimate of the expected return μ. For example, if the equity fund is expected to earn a mere $\mu = 6\%$, which is 100 basis point more than the CPI-linked fund, the initial optimal allocation to the CPI-linked fund is 72.6%. The other 27.4% is allocated to the equity fund. Obviously, since the expected return μ from the equity fund is low the majority of the nest egg should rationally be placed in the CPI-linked fund. In this case, under the optimal strategy, the probability of ruin is 24.4%. This is the lowest possible ruin probability – from the set of all possible dynamic investment strategies – given the specified parameters. Now, if the retiree expects the return on the equity fund to be $\mu = 11\%$, which is 500 basis points higher than the previous example, then the optimal allocation the CPI-linked fund is a (much) lower 56%. This means that 64% should be allocated to equity. Naturally, if one is more bullish about equity, the allocation to equity is greater. In this case, the minimal ruin probability is a mere 6.5% as opposed to 24.4%.

Table #2b Placed Here

All of the numbers in Table #2a assume that the correlation between the (shocks to) retirement liabilities L_t and the CPI-linked fund returns are quite high, at $\rho_{\pi r} = 95\%$. To contrast this case, in Table # 2b we display the results of the minimum ruin probability ψ and optimal asset allocation percentage α_* for a lower correlation of $\rho_{\pi r} = 40\%$ between the personal inflation rate shocks and the returns to the CPI-linked fund. We still assume the (high) volatility of personal inflation, $\xi = 3\%$. In this case, notice that for the same earlier-mentioned $W_0 = 25$, which is a spending rate of 4%, the optimal allocations to the CPI-linked fund are (generally, although not always) lower. The minimal ruin probabilities ψ are uniformly higher in Table #2b compare to Table #2a. The intuition is simple. If the CPI-linked fund is a "worse" hedge for the liabilities, the probability of ruin is higher, even under the most efficient or optimal investment strategy.

Table #3 Placed Here

In Table #3 we present a different perspective on the results, by comparing the allocations as a function of perceived equity-market volatility. For example, a retiree who expects equities to earn $\mu = 10\%$ and market volatility to be on the order of $\sigma = 20\%$, should allocate 72% of his nest egg to CPI-linked bonds. However, if the volatility can be controlled and is reduced to a mere $\sigma = 10\%$ annualized, then the optimal allocation to the CPI-linked fund falls to 48%. And, these results are obtained when the correlation between personal inflation and returns to the CPI-linked fund are high at $\rho_{\pi r} = 95\%$. When the correlation is much weaker at $\rho_{\pi r} = 25\%$, the optimal allocations to the CPI-linked fund drop to between 32% and 65%.

Finally, in order to provide insight on how a dynamic (versus static) asset allocation strategy helps to lower the ruin probability ψ, we have plotted several key quantities in Figure #4. The underlying parameter values are the same as those in Table #2a with an expected equity return of $\mu = 8\%$. First, the minimum ruin probability ψ as well as the ruin probabilities assuming a 100% allocation to the ILBF ($\alpha = 1$) or 100% allocation to the equity-based fund ($\alpha = 0$) are presented in Figure #4a. Notice how the optimal strategy results in a ruin probability that is lower than either of the extreme allocations. Then, the variation of the minimum ruin probability with age is plotted in Figure #4b and the dynamics of the optimal asset allocation ratio α_* at different ages is plotted in Figure #4c. To gain some intuition for Figures #4b and #4c one should focus on the x-axis point where w/l is 20 units. This represents an individual with assets worth 20 times their current level of consumption. At the age of 80, this is a very fortunate situation since the ruin probability is less than 5% in Figure #4b. However, at the age of 65, this wealth to liability ratio induces a ruin probability of approximately 20%. Obviously, as the wealth to liability ratio increases, the entire vector of ruin probabilities declines, and reaches zero at approximately 50-to-1.

Figure #4a, #4b, #4c Placed Here

Notice in Figure #4c, the optimal allocation to the CPI-linked fund, which is represented in the y-axis, depends on age but asymptotes to about 80% when the wealth ratios are very high. At lower levels of wealth, the allocation to the CPI-linked fund is zero, especially at younger ages.

