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Discrete time hedging with liquidity risk

Hyejin Ku [a], Kiseop Lee [b,c,*], Huaiping Zhu [a]

[a] Department of Mathematics and Statistics, York University, Toronto, ON, Canada M3J 1P3
[b] Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
[c] Graduate Department of Financial Engineering, Ajou University, Suwon, Republic of Korea

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ABSTRACT

We study a discrete time hedging and pricing problem in a market with liquidity costs. Using Leland's discrete time replication scheme [Leland, H.E., 1985. Journal of Finance, 1283–1301], we consider a discrete time version of the Black–Scholes model and a delta hedging strategy. We derive a partial differential equation for the option price in the presence of liquidity costs and develop a modified option hedging strategy which depends on the size of the parameter for liquidity risk. We also discuss an analytic method of solving the pricing equation using a series solution.

1. Introduction

Models in mathematical finance theory have been based on the simplified assumption of the competitive and frictionless market. A frictionless market is one without transaction costs and trade restrictions, and a competitive market is a market where traders act as price takers so that their trades do not have any impact on the price process. Although these simplified models are useful first steps for analyzing markets, we need more realistic models without these assumptions to fit the market phenomena better. Recently, models considering market microstructure effects such as liquidity risk, microstructure noises, and information asymmetry have been popular topics.

Liquidity risk is the additional risk in the market due to the timing and size of a trade. The price process may depend on the activities of traders, especially the trading volume. In the past decade or

* Corresponding author at: Department of Mathematics, University of Louisville, Louisville, KY 40292, USA.
E-mail addresses: hku@mathstat.yorku.ca (H. Ku), kiseop.lee@louisville.edu, kiseop@ajou.ac.kr (K. Lee), huaiping@mathstat.yorku.ca (H. Zhu).

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two, the literature on liquidity risk has been growing rapidly; for example, see Jarrow (1992, 1994, 2001), Back (1993), Frey (1998), Frey and Stremme (1997), Cvitanic and Ma (1996), Subramanian and Jarrow (2001), Duffie and Ziegler (2003), Bank and Baum (2004), and Çetin et al. (2004).

Among these works, Çetin et al. (2004) developed a rigorous model incorporating liquidity risk into the arbitrage pricing theory. They established a mathematical formulation of liquidity costs, admissible strategies, self-financing strategies, and an approximately complete market. They also showed the two (approximately modified) fundamental theorems of finance hold under the existence of liquidity risk. They also studied an extension of the Black–Scholes economy incorporating liquidity risk as an illustration of the theory.

Built on the asset pricing theory developed in Çetin et al. (2004), we study how the classical hedging strategies should be modified and how the prices of derivatives should be changed in a financial market with liquidity costs, especially when we hedge only at discrete time points. We consider a discrete time version of the Black–Scholes model and a multiplicative supply curve. Using the Leland approximation scheme (Leland, 1985), we obtain a nonlinear partial differential equation which requires the expected hedging error to be zero. We provide an approximate method for solving this equation using a series solution.

The remaining of the paper is organized as follows. Section 2 introduces the Çetin–Jarrow–Protter model, and explains our model in detail. Section 3 derives a nonlinear partial differential equation for the option price which includes liquidity costs. Numerical results are also provided. Section 4 studies an analytic solution for the PDE given in Section 3. Section 5 presents some conclusions.

2. Model

2.1. Background on liquidity costs

This subsection recalls the concepts introduced in the work of Çetin et al. (2004). A basic idea is that a buy-initiated order drives the price up since it removes the best ask prices in the limit order book; similarly, a sell-initiated order drives it down.

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbf{P})$ satisfying the usual conditions where T is a fixed time. $\mathbf{P}$ represents the statistical or empirical probability measure. We consider a market with a risky asset (stock) and a money market account. The stock pays no dividend and we assume that the interest rate is zero, without loss of generality. $S(t, x, \omega)$ represents the stock price per share at time $t \in [0, T]$ that the trader pays/receives for an order of size $x \in \mathbf{R}$ given state $\omega \in \Omega$. A positive order ($x > 0$) represents a buy, a negative order ($x < 0$) represents a sale, and the order zero ($x = 0$) corresponds to the marginal trade. For the detailed structure of the supply curve, we refer to Section 2 of Çetin et al. (2004).

