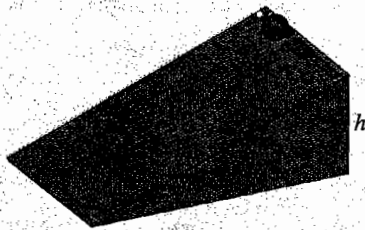


APPLIED  
PROJECT

## ROLLER DERBY

Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question, we consider a ball or cylinder with mass  $m$ , radius  $r$ , and moment of inertia  $I$  (about the axis of rotation). If the vertical drop is  $h$ , then the potential energy at the top is  $mgh$ . Suppose the object reaches the bottom with velocity  $v$  and angular velocity  $\omega$ , so  $v = r\omega$ . The kinetic energy at the bottom consists of two parts:  $\frac{1}{2}mv^2$  from translation (moving down the slope) and  $\frac{1}{2}I\omega^2$  from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

1. Show that

$$v^2 = \frac{2gh}{1 + I^*} \quad \text{where } I^* = \frac{I}{mr^2}$$

2. If  $y(t)$  is the vertical distance traveled at time  $t$ , then the same reasoning as used in Problem 1 shows that  $v^2 = 2gy/(1 + I^*)$  at any time  $t$ . Use this result to show that  $y$  satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}} (\sin \alpha) \sqrt{y}$$

where  $\alpha$  is the angle of inclination of the plane.

3. By solving the differential equation in Problem 2, show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g \sin^2 \alpha}}$$

This shows that the object with the smallest value of  $I^*$  wins the race.

4. Show that  $I^* = \frac{1}{2}$  for a solid cylinder and  $I^* = 1$  for a hollow cylinder.  
 5. Calculate  $I^*$  for a partly hollow ball with inner radius  $a$  and outer radius  $r$ . Express your answer in terms of  $b = a/r$ . What happens as  $a \rightarrow 0$  and as  $a \rightarrow r$ ?  
 6. Show that  $I^* = \frac{2}{5}$  for a solid ball and  $I^* = \frac{2}{3}$  for a hollow ball. Thus the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.

## 15.9

## CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of  $x$  and  $u$ , we can write the Substitution Rule (5.5.6) as

$$\boxed{1} \quad \int_a^b f(x) dx = \int_c^d f(g(u))g'(u) du$$

where  $x = g(u)$  and  $a = g(c)$ ,  $b = g(d)$ . Another way of writing Formula 1 is as follows:

$$\boxed{2} \quad \int_a^b f(x) dx = \int_c^d f(x(u)) \frac{dx}{du} du$$

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables  $r$  and  $\theta$  are related to the old variables  $x$  and  $y$  by the equations

$$x = r \cos \theta \quad y = r \sin \theta$$

and the change of variables formula (15.4.2) can be written as

$$\iint_R f(x, y) \, dA = \iint_S f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

where  $S$  is the region in the  $r\theta$ -plane that corresponds to the region  $R$  in the  $xy$ -plane.

More generally, we consider a change of variables that is given by a **transformation**  $T$  from the  $uv$ -plane to the  $xy$ -plane:

$$T(u, v) = (x, y)$$

where  $x$  and  $y$  are related to  $u$  and  $v$  by the equations

$$\boxed{3} \quad x = g(u, v) \quad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \quad y = y(u, v)$$

We usually assume that  $T$  is a  $C^1$  **transformation**, which means that  $g$  and  $h$  have continuous first-order partial derivatives.

A transformation  $T$  is really just a function whose domain and range are both subsets of  $\mathbb{R}^2$ . If  $T(u_1, v_1) = (x_1, y_1)$ , then the point  $(x_1, y_1)$  is called the **image** of the point  $(u_1, v_1)$ . If no two points have the same image,  $T$  is called **one-to-one**. Figure 1 shows the effect of a transformation  $T$  on a region  $S$  in the  $uv$ -plane.  $T$  transforms  $S$  into a region  $R$  in the  $xy$ -plane called the **image of  $S$** , consisting of the images of all points in  $S$ .