One of the main objectives of this study is to investigate the "correlation impact." To provide a graphical illustration we have plotted α_* (asset allocation) and ψ (ruin probability) against the wealth to liability ratio w/l for three different correlation scenarios: $\rho_{\pi r} = 0$, 0.4, 0.95. The results are displayed in Figure #5. This allows us to directly observe (only) the impact of the correlation between the personal inflation rate and the returns to the CPI-linked fund. One of the more noticeable results is the reversal of the asset allocation strategy – from safety to gambling – at a critical value of w/l. This is typical of all ruin minimization models, see Browne (1999) for example, where at some point the retiree decides to gamble in order to make ends meet.

Figure #5a, #5b Placed Here

In sum, we obviously don't expect the normative model we developed here to drive actual asset allocation recommendations at retirement, especially given the rather restricted nature of the assumed price dynamics (Geometric Brownian Motion) and investment choices. Furthermore, our selection of the *minimum ruin* criteria, as opposed to a more economically-driven *utility maximization* criteria (which we derive in the appendix) is meant to flush-out the following point. Even the most risk averse preferences – i.e. "my main goal is to never run out of money" – do not necessarily induce a high allocation to the so-called safest asset, *when the hedging and insurance aren't perfect.* To some readers this result might be obvious in an incomplete market, but we believe that it is worth emphasizing and illustrating in the context of inflation and the cost of living in retirement. There is no guarantee that an ILBF will hedge personal liabilities. In our opinion, it should be treated as just another asset class in a liability-driven investment portfolio stategy, but with a unique set of risk, return and correlation parameters.

5 Final Words

It is not widely known outside the specialized world of bond-traders and fed-watchers, that inflation as measured for retirees is different from and mostly higher than the macro-economic inflation rate of the aggregate population. In the U.S. the consumer price index (CPI) has a lesser-known relative, which is called the CPI-E (for the elderly) in which the sub-component weights are based on the consumption patterns of Americans above the age of 62. The CPI-E has consistently outpaced the CPI (both the W and U version) over the last 25 years that it has been measured, primarily due to the greater weight placed on health-care expenses, long-term care facilities, the utilization of nursing services, etc.

Indeed, age specific inflation is not a new phenomena for researchers, however it does suggest that investments linked to the CPI – such as retail mutual funds – might not be a good hedge for individual retirees' unique cost of living. To focus on this idea in a more rigorous manner, this paper extends lifetime ruin minimization (LRM) techniques – recently studied by Moore and Young (2006), Young (2004) as well as Browne (1995) – to investigate the optimal asset allocation between a ILBF and a generic investment fund for a retiree facing an exogenous liability stream that is imperfectly correlated to the CPI-linked fund. Our simple model trades-off the benefits of an imperfect insurance hedge against the risky potential for investment growth. In the appendix we examine a similar model assuming the maximization of the utility of lifetime consumption. Our objective is to help understand the tradeoff between insurance and speculation.

Either way our results suggest that CPI-linked funds should be treated as just another asset class in the broad optimization problem, as opposed to a special or unique category. Thus, retirees should obviously include CPI-linked mutual funds and investments in their portfolio optimization universe, but not disproportionately so, since they sacrifice the long-run equity risk premium if they are over-exposed. In other words, your inflation-linked assets might not keep-up with your liability's inflation rate.

References

[1] Amble, A. and Stewart, K. (1994), Experimental price index for elderly consumers, *Monthly Labor Review*, May 1994, pg. 11-16.

[2] Aziz, A., Katz, E. and Prisman, E. Z. (2001), Managing the risk of relative price changes by splitting index-linked bonds, *Journal of Risk,* Vol. 3(4), pg. 69-88.

[3] Bayraktar, E. and V.R. Young (2007), Correspondence between lifetime minimum wealth and utility of consumption, *Finance and Stochastics*, Vol. 11(2), page 213-236.

[4] Bodie, Z. and C. Clowes (2005), *Risk Free Investing,* Pearson Education, FT/Press

[5] Browne, S. (1995), Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, Vol. 20(4), pg. 937-958.

[6] Browne, S. (1999), The risk and reward of minimizing shortfall probability, *Journal of Portfolio Management*, Vol. 25(4): 76-85.

[7] Cairns, A.J.G., Blake, D. and Dowd, K. (2006), Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans, *Journal of Economic Dynamics and Control*, Vol. 30, pg. 843-877

[8] Campbell, J. and Viciera, L. (2002), *Strategic Asset Allocation: Portfolio Choice for Long Term Investors*, Oxford University Press.