A *trading strategy*(portfolio) is a triplet $((X_t, Y_t{:}t \in [0, T]), \tau)$ where X_t represents the trader's aggregate stock holding at time t (units of the stock), Y_t represents the trader's aggregate money market account position at time t (units of money market account), and τ represents the liquidation time of the stock position.[1] Here, X_t and Y_t are predictable and optional processes, respectively, with $X_{0-} \equiv Y_{0-} \equiv 0$.

A *self-financing strategy* is a trading strategy $((X_t, Y_t{:}t \in [0, T]), \tau)$ where X_t is cadlag if $\partial S(t, 0)/\partial x = 0$ for all t,X_t is cadlag with finite quadratic variation ($[X, X]_T < \infty$) otherwise, and

$$Y_t = Y_0 + X_0S(0,X_0) + \int_0^t X_{u-}dS(u,0) - X_tS(t,0) - \sum_{0\leqslant u\leqslant t} \Delta X_u[S(u,\Delta X_u) - S(u,0)] - \int_0^t \frac{\partial S}{\partial x} \times (u,0)d[X,X]_u^c. \tag{2.1}$$

The first line of Eq. (2.1) is the same as the usual self-financing condition without liquidity risk, and the second line of (2.1) represents the loss due to liquidity costs. Therefore, it is natural to define *the liquidity cost* of a self-financing trading strategy (X, Y, τ) by

[1] We do not use the liquidation time τ in this study, and we assume that we have no such obligation. A self financing strategy and the liquidity cost below are still well defined without τ.

$$L_t = \sum_{0\leqslant u\leqslant t} \Delta X_u[S(u,\Delta X_u) - S(u,0)] + \int_0^t \frac{\partial S}{\partial x}(u,0)d[X,X]_u^c, \qquad 0 < t \leqslant T, \tag{2.2}$$

where $L_{0-} = 0$, and $L_0 = X_0[S(0,X_0) - S(0,0)]$. The first term in (2.2) denotes the price impact costs of discrete changes in share holdings and the second term denotes the price impact costs of continuous changes in share holdings. Note that the liquidity cost is always non-negative, since $S(t,x)$ is an increasing function of x and $[X,X]_t^c$ is an increasing process. We define *the marked-to-market value* of the self-financing portfolio as the value of the portfolio calculated under the marginal trade assumption, which is $Y_t + X_tS(t,0)$.

2.2. Model

Let $S(t,0) = S_t$ be the marginal price of the supply curve. We assume the price process S_t follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad 0 \leqslant t \leqslant T, \tag{2.3}$$

where the drift μ is a constant, the volatility σ is a positive real number, W is a standard Brownian motion, and T is the terminal time of an European contingent claim $C(C = g(S_T)$ for some function $g)$ of interest.

In this study, we are concerned with discrete time hedging and pricing. Let us consider equally spaced times $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n = T$. Set $\Delta t = t_i - t_{i-1}$ for $i = 1,\ldots,n$. We consider the following discrete time version of (2.3)

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma Z\sqrt{\Delta t},$$

where Z is a standard normal random variable, and assume a multiplicative supply curve

$$S(t,x) = f(x)S(t,0),$$

where f is a smooth and increasing function with $f(0) = 1$.

Since we consider discrete time trading, (2.2) implies that the total liquidity costs up to time T becomes

$$L_T = \sum_{0\leqslant u\leqslant T} \Delta X_u[S(u,\Delta X_u) - S(u,0)] = \sum_{i=1}^n \Delta X_i[S(t_i,\Delta X_i) - S(t_i,0)] + X_0[S(0,X_0) - S(0,0)], \tag{2.4}$$

where $\Delta X_i = X_{t_i} - X_{t_{i-1}}$.

We let C_0 denote the value at time 0 of contingent claim C so that the hedging error inclusive of liquidity costs is

$$\sum_{i=0}^{n-1} X_{t_i}(S_{t_{i+1}} - S_{t_i}) - C + C_0 - L_T. \tag{2.5}$$

Notice that the above hedging error is obtained by subtracting the aggregate liquidity costs from the usual hedging error without liquidity costs for which the marked-to-market value of the strategy (X,Y),$Y + XS_t$, is being considered. This implies that a trading strategy (X,Y) costs more to hedge the contigent claim C due to liquidity costs.