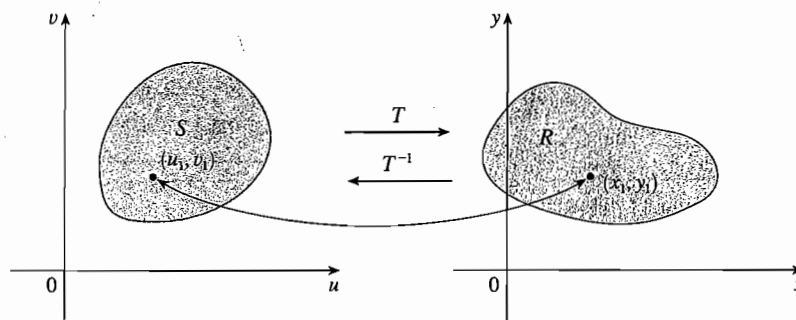


FIGURE 1

If  $T$  is a one-to-one transformation, then it has an **inverse transformation**  $T^{-1}$  from the  $xy$ -plane to the  $uv$ -plane and it may be possible to solve Equations 3 for  $u$  and  $v$  in terms of  $x$  and  $y$ :

$$u = G(x, y) \quad v = H(x, y)$$

▣ **EXAMPLE 1** A transformation is defined by the equations

$$x = u^2 - v^2 \quad y = 2uv$$

Find the image of the square  $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$ .

**SOLUTION** The transformation maps the boundary of  $S$  into the boundary of the image. So we begin by finding the images of the sides of  $S$ . The first side,  $S_1$ , is given by  $v = 0$

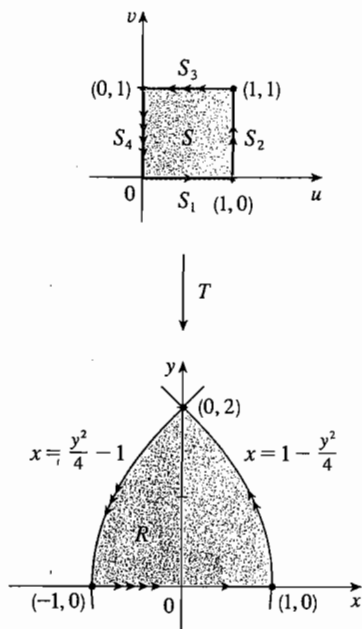


FIGURE 2

( $0 \leq u \leq 1$ ). (See Figure 2.) From the given equations we have  $x = u^2$ ,  $y = 0$ , and so  $0 \leq x \leq 1$ . Thus  $S_1$  is mapped into the line segment from  $(0, 0)$  to  $(1, 0)$  in the  $xy$ -plane. The second side,  $S_2$ , is  $u = 1$  ( $0 \leq v \leq 1$ ) and, putting  $u = 1$  in the given equations, we get

$$x = 1 - v^2 \quad y = 2v$$

Eliminating  $v$ , we obtain

$$\boxed{4} \quad x = 1 - \frac{y^2}{4} \quad 0 \leq x \leq 1$$

which is part of a parabola. Similarly,  $S_3$  is given by  $v = 1$  ( $0 \leq u \leq 1$ ), whose image is the parabolic arc

$$\boxed{5} \quad x = \frac{y^2}{4} - 1 \quad -1 \leq x \leq 0$$

Finally,  $S_4$  is given by  $u = 0$  ( $0 \leq v \leq 1$ ) whose image is  $x = -v^2$ ,  $y = 0$ , that is,  $-1 \leq x \leq 0$ . (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of  $S$  is the region  $R$  (shown in Figure 2) bounded by the  $x$ -axis and the parabolas given by Equations 4 and 5.

Now let's see how a change of variables affects a double integral. We start with a small rectangle  $S$  in the  $uv$ -plane whose lower left corner is the point  $(u_0, v_0)$  and whose dimensions are  $\Delta u$  and  $\Delta v$ . (See Figure 3.)