[9] Ernst, B., Cockburn, I., Douglas, C., Epstein, A. and Grilliches, Z. (1997), Is price inflation different for the elderly? A empirical analysis of prescription drugs, *National Bureau of Economic Research (NBER) Working Paper 6182.*

[10] Freedman, B. (2008), Efficient post-retirement asset allocation, *North American Actuarial Journal*, Vol. 12(3), pg. 228-241.

[11] Gupta, A. and Z. Li (2007), Integrated optimal annuity planning with consumption-investment selections in retirement planning, *Insurance: Mathematics and Economics*, Vol. 41(1), 96-110

[12] Haberman, S. and E. Vigna (2002), Optimal investment strategies and risk measures in defined contribution pension schemes, *Insurance: Mathematics and Economics*, Vol. 31(1), 35-69.

[13] Hoevenaars, P.M.M., Molenaar, R.D.J, Schotman, P.C. and T.B.M. Steenkamp (2008), Strategic asset allocation with liabilities: Beyond stocks and bonds, *Journal of Economic Dynamics and Control*, Vol. 32, pg. 2939-2970.

[14] Hobijn, B. and Lagakos, D. (2003), Social Security and the consumer price index for the elderly, *Current Issues in Economics and Finance*, Vol. 9(5), Federal Reserve Bank of New York

[15] Horneff, W.J., Maurer, R.H. and Stamos, M.Z. (2008), Lifecycle asset allocation with annuity markets, *Journal of Economic Dynamics and Control*, Vol. 32, pg. 3590-3612.

[16] Ibbotson, R.G., M.A. Milevsky, P. Chen and K. Zhu (2007), *Lifetime Financial Advice: Human Capital, Asset Allocation and Insurance*, CFA Institute.

[17] Jarrow, R. and Yildirim, Y. (2003), Pricing treasury inflation protected securities and relative derivatives using HJM model, *Journal of Financial and Quantitative Analysis*, Vol. 38(2), pg. 337-358.

[18] Jennings, W. (2006), Disaggregated TIPS: The case for disaggregating inflation linked bonds into bonds linked to narrower CPI components, *Journal of Pension Economics and Finance,* Vol. 5(3), pg. 325-343

[19] Kothari, S.P. and J. Shanken (2004), Asset allocation with inflation-protected bonds, *Financial Analysts Journal*, Vol. 60(1), pg. 54-70.

[20] Merton, R. C. (1971), Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory*, Vol. 3 (December): 373-413.

[21] Milevsky, M.A. and C. Robinson (2000), Self-annuitization and ruin in retirement, *North American Actuarial Journal*, Vol. 4(4), pg. 112-129.

[22] Milevsky, M.A. (2006), *The Calculus of Retirement Income: Financial Models for Life Insurance and Pension Annuities*, Cambridge University Press, UK

[23] Moore, K.S. and V.R. Young (2006), Optimal and simple: Nearly optimal rules for minimizing the probability of financial ruin in retirement, *North American Actuarial Journal*, Vol. 10(4), pg. 145-162.

[24] Robinson, C. and N. Tahani (2007), Sustainable retirement income for the socialite, the gardener and the uninsured, *Atkinson College Working Paper.* Available at SSRN: http://ssrn.com/abstract=989165

[25] Young, V.R. (2004), Optimal investment strategy to minimize the probability of lifetime ruin, *North American Actuarial Journal*, Vol. 8(4): 106-126.

[26] House of Representatives (2001), Consumer Price Index for the Elderly Consumers Act, H.R. 2035 (107th Congress)

Table #1

Table #1: Components of the Consumer Price Index (CPI)			
Wage-earners (W) vs. Urban (U) vs. Elderly (E)			
Spending on ...	**% of CPI-W**	**% of CPI-U**	**% of CPI-E**
Apparel	3.57%	3.86%	2.42%
Education & Communication	5.97%	5.51%	3.19%
Food & Beverage	15.10%	16.56%	12.87%
Housing	42.24%	39.96%	47.51%
Medical Care	6.35%	5.27%	10.81%
Recreation	5.38%	4.84%	4.62%
Transportation	17.95%	20.37%	14.99%
Other Goods & Services	3.45%	3.64%	3.59%
Total	100%	100%	100%

These weights are from December 2007 and computed from the Bureau of Labour Statistics (BLS) disclosures in April 2008. Note that weights vary over time based on the consumer expenditure survey conducted by the BLS and with inflation adjustments.