3. The pricing differential equation

Let us consider a European call option C expiring at T with strike price K, and the delta hedging strategy

$$X_{t_i} = C_S|_{S=S_{t_i}}. \tag{3.1}$$

Let $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$ be equally spaced trading times and $\Delta t = t_i - t_{i-1}$ for $i = 1,\ldots,n$. A perfect hedging strategy will produce a zero hedging error with probability 1. But, considering discrete trading and liquidity costs, it is not possible to produce a strategy whose hedging error equals 0.

The following theorem presents a partial differential equation which provides discrete time hedging strategies whose expected hedging error approaches zero as the length of the revision interval goes to zero.

Theorem 3.1. *Let C(x, t) denote the solution*[2] *of the following partial differential equation*

$$C_t(x,t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x,t)(1 + 2f'(0)xC_{xx}(x,t)) = 0 \qquad \text{for all} \quad t \in [0,T), \quad x \geqslant 0, \tag{3.2}$$

with the terminal condition

$$C(x,T) = (x-K)^+.$$

Then, the expected hedging error, using the strategy given by (3.1), over the period [0, T] approaches 0 as Δt goes to 0.

Proof. See Appendix A. □

Remark 3.1. The hedging error over each revision interval is expressed similarly to the one studied by Leland (1985). The liquidity cost given in (A.1) corresponds to the transaction costs term TC of Leland (1985, p. 1290). However, our approach to the problem differs from that of Leland because instead of modifying the parameters of the Black–Scholes price, we modify the Black–Scholes PDE in order to obtain suitable discrete time delta-hedging strategies with the presence of liquidity costs.

Remark 3.2. As discussed in Section 4 of Leland (1985), the value C in Theorem 3.1 only provides an upper bound for the price of the option (see also Denis et al., in press), but we use it as our price since we study the price from the seller's point of view.

We now apply the Law of Large Numbers for Martingales to obtain almost sure convergence of the values of the discrete hedging strategies to the payoff of the option as the revision interval approaches zero.

Let ΔH_i denote by the hedging error over time interval $[t_{i-1}, t_i]$. We note that (after dropping the subscripts i's of ΔH_i)

$$E[(\Delta H)^2] = E\left[\left(-C_t\Delta t - \frac{1}{2}C_{SS}(\Delta S)^2 - f'(0)C_{SS}^2\sigma^2 Z^2 S^3 \Delta t\right)^2\right] \leqslant M(\Delta t)^2,$$

for some constant M over all $t \in [0, T]$ because of the smoothness condition on C (see footnote 2).

By the Law of Large Numbers for Martingales (see Feller, 1970, p. 243),

$$E\left[\frac{\Delta H_i}{\Delta t}|\mathcal{F}_{t_{i-1}}\right] = 0,$$

for all i and

$$\sum_{i=1}^{\infty}\frac{1}{i^2}E\left[\left(\frac{\Delta H_i}{\Delta t}\right)^2\right] \leqslant M\sum_{i=1}^{\infty}\frac{1}{i^2} < \infty$$

imply that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta H_i}{\Delta t} = \frac{\Delta t}{T}\sum_{i=1}^{n}\frac{\Delta H_i}{\Delta t} = \frac{1}{T}\sum \Delta H_i \to 0$$

almost surely as $\Delta t \to 0$. This leads to almost sure convergence of the total hedging error $\sum \Delta H_i$ as Δt tends to 0. Thus we have obtained the following result.

[2] We assume smoothness conditions on C(x, *t*), *i.e.*, $\|C\|_{m,n,p} = \sup\left[x^m \frac{\partial^{n+p}C(x,t)}{\partial x^n \partial t^p}\right]$ is finite for all non-negative m, m and p.

Table 1
Option prices with liquidity costs.

Initial spot	Liquidity cost $f'(0)$				
	0 (Black–Scholes)	0.0001	0.0005	0.001	0.002
80	1.1859	1.1869	1.1910	1.1959	1.2058
85	2.1613	2.1621	2.1680	2.1754	2.1899
90	3.5891	3.5896	3.5973	3.6069	3.6258
95	5.5195	5.5198	5.5288	5.5400	5.5620
100	7.9656	7.9657	7.9753	7.9872	8.0106
105	10.9056	10.9058	10.9152	10.9268	10.9498
110	14.2920	14.2925	14.3011	14.3117	14.3327
115	18.0619	18.0627	18.0701	18.0792	18.0973

Table 2
Hedging errors of the modified strategy.