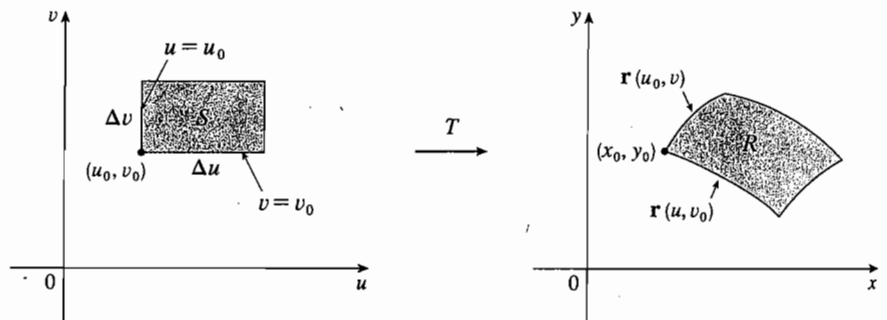


FIGURE 3

The image of  $S$  is a region  $R$  in the  $xy$ -plane, one of whose boundary points is  $(x_0, y_0) = T(u_0, v_0)$ . The vector

$$\mathbf{r}(u, v) = g(u, v) \mathbf{i} + h(u, v) \mathbf{j}$$

is the position vector of the image of the point  $(u, v)$ . The equation of the lower side of  $S$  is  $v = v_0$ , whose image curve is given by the vector function  $\mathbf{r}(u, v_0)$ . The tangent vector at  $(x_0, y_0)$  to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0) \mathbf{i} + h_u(u_0, v_0) \mathbf{j} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j}$$

Similarly, the tangent vector at  $(x_0, y_0)$  to the image curve of the left side of  $S$  (namely,  $u = u_0$ ) is

$$\mathbf{r}_v = g_v(u_0, v_0) \mathbf{i} + h_v(u_0, v_0) \mathbf{j} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j}$$

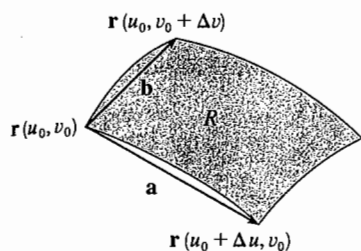


FIGURE 4

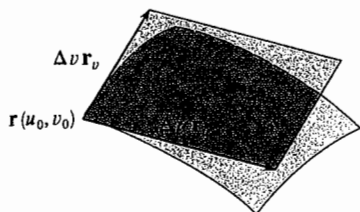


FIGURE 5

We can approximate the image region  $R = T(S)$  by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \quad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

shown in Figure 4. But

$$\mathbf{r}_u = \lim_{\Delta u \rightarrow 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u$$

Similarly

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v$$

This means that we can approximate  $R$  by a parallelogram determined by the vectors  $\Delta u \mathbf{r}_u$  and  $\Delta v \mathbf{r}_v$ . (See Figure 5.) Therefore we can approximate the area of  $R$  by the area of this parallelogram, which, from Section 12.4, is

$$\boxed{6} \quad |(\Delta u \mathbf{r}_u) \times (\Delta v \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

**7 DEFINITION** The **Jacobian** of the transformation  $T$  given by  $x = g(u, v)$  and  $y = h(u, v)$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area  $\Delta A$  of  $R$ :

$$\boxed{8} \quad \Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

where the Jacobian is evaluated at  $(u_0, v_0)$ .

■ The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

Next we divide a region  $S$  in the  $uv$ -plane into rectangles  $S_{ij}$  and call their images in the  $xy$ -plane  $R_{ij}$ . (See Figure 6.)

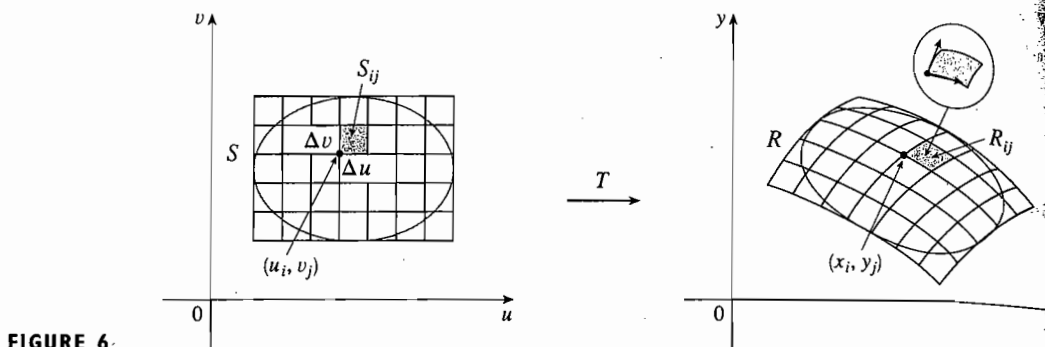


FIGURE 6

Applying the approximation (8) to each  $R_{ij}$ , we approximate the double integral of  $f$  over  $R$  as follows:

$$\begin{aligned} \iint_R f(x, y) \, dA &\approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta A \\ &\approx \sum_{i=1}^m \sum_{j=1}^n f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v \end{aligned}$$

where the Jacobian is evaluated at  $(u_i, v_j)$ . Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$$

The foregoing argument suggests that the following theorem is true. (A full proof is given in books on advanced calculus.)

**9 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL** Suppose that  $T$  is a  $C^1$  transformation whose Jacobian is nonzero and that maps a region  $S$  in the  $uv$ -plane onto a region  $R$  in the  $xy$ -plane. Suppose that  $f$  is continuous on  $R$  and that  $R$  and  $S$  are type I or type II plane regions. Suppose also that  $T$  is one-to-one, except perhaps on the boundary of  $S$ . Then

$$\iint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$$

Theorem 9 says that we change from an integral in  $x$  and  $y$  to an integral in  $u$  and  $v$  by expressing  $x$  and  $y$  in terms of  $u$  and  $v$  and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2. Instead of the derivative  $dx/du$ , we have the absolute value of the Jacobian, that is,  $|\partial(x, y)/\partial(u, v)|$ .

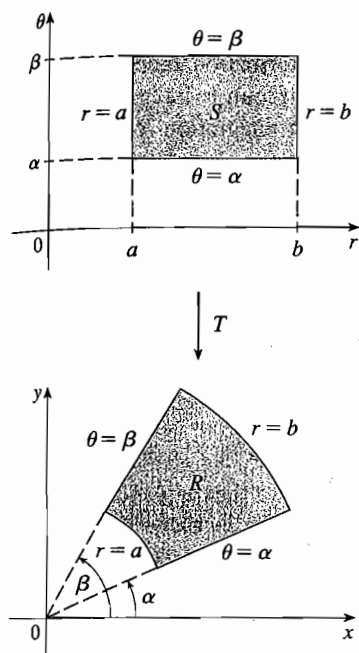


FIGURE 7  
The polar coordinate transformation

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation  $T$  from the  $r\theta$ -plane to the  $xy$ -plane is given by

$$x = g(r, \theta) = r \cos \theta \quad y = h(r, \theta) = r \sin \theta$$

and the geometry of the transformation is shown in Figure 7.  $T$  maps an ordinary rectangle in the  $r\theta$ -plane to a polar rectangle in the  $xy$ -plane. The Jacobian of  $T$  is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r > 0$$

Thus Theorem 9 gives

$$\begin{aligned} \iint_R f(x, y) \, dx \, dy &= \iint_S f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta \\ &= \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \end{aligned}$$

which is the same as Formula 15.4.2.

**EXAMPLE 2** Use the change of variables  $x = u^2 - v^2$ ,  $y = 2uv$  to evaluate the integral  $\iint_R y \, dA$ , where  $R$  is the region bounded by the  $x$ -axis and the parabolas  $y^2 = 4 - 4x$  and  $y^2 = 4 + 4x$ ,  $y \geq 0$ .

**SOLUTION** The region  $R$  is pictured in Figure 2 (on page 1014). In Example 1 we discovered that  $T(S) = R$ , where  $S$  is the square  $[0, 1] \times [0, 1]$ . Indeed, the reason for making the change of variables to evaluate the integral is that  $S$  is a much simpler region than  $R$ . First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\begin{aligned} \iint_R y \, dA &= \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2) \, du \, dv \\ &= 8 \int_0^1 \int_0^1 (u^3v + uv^3) \, du \, dv = 8 \int_0^1 \left[ \frac{1}{4}u^4v + \frac{1}{2}u^2v^3 \right]_{u=0}^{u=1} \, dv \\ &= \int_0^1 (2v + 4v^3) \, dv = [v^2 + v^4]_0^1 = 2 \end{aligned}$$

□

**NOTE** Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If  $f(x, y)$  is difficult to integrate, then the form of  $f(x, y)$  may suggest a transformation. If the region of integration  $R$  is awkward, then the transformation should be chosen so that the corresponding region  $S$  in the  $uv$ -plane has a convenient description.