Table #2a: OPTIMAL ALLOCATION TO ILBF								
High ($\rho_{\pi r} = 95\%$) Correlation with Personal Inflation Rate								
High ($\xi = 3\%$) Volatility of Personal Inflation Rate; Expected $\pi = 4\%$								
Expected	$W_0 = 18$		$W_0 = 20$		$W_0 = 22$		$W_0 = 25$	
Investment	**Spend = 5.5%**		**Spend = 5.0%**		**Spend = 4.54%**		**Spend = 4.0%**	
Fund Return	α_*	ψ	α_*	ψ	α_*	ψ	α_*	ψ
$\mu = 6\%$	0	0.478	0.300	0.407	0.541	0.339	0.726	0.244
$\mu = 7\%$	0.076	0.430	0.306	0.359	0.477	0.293	0.642	0.205
$\mu = 8\%$	0.127	0.378	0.312	0.307	0.453	0.244	0.599	0.165
$\mu = 9\%$	0.165	0.324	0.323	0.256	0.445	0.197	0.576	0.126
$\mu = 10\%$	0.201	0.271	0.339	0.206	0.447	0.153	0.564	0.093
$\mu = 11\%$	0.233	0.219	0.357	0.160	0.454	0.114	0.560	0.065
Note: Assets that are not invested in the ILBF								
are allocated to Equity Fund with mean return μ, and volatility $\sigma = 20\%$								
The expected return from ILBF is $r = 5\%$, with volatility $\eta = 8\%$.								
Correlation between ILBF and Equity Fund return is $\rho_{ry} = 0\%$.								
Correlation between personal inflation and Equity Fund return is $\rho_{\pi y} = 25\%$.								

Note that there is a constraint on the third correlation figure, $\rho_{\pi y}$ needed for the positivity of the covariance matrix. For $\rho_{ry} = 0$ and $\rho_{\pi r} = 0.95$, we find that $\rho_{\pi y} < 31.225\%$. In the computations, we have chosen $\rho_{\pi y} = 25\%$. All of these numbers are consistent with – although not identical to – the Ibbotson Associates estimates for the behavior of TIPS-based funds which invest in long maturity CPI-linked bonds. The underlying model assumes the retiree starts with W_0 in liquid assets and spends $L_0 = 1$ per year, which increases by an expected $\pi = 4\%$ with a volatility of ξ per year. The column headed by α_* denotes the optimal allocation and the column headed by ψ denotes the minimum ruin probability.

Table #2b: **OPTIMAL ALLOCATION TO ILBF**								
Low ($\rho_{\pi r} = 40\%$) Correlation with Personal Inflation Rate								
High ($\xi = 3\%$) Volatility of Personal Inflation Rate; Expected $\pi = 4\%$								
Expected	$W_0 = 18$		$W_0 = 20$		$W_0 = 22$		$W_0 = 25$	
Investment	**Spend = 5.5%**		**Spend = 5.0%**		**Spend = 4.54%**		**Spend = 4.0%**	
Fund Return	α_*	ψ	α_*	ψ	α_*	ψ	α_*	ψ
$\mu = 6\%$	0.089	0.479	0.435	0.408	0.624	0.340	0.735	0.249
$\mu = 7\%$	0.143	0.432	0.366	0.361	0.517	0.296	0.645	0.212
$\mu = 8\%$	0.164	0.381	0.341	0.311	0.468	0.249	0.590	0.172
$\mu = 9\%$	0.185	0.328	0.334	0.260	0.444	0.203	0.556	0.135
$\mu = 10\%$	0.206	0.275	0.337	0.211	0.434	0.159	0.535	0.101
$\mu = 11\%$	0.229	0.223	0.344	0.166	0.431	0.120	0.523	0.072
Note: Assets that are not invested in the ILBF are allocated to Equity Fund with mean return μ, and volatility $\sigma = 20\%$								
The expected return from ILBF is $r = 5\%$, with volatility $\eta = 8\%$.								
Correlation between ILBF and Equity Fund return is $\rho_{ry} = 0\%$.								
Correlation between personal inflation and Equity Fund return is $\rho_{\pi y} = 25\%$.								

Note that the third correlation figure, $\rho_{\pi y} < 91.66\%$ is forced by the two other values $\rho_{ry} = 0$ and $\rho_{\pi r} = 0.40$. For consistency, we have chosen the same value as before, $\rho_{\pi y} = 25\%$.