Liquidity cost $f'(0)$	Option price	Expected hedging error	
		Weekly revision	Daily revision
0	7.9656	−0.0099	−0.0017
0.0001	7.9657	−0.0076	0.0006
0.0005	7.9753	0.0016	0.0097
0.001	7.9872	0.0128	0.0210
0.002	8.0106	0.0351	0.0432

Theorem 3.2. *The values of the discrete time delta-hedging strategy ($X = C_S$, $Y = C - C_S$) where C is the solution of the PDE in Theorem 3.1 converge almost surely to the payoff of the option $(S - K)^+$ including liquidity costs, as $\Delta t \to 0$.*

We next present some numerical simulations to examine the effect of liquidity costs on option prices and on the expected hedging error. Table 1 presents the option prices inclusive of liquidity costs with varying $f'(0)$ and initial stock price S_0. The options prices are obtained by solving the PDE given in Theorem 3.1 numerically. The parameter values that we used are strike price $K = 100$, $\sigma = 0.2$ and $T = 1$ year. Consistent with intuition, we observe that the option prices increase slightly when the parameter of liquidity costs $f'(0)$ (the slope at 0 of supply curve) increases.

Table 2 shows the accuracy of hedging strategies for different values of $f'(0)$ and revision period Δt. The hedging errors are computed by the Monte Carlo simulation. When we follow the modified hedging strategy using our solution with realistic values[3] of $f'(0)$ and Δt, the expected hedging errors are very small, similar to the no liquidity costs case. We have observed when one follows the delta hedging strategy with the usual Black–Scholes prices, one would face a significant amount of loss with the presence of liquidity costs. The option parameters in this computation are initial stock price $S_0 = 100$, strike price $K = 100$, $\sigma = 0.2$ and $T = 1$ year. The standard deviations of hedging error are 0.0128 (weekly revision) and 0.005 (daily revision).

Remark 3.3. In this section, we treat the case of a call option. The argument and methodology we develop here also work well for general European contingent claims.

[3] Realistic values of $f'(0)$ are studied in Çetin et al., 2006. Table 1 in Çetin et al. (2006) shows that estimated values for $f'(0)$ are about 0.00005–0.0001. Weekly and daily revisions are used as the values of revision interval Δt in our paper, while Çetin et al. (2006) uses one or two days as Δt.

4. Approximate solution

In this section, we discuss a possible analytic solution for the pricing equation derived in Section 3. We explain briefly how to approximate the option value including liquidity costs using a series solution. The techniques used here come from perturbation theory. (See, for example, Shivamoggi, 2002.) In other words, our approximate solutions are represented by a series solution in terms of a small parameter $\alpha = f'(0)$ and coefficients $C_n = C_n(x,t)$ of α^n which are given as the solutions of inhomogeneous Black–Scholes equations. This kind of argument is similar to Hilbert's expansion for the Boltzmann equation (Cercignani et al., 1994) in kinetic theory of dilute gases. The well-posedness issue such as the existence of smooth solutions and uniqueness to Eq. (A.2) is beyond our scope, so we will leave this interesting issue for a future work.

Let $\alpha = f'(0)$. Now (3.2) is written as

$$C_t + \frac{1}{2}\sigma^2 x^2 C_{xx} + \alpha\sigma^2 x^3 C_{xx}^2 = 0 \qquad \text{for all } t \in [0,T), \quad x \geqslant 0, \tag{4.1}$$

with the boundary condition $C(x,T) = (x-K)^+$.

For sufficiently small $\alpha > 0$, we seek a solution in the form of

$$C(x,t) = C_0(x,t) + \alpha C_1(x,t) + \alpha^2 C_2(x,t) + \cdots = C_0(x,t) + \alpha C_1(x,t) + O(\alpha^2) \tag{4.2}$$

Inserting (4.2) into (4.1), we obtain the following equations for $C_0(x,t)$ and $C_1(x,t)$:

$$\frac{\partial C_0}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_0}{\partial x^2} = 0 \qquad \text{for all } t \in [0,T), \quad x \geqslant 0, \tag{4.3}$$

with the condition $C_0(x,T) = (x-K)^+$, and

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_1}{\partial x^2} + \sigma^2 x^3 \left(\frac{\partial^2 C_0}{\partial x^2}\right)^2 = 0 \qquad \text{for all } t \in [0,T), \quad x \geqslant 0, \tag{4.4}$$

with the condition $C_1(x,T) = 0$.