**EXAMPLE 3** Evaluate the integral  $\iint_R e^{(x+y)/(x-y)} dA$ , where  $R$  is the trapezoidal region with vertices  $(1, 0)$ ,  $(2, 0)$ ,  $(0, -2)$ , and  $(0, -1)$ .

**SOLUTION** Since it isn't easy to integrate  $e^{(x+y)/(x-y)}$ , we make a change of variables suggested by the form of this function:

$$\text{[10]} \quad u = x + y \quad v = x - y$$

These equations define a transformation  $T^{-1}$  from the  $xy$ -plane to the  $uv$ -plane. Theorem 9 talks about a transformation  $T$  from the  $uv$ -plane to the  $xy$ -plane. It is obtained by solving Equations 10 for  $x$  and  $y$ :

$$\text{[11]} \quad x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v)$$

The Jacobian of  $T$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

To find the region  $S$  in the  $uv$ -plane corresponding to  $R$ , we note that the sides of  $R$  lie on the lines

$$y = 0 \quad x - y = 2 \quad x = 0 \quad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the  $uv$ -plane are

$$u = v \quad v = 2 \quad u = -v \quad v = 1$$

Thus the region  $S$  is the trapezoidal region with vertices  $(1, 1)$ ,  $(2, 2)$ ,  $(-2, 2)$ , and  $(-1, 1)$  shown in Figure 8. Since

$$S = \{(u, v) \mid 1 \leq v \leq 2, -v \leq u \leq v\}$$

Theorem 9 gives

$$\begin{aligned} \iint_R e^{(x+y)/(x-y)} dA &= \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_1^2 \int_{-v}^v e^{u/v} \left(\frac{1}{2}\right) du dv = \frac{1}{2} \int_1^2 [ve^{u/v}]_{u=-v}^{u=v} dv \\ &= \frac{1}{2} \int_1^2 (e - e^{-1})v dv = \frac{3}{4}(e - e^{-1}) \end{aligned}$$

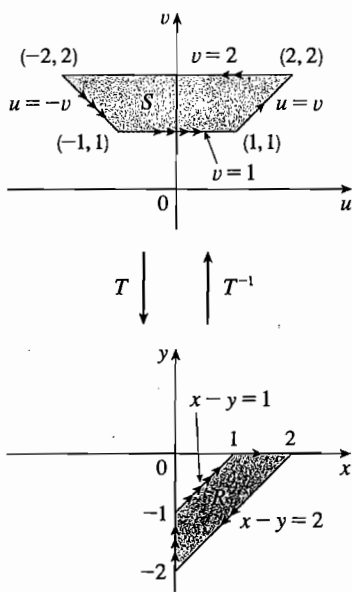


FIGURE 8

## TRIPLE INTEGRALS

There is a similar change of variables formula for triple integrals. Let  $T$  be a transformation that maps a region  $S$  in  $uvw$ -space onto a region  $R$  in  $xyz$ -space by means of the equations

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

The **Jacobian** of  $T$  is the following  $3 \times 3$  determinant:

$$\boxed{12} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

$$\boxed{13} \quad \iiint_R f(x, y, z) \, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw$$

**EXAMPLE 4** Use Formula 13 to derive the formula for triple integration in spherical coordinates.

**SOLUTION** Here the change of variables is given by

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

We compute the Jacobian as follows:

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi \begin{vmatrix} -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \end{vmatrix} - \rho \sin \phi \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\ &= \cos \phi (-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta) \\ &\quad - \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \\ &= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin \phi \sin^2 \phi = -\rho^2 \sin \phi \end{aligned}$$

Since  $0 \leq \phi \leq \pi$ , we have  $\sin \phi \geq 0$ . Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = |-\rho^2 \sin \phi| = \rho^2 \sin \phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z) \, dV = \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

which is equivalent to Formula 15.8.3.  $\square$