	Table #3: OPTIMAL ALLOCATION TO ILBF								
	Assuming $\underline{W_0 = 33.3}$, initial spending rate of $1/33.3 = 3\%$								
Expected	$\rho_{\pi r} = 0.25$			$\rho_{\pi r} = 0.50$			$\rho_{\pi r} = 0.95$		
Investment	Volatility σ			Volatility σ			Volatility σ		
Fund Return	10%	15%	20%	10%	15%	20%	10%	15%	20%
$\mu = 6\%$	0.485	0.686	0.795	0.521	0.707	0.808	0.590	0.746	0.832
$\mu = 8\%$	0.365	0.577	0.710	0.410	0.604	0.728	0.503	0.660	0.763
$\mu = 10\%$	0.325	0.520	0.654	0.377	0.552	0.676	0.486	0.621	0.721
Cost of living to increase by $\pi = 4\%$ per year, with a volatility of $\xi = 3\%$.									
Expected return from the ILBF is $r = 5\%$ per annum with a volatility of $\eta = 8\%$									
The correlation between the equity fund and the ILBF is $\rho_{ry} = 0\%$									
Correlation between the liability and the investment fund is $\rho_{\pi y} = 25\%$									

This table examines the low (3%) spending rate case.

Table #4a: **OPTIMAL ALLOCATION TO ILBF** ($\beta = 0.5$)								
High ($\rho_{\pi r} = 95\%$) Correlation with Spending/Liabilities								
High ($\xi = 3\%$) Volatility of Spending/Liabilities; $\pi = 4\%$								
Expected	$W_0 = 18$		$W_0 = 20$		$W_0 = 22$		$W_0 = 25$	
Investment	L_0/W_0 = **5.5%**		L_0/W_0 = **5.0%**		L_0/W_0 = **4.54%**		L_0/W_0 = **4.0%**	
Fund Return	α^*	c^*/W_0	α^*	c^*/W_0	α^*	c^*/W_0	α^*	c^*/W_0
$\mu = 6\%$	0.7647	0.0523	0.7600	0.0584	0.7562	0.0633	0.7517	0.0692
$\mu = 7\%$	0.6620	0.0530	0.6532	0.0590	0.6460	0.0640	0.6374	0.0699
$\mu = 8\%$	0.5587	0.0539	0.5458	0.0599	0.5353	0.0649	0.5227	0.0708
$\mu = 9\%$	0.4549	0.0550	0.4381	0.0610	0.4243	0.0660	0.4077	0.0719
$\mu = 10\%$	0.3507	0.0563	0.3298	0.0623	0.3128	0.0673	0.2924	0.0733
$\mu = 11\%$	0.2459	0.0578	0.2211	0.0639	0.2009	0.0689	0.1767	0.0749
Note: Assets that are not invested in the ILBF								
are allocated to Investment Fund with mean return μ, and volatility $\sigma = 20\%$								
The expected return from ILBF is $r = 5\%$, with volatility $\eta = 8\%$.								
Correlation between ILBF and Investment Fund return is $\rho_{ry} = 0\%$.								
Correlation between liability and Investment Fund return is $\rho_{\pi y} = 25\%$.								

Table #4b: **OPTIMAL ALLOCATION TO ILBF** ($\beta = 1.5$)								
High ($\rho_{\pi r} = 95\%$) Correlation with Spending/Liabilities								
High ($\xi = 3\%$) Volatility of Spending/Liabilities; $\pi = 4\%$								
Expected	$W_0 = 18$		$W_0 = 20$		$W_0 = 22$		$W_0 = 25$	
Investment	L_0/W_0 = **5.5%**		L_0/W_0 = **5.0%**		L_0/W_0 = **4.54%**		L_0/W_0 = **4.0%**	
Fund Return	α^*	c^*/W_0	α^*	c^*/W_0	α^*	c^*/W_0	α^*	c^*/W_0
$\mu = 6\%$	0.6733	0.0607	0.6778	0.0659	0.6815	0.0701	0.6859	0.0752
$\mu = 7\%$	0.4898	0.0614	0.4982	0.0666	0.5052	0.0708	0.5135	0.0759
$\mu = 8\%$	0.3067	0.0624	0.3191	0.0676	0.3293	0.0718	0.3415	0.0769
$\mu = 9\%$	0.1242	0.0638	0.1405	0.0689	0.1538	0.0732	0.1698	0.0782
$\mu = 10\%$	-0.0579	0.0655	-0.0378	0.0707	-0.0213	0.0749	-0.0015	0.0799
$\mu = 11\%$	-0.2395	0.0676	-0.2156	0.0727	-0.1960	0.0769	-0.1725	0.0819
Note: Assets that are not invested in the ILBF								
are allocated to Investment Fund with mean return μ, and volatility $\sigma = 20\%$								
The expected return from ILBF is $r = 5\%$, with volatility $\eta = 8\%$.								
Correlation between ILBF and Investment Fund return is $\rho_{ry} = 0\%$.								
Correlation between liability and Investment Fund return is $\rho_{\pi y} = 25\%$.								