The solution to the Black–Scholes partial differential Eq. (4.3) is well-known as

$$C_0(x,t) = xN(d+) - KN(d-),$$

where

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}}\, dz,$$
$$d+ = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{x}{K} + \frac{\sigma^2}{2}(T-t)\right),$$
$$d- = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{x}{K} - \frac{\sigma^2}{2}(T-t)\right).$$

Also, it is known that

$$\frac{\partial^2 C_0}{\partial x^2} = \frac{1}{\sigma x\sqrt{T-t}} N'(d+) = \frac{1}{\sigma x\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d+)^2}. \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_1}{\partial x^2} + \frac{x}{2\pi(T-t)} e^{-\frac{\left(\ln\frac{x}{K} + \frac{\sigma^2}{2}(T-t)\right)^2}{\sigma^2(T-t)}} = 0, \tag{4.6}$$

with the condition $C_1(x,T) = 0$. By change of variables

$$y = \ln\frac{x}{K} \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T-t),$$

we rewrite (4.6) as

$$-\frac{\partial C_1}{\partial \tau}+\frac{\partial^2 C_1}{\partial y^2}-\frac{\partial C_1}{\partial y}+\frac{K}{2\pi\tau}e^{\left(y-\frac{(y+\tau)^2}{2\tau}\right)}=0,\quad -\infty<y<\infty \tag{4.7}$$

with the initial condition $C_1(y, 0) = 0$. Letting

$$C_1(y,\tau)=e^{\left(\frac{y}{2}-\frac{\tau}{4}\right)}\omega(y,\tau),$$

(4.7) can be expressed as

$$\frac{\partial \omega}{\partial \tau}=\frac{\partial^2\omega}{\partial y^2}+\frac{K}{2\pi\tau}e^{-\frac{2y^2+2y\tau+\tau^2}{4\tau}},\qquad -\infty<y<\infty, \tag{4.8}$$

with the initial condition $\omega(y,0) = 0$.

The solution to a Cauchy problem for the nonhomogeneous heat equation can be represented in terms of the Green's function (see, for example, Friedman, 1983). Therefore, the solution of (4.8) is written as

$$w(y,\tau)=\int_0^\tau\int_{-\infty}^{\infty}\frac{K}{2\pi u}e^{-\frac{2\xi^2+2\xi u+u^2}{4u}}\frac{1}{2\sqrt{\pi(\tau-u)}}e^{-\frac{(y-\xi)^2}{4(\tau-u)}}d\xi\,du$$

and the solution to (4.7) is

$$C_1(y,\tau)=e^{\left(\frac{y}{2}-\frac{\tau}{4}\right)}\int_0^\tau\int_{-\infty}^{\infty}\frac{K}{2\pi u}e^{-\frac{2\xi^2+2\xi u+u^2}{4u}}\frac{1}{2\sqrt{\pi(\tau-u)}}e^{-\frac{(y-\xi)^2}{4(\tau-u)}}d\xi\,du.$$

To achieve one additional order of accuracy in (4.2), we can derive the equation for $C_2(x, t)$ from (4.1):

$$\frac{\partial C_2}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 C_2}{\partial x^2}+2\sigma^3x^3\left(\frac{\partial^2 C_0}{\partial x^2}\right)\left(\frac{\partial^2 C_1}{\partial x^2}\right)=0$$

with the condition $C_2(x, T) = 0$. We then use the similar techniques to determine a formula for C_2 as we did for C_1. By repeating the above procedures, the equation for $C_n(x, t)$ for $n = 3, 4, \ldots$ can be obtained recursively.

However, we note that the estimated $\alpha = f'(0)$ is extremely small in most cases, so typically the values of C_0 and C_1 will give a good approximation for the option price under liquidity risk.