6 Appendix: Model of Utility Maximization

As we mentioned in the body of the paper, some readers might be interested in the optimal investment and portfolio choice policy under a more traditional or econonomically-driven objective function in which BOTH consumption and asset allocation is optimized. And, while we have argued for (and believe in) the merits of a Lifetime Ruin Minimization (LRM) approach, in this appendix we derive similar control policies under the alternative Utility of Consumption Maximization (UCM) with constant relative risk aversion (CRRA) preferences. We briefly review the model and introduce new variables before we derive the control policy.

As before, our model starts with initial wealth denoted by $W_0 = w$ The retiree's required and exogenous annual rate of consumption – which is measured in nominal terms – evolved according to the diffusion process:

$$dL_t = \pi L_t dt + \xi L_t dB_t^{\pi}, \quad L_0 = 1 \tag{32}$$

where the parameter $\pi > 0$ is the expected cost of living adjustment (COLA) which is unique to the retiree, while $\xi > 0$ is the volatility rate and B_t^{π} is the Brownian motion driving the uncertainty. As before, the retiree's net-worth of $W_0 = w$ can only be allocated between two types of risky investment funds. The first is a low-risk ILBF:

$$dI_t = rI_t dt + \eta I_t dB_t^r, \quad I_0 = 1 \tag{33}$$

where r is the expected return – in nominal terms – and $\eta > 0$ is the volatility. In this case B_t^r denotes the Brownian motion driving the ILBF. The second investment alternative is a diversified mutual fund that evolves according to the diffusion process:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t^y, \quad Y_0 = 1 \tag{34}$$

where $\mu > 0$ is the expected return and $\sigma > 0$ is the volatility of this risky asset. The correlation structure between the Brownian motions driving the uncertainty is similar to the LRM model. First, the correlation between B_t^{π} (driving the retiree's liability) and B_t^r (driving the ILBF), which is relatively high but not 100%, is denoted by $\rho_{\pi r}$. Likewise, the correlation between B_t^{π} (retiree liabilities) and B_t^y (the equity fund) is obviously lower than $\rho_{\pi r}$, and will be denoted by $\rho_{\pi y}$. Finally, the correlation between B_t^y and B_t^r is denoted by

ρ_{ry}. The dynamics for the retirees investment portfolio W_t will now obey:

$$dW_t = \alpha_t W_t \frac{dI_t}{I_t} + (1-\alpha_t)W_t \frac{dY_t}{Y_t} - (L_t + C_t)dt, \quad W_0 = w \tag{35}$$

where $\alpha_t \geq 0$ is the fraction of the portfolio that is allocated to the ILBF and $(1-\alpha_t)$ is the fraction allocated to the investment mutual fund. Note that in the appendix, and in contrast to the LRM case, $L_t > 0$ is defined as a minimal liability consumption rate and $C_t \geq 0$ is an additional consumption rate above and beyond the required liabilities. This variable C_t is the new ingredient in a UCM vs. LRM approach. The retiree's optimization problem now is to maximize the discounted utility of consumption over a (random) retirement horizon until the time of death τ_d; under a particular law of mortality, or a preset time T, which ever comes earlier.

We assume instantaneous utility of consumption of the CRRA form:

$$u(c) = \frac{c^{(1-\gamma)}}{1-\gamma}. \tag{36}$$

Our most general objective functions is:

$$J = \max_{\{\alpha_s, c_s\}} E_t \left[\int_t^{\min\{T,\tau_d\}} e^{-\delta s} u(C_s + \beta L_s) ds \right], \tag{P}$$

where β is a (subjective) parameter and where T is the time horizon, δ denotes a subjective discount rate on the order of 2% to 4% for the numerical examples. The economic interpretation is as follows. The retiree must consume at least the liability L_t, which can be viewed as substinance (survival, medical needs) consumption. Then, in addition to L_t the retiree can decide to consume $C_t \geq 0$, which would lead to a total instantaneous utility of $u(L_t + \beta C_t)$.