5. Conclusion

We investigated hedging and pricing problems of a contingent claim under liquidity risk. We derived a partial differential equation for the option value and proposed a hedging strategy in discrete time with the presence of liquidity costs. The hedging error analysis over each revision interval is similar to Leland in which the transaction costs term plays the same role to that of liquidity costs. However, in contrast to modifying the parameter of the Black–Scholes price, we proposed the modified PDE to provide discrete time hedging strategies. The option price including liquidity costs as the solutions to the derived PDE has increased as the parameter of liquidity risk, the slope at 0 of supply curve $f'(0)$, becomes large. Finally, we showed that following the discrete time delta-hedging strategy ($X = C_S$, $Y = C - C_S$) where C is the solution of the PDE in Theorem 3.1 yields the payoff of the option $(S - K)^+$ including liquidity costs, as the revision interval tends to 0. In addition to call options, the same argument can be used for general European contingent claims.

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Appendix A

Proof of Theorem 3.1. Over the small time interval $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$, the change in the call option value is

$$\Delta C = C(S + \Delta S, t + \Delta t) - C(S,t) = C_S \Delta S + C_t \Delta t + \frac{1}{2} C_{SS} (\Delta S)^2 + O(\Delta t^{3/2}),$$

and the change of the hedging strategy is

$$\Delta X = C_S(S + \Delta S, t + \Delta t) - C_S(S,t) = C_{SS} \Delta S + C_{St} \Delta t + \frac{1}{2} C_{SSS} (\Delta S)^2 + O(\Delta t^{3/2}).$$

From (2.4), the liquidity cost at each interval is

$$\Delta X(S(t, \Delta X) - S(t, 0)) = \Delta X(f(\Delta X) - 1)S(t, 0). \tag{A.1}$$

Since $f(0) = 1$, by Taylor series expansion, we have

$$f(x) - 1 = f'(0)x + \frac{f''(0)x^2}{2} + O(x^3).$$

Recall that

$$\begin{aligned} &\Delta S = S(\mu \Delta t + \sigma Z \sqrt{\Delta t}), \\ &(\Delta S)^2 = S^2 \sigma^2 Z^2 \Delta t + O(\Delta t^{3/2}), \\ &(\Delta S)^k = O(\Delta t^{3/2}), \qquad k = 3, 4, 5, \ldots \end{aligned}$$

and

$$\begin{aligned} &\Delta X = C_{SS}\Delta S + C_{St}\Delta t + \frac{1}{2} C_{SSS}(\Delta S)^2 + O(\Delta t^{3/2}) = C_{SS} S(\mu \Delta t + \sigma Z \sqrt{\Delta t}) + C_{St}\Delta t + \frac{1}{2} C_{SSS} S^2 \sigma^2 Z^2 \Delta t + O(\Delta t^{3/2}), \\ &(\Delta X)^2 = C_{SS}^2 (\Delta S)^2 + O(\Delta t^{3/2}) = C_{SS}^2 S^2 \sigma^2 Z^2 \Delta t + O(\Delta t^{3/2}), \\ &(\Delta X)^k = O(\Delta t^{3/2}), \ k = 3, 4, 5 \ldots \end{aligned}$$

Then, Eq. (A.1) becomes

$$\begin{aligned} \Delta X(S(t, \Delta X) - S(t, 0)) = \Delta X(f(\Delta X) - 1)S(t, 0) &= \Delta X(f'(0)\Delta X + \frac{f''(0)}{2}(\Delta X)^2)S + O(\Delta t^{3/2}) \\ &= f'(0) C_{SS}^2 \sigma^2 Z^2 S^3 \Delta t + O(\Delta t^{3/2}). \end{aligned}$$

Therefore, the hedging error over each revision interval is

$$\Delta H = X \Delta S - \Delta C - \Delta X(S(t, \Delta X) - S(t, 0)) = -C_t \Delta t - \frac{1}{2} C_{SS} (\Delta S)^2 - f'(0) C_{SS}^2 \sigma^2 Z^2 S^3 \Delta t + O(\Delta t^{3/2}).$$

If C satisfies

$$C_t + \frac{1}{2} \sigma^2 S^2 C_{SS} (1 + 2f'(0) S C_{SS}) = 0, \tag{A.2}$$

then, by taking expectations, the expected hedging error over each revision interval is

$$E[\Delta H] = O(\Delta t^{3/2}).$$

The total hedging error over the entire interval $[0, T]$ is the sum of ΔH's (see (2.5)). Thus $E[\sum \Delta H] = O(\Delta t^{1/2})$. Therefore, the expected hedging error over the period $[0, T]$ approaches 0 as Δt becomes small. $\square$

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