6.1 Optimal Allocation and Consumption

Suppose that at time t the wealth and liability are $W_t = w$ and $L_t = l$, the problem at hand is to find $J(t, w, l)$, subject to the budget constraint, rewritten as

$$dW_t = W_t[\alpha_t r + (1-\alpha_t)\mu - L_t - C_t]dt + W_t[\alpha_t \eta dB_t^r + (1-\alpha_t)\sigma dB_t^y]. \tag{37}$$

Using Bellman's principle, the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\lambda_{a+t} J = J_t + \max_{\{c_t, \alpha_t\}} \mathcal{H}^J, \tag{38a}$$

where λ_{a+t} is the hazard rate for an individual of age a at current t, and

$$\begin{aligned}\mathcal{H}^J &= e^{-\delta t}u(c+\beta l) - cJ_w + [\alpha\mu_x w + (1-\alpha)\mu w - l]J_w + \pi l J_l \\ &+ \frac{1}{2}\left\{(\alpha\eta)^2 + [(1-\alpha)\sigma]^2 + 2\alpha(1-\alpha)\rho_{ry}\eta\sigma\right\} w^2 J_{ww} \\ &+ \frac{1}{2}\xi^2 l^2 J_{ll} + [\alpha\rho_{\pi r}\eta + (1-\alpha)\rho_{\pi y}\sigma]\xi w l J_{wl}.\end{aligned} \tag{38b}$$

If the retiree survives to the end of the time horizon T, we set $J(T,w,l) = \theta u(w)$, where θ is a constant parameter representing the relative importance of the end-of-the-period wealth. We could choose $\theta = 0$ (zero end-of-the-period wealth), $\theta = 1$ or $\theta = \exp(-\delta T)$ (adjusted by the discount rate).

6.2 A special case

We first consider a special case where $\rho_{\pi y} = \rho_{\pi r}\rho_{ry}$, $\rho_{ry} = -1$ and $\rho_{\pi r} = 1$. It can be shown that the value function has an closed form representation

$$J = \frac{h(w+kl)^{1-\gamma}}{1-\gamma} \tag{39}$$

where h and k are functions of t only, which are given by the following *ordinary differential equations*

$$\dot{h} + d_h h + \gamma e^{-\frac{\delta t}{\gamma}} h^{1-\frac{1}{\gamma}} = 0, \tag{40a}$$

$$\dot{k} + d_k k + \beta - 1 = 0, \tag{40b}$$

where

$$d_h = (1-\gamma)\left[\frac{(r-\mu)^2}{2\gamma(\eta-\rho_{ry}\sigma)^2} + \frac{\mu\eta - r\rho_{ry}\sigma}{\eta-\rho_{ry}\sigma}\right] - \lambda_{a+t}, \tag{40c}$$

$$d_k = \theta + \frac{r(\rho_{ry}\sigma - \rho_{\pi r}\xi) - \mu(\eta - \rho_{\pi r}\xi)}{\eta - \rho_{ry}\sigma}. \tag{40d}$$

The terminal conditions for h and k are $h(T) = \theta$ and $k(T) = 0$.

The optimal consumption rate and allocation are given by

$$c_* = e^{-\frac{\delta t}{\gamma}} h^{-\frac{1}{\gamma}}(w+kl) - \beta l, \tag{41a}$$

$$\alpha_* = \frac{r-\mu}{\gamma w(\eta-\rho_{ry}\sigma)^2}(w+kl) - \frac{\rho_{ry}\sigma w + \rho_{\pi r}\xi k l}{w(\eta-\rho_{ry}\sigma)}. \tag{41b}$$

Remark. Note (40a) and (40b) can be solved in closed form when λ_{a+t} is a constant, and the solutions are

$$h = \left[\theta^{\frac{1}{\gamma}} e^{\frac{d_h}{\gamma}(T-t)} + \frac{\gamma}{d_h - \gamma} e^{-\frac{d_h t}{\gamma}} \left(e^{\frac{d_h - \delta}{\gamma} T} - e^{\frac{d_h - \delta}{\gamma} t} \right) \right]^{\gamma}, \tag{42a}$$

$$k = \frac{1-\beta}{d_k} \left(1 - e^{d_k (T-t)}\right). \tag{42b}$$

An immediate observation is that $k = 0$ when $\beta = 1$, in which case the solution is *independent* of liability l. Another observation is that k could be negative for sufficiently large T when $\beta < 1$ and $d_k > 0$, which implies that the solution exists only for the liability to wealth ratio below a critical value, given by $-k(0)$.

6.3 General case: similarity reduction

We postulate that the value function can be written as

$$J = \frac{w^{1-\gamma}}{1-\gamma} F(t, z) \tag{43}$$

where $z = l/w$. The HJB equation (38a) becomes

$$\lambda_{t+a} F = F_t + \min_{\{c_t, \alpha_t\}} \mathcal{H}^F, \tag{44a}$$

where

$$\begin{aligned} \mathcal{H}^F &= e^{-\delta t} (\hat{c} + \beta z)^{1-\gamma} - (\hat{c} + \beta z)[-zF_z + (1-\gamma)F] \\ &+ [\alpha r + (1-\alpha)\mu + (\beta - 1) z] [-zF_z + (1-\gamma)F] + \pi z F_z \\ &+ \frac{1}{2} \left\{ (\alpha\eta)^2 + [(1-\alpha)\sigma]^2 + 2\alpha(1-\alpha)\rho_{ry}\eta\sigma \right\} [z^2 F_{zz} + 2\gamma z F_z - \gamma(1-\gamma)F] \\ &+ \frac{1}{2}\xi^2 z^2 F_{zz} + [\alpha\rho_{\pi r}\eta\xi + (1-\alpha)\rho_{\pi y}\sigma\xi](-z^2 F_{zz} - \gamma z F_z) \end{aligned} \tag{44b}$$

with terminal condition $F = \theta$ at $t = T$. Note that we have scaled the consumption rate by the wealth, i.e., $\hat{c} = c/w$. The scaled optimal consumption $\hat{c}_*$ is given by the first order condition

$$\hat{c}_* = -\beta z + e^{-\frac{\delta t}{\gamma}} \left(F - \frac{zF_z}{1-\gamma} \right)^{-\frac{1}{\gamma}} \tag{45}$$

and the optimal allocation α_* is also given by the first order condition

$$\begin{aligned} \alpha_* = &\frac{\sigma^2 - \rho_{ry}\eta\sigma}{\eta^2 + \sigma^2 - 2\rho_{ry}\eta\sigma} \\ &- \frac{(r-\mu)[-zF_z + (1-\gamma)F] - (\rho_{\pi r}\xi\eta - \rho_{\pi y}\xi\sigma)(z^2 F_{zz} + \gamma z F_z)}{(\eta^2 + \sigma^2 - 2\rho_{ry}\eta\sigma)[z^2 F_{zz} + 2\gamma z F_z - \gamma(1-\gamma)F]}. \end{aligned} \tag{46}$$

6.4 Numerical Results

We apply the method of lines to solve the one-dimensional HJB (44a): we approximate the "spatial" derivatives F_z and F_{zz} in (44a), (45) and (46) by finite difference formula and solve the resulting system of ordinary differential equations in time by the Matlab solver `ode45.m`.

We first compare the results for the special case when liability L_t and ILBF I_t are perfectly correlated while the rest are anti-correlated. The numerical results and the closed form solution agree with each other. In Tables 4a and 4b we present the results on consumption and allocation for a typical set of parameter values: $\pi = 5\%$, $\xi = 0.03$, $r = 5\%$, $\eta = 0.08$, $\mu = 6 - 11\%$, $\sigma = 0.2$, $\rho_{ry} = 0$, $\rho_{\pi y} = 0.25$, $\rho_{\pi r} = 0.4 - 0.95$, $\delta = 2\%$, $a = 65$, $T = 75$, $\lambda_0 = 0.003$, $b = 9.5$, $m = 86.3$, $\beta = 0.5 - 1.5$, $\gamma = 1.5$ and $\theta = 1$.

Table #4a, #4b Placed Here

Note that there is a constraint on the third correlation figure, $\rho_{\pi y}$ due to the positivity of the covariance matrix. For $\rho_{ry} = 0$ and $\rho_{\pi r} = 0.95$, we find that $\rho_{\pi y} < 31.225\%$. In the computation, we have chosen $\rho_{\pi y} = 25\%$. The underlying model assumes the retiree starts with W_0 in liquid assets and spends $L_0 = 1$ per year, which increases by an expected $\pi = 4\%$ with a volatility of ξ per year. The column headed by α^* denotes the optimal allocation and the column headed by c^*/W_0 denotes the scaled consumption (in addition to liability).