4. Greek Letters, Value-at-Risk

4.2 Value-at-Risk
(Hull's, Chapter 18)

Outline
(Hull, Chap 18)

• What is Value at Risk (VaR)?
• Historical simulations
• Monte Carlo simulations
• Model based approach
  – Variance-covariance method
• Comparisons of methods
• Testing

The Question Being Asked in VaR

“What loss level is such that we are X% confident it will not be exceeded in next N business days?”
VaR and Regulatory Capital
(Hull's Business Snapshot 18.1, page 436)

- We are X% certain that we will not lose more than V dollars in the next N days
- VaR is the loss level V that will not be exceeded with a specified probability
- Regulators use VaR in determining the capital a bank is required to keep to reflect the market risks it is bearing
- The market-risk capital is k times the 10-day 99% VaR where k is at least 3.0

Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”

Time Horizon N

- Instead of calculating the N-day, X% VaR directly analysts usually calculate a 1-day X% VaR and assume
  \[ \text{N-day VaR} = \sqrt{N} \times \text{1-day VaR} \]
- This is exactly true when portfolio changes on successive days come from independent identically distributed normal distributions
Methods of Estimating VaR

- Historical simulations
- Monte Carlo simulations
- Model building approaches

Historical Simulation
(Hull, See Tables 16.1 and 16.2, page 438-439))

- Create a database of the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day.
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day.
- and so on

Historical Simulation continued

- Suppose we use \( m \) days of historical data.
- Let \( v_i \) be the value of a variable on day \( i \).
- There are \( m-1 \) simulation trials.
- The \( i \)th trial assumes that the value of the market variable tomorrow (i.e., on day \( m+1 \)) is

\[ v_{m+1} = v_m \frac{v_i}{v_{m+i}} \]
Example A
(Hull, page 438)

<table>
<thead>
<tr>
<th>Day</th>
<th>Market Variable 1</th>
<th>Market Variable n</th>
<th>Portfolio value ($ millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.33</td>
<td>65.37</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20.78</td>
<td>64.91</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>21.44</td>
<td>65.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20.97</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>25.75</td>
<td>61.99</td>
<td>23.50</td>
</tr>
</tbody>
</table>

Example A
(Hull, page 439)

<table>
<thead>
<tr>
<th>Scenario Number</th>
<th>Market Variable 1</th>
<th>Market Variable n</th>
<th>Portfolio value ($ millions)</th>
<th>Change in value ($ millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.42</td>
<td>61.66</td>
<td>23.71</td>
<td>0.21</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>62.21</td>
<td>23.12</td>
<td>-0.38</td>
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<tr>
<td></td>
<td>26.67</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>499</td>
<td>25.88</td>
<td>23.63</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>25.95</td>
<td>22.87</td>
<td>-0.63</td>
</tr>
</tbody>
</table>

Example B

- A portfolio consist of 9 different stocks
- The number of each stock in the portfolio is 905 569 632 234 549 932 335 656 392, respectively.
- We have 3 year historical data of each stock
- Estimate 1-Day 95% VaR using historical simulations
Example B

- 787 simulations; VaR = 9.7388e+004 (the 39th)

Monte Carlo Simulation (page 448-449)

To calculate VaR using M.C. simulation we
- Value portfolio today
- Sample once from the multivariate distributions of the $\Delta x_i$
- Use the $\Delta x_i$ to determine market variables at end of one day
- Revalue the portfolio at the end of day

Monte Carlo Simulation

- Calculate $\Delta P$
- Repeat many times to build up a probability distribution for $\Delta P$
- VaR is the appropriate fractile of the distribution times square root of $N$
- For example, with 1,000 trials the 1 percentile is the 10th worst case.
Example B

- A portfolio consists of 9 different stocks
- The number of each stock in the portfolio is 905 569 632 234 549 932 335 656 392, respectively.
- We have 3 year historical data of each stock
- Estimate 1-Day 95% VaR using historical simulations

1000 simulations: VaR = 1.2656e+005 (the 50th)

The Model-Building Approach

- Another alternative approach is to make assumptions about the probability distributions of return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically
- This is known as the model building approach or the variance-covariance approach
Daily Volatilities

- In option pricing we measure volatility “per year”
- In VaR calculations we measure volatility “per day”

\[ \sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}} \approx \frac{6\%}{\sqrt{252}} \]

Daily Volatility continued

- Strictly speaking we should define \( \sigma_{\text{day}} \) as the standard deviation of the continuously compounded return in one day
- In practice we assume that it is the standard deviation of the percentage change in one day

Microsoft Example (page 440)

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use \( N=10 \) and \( \lambda=99 \)
Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- The standard deviation of the change in the portfolio in 1 day is $200,000
- We assume that the change in the value of the portfolio is normally distributed
- Since $N(-2.33) = 0.01$, the 1-day 99% VaR is $2.33 \times 200,000 = $466,000

Microsoft Example continued

- The 10-day 99% VaR
  $$466,000 \sqrt{10} = $1,473,621$$

AT&T Example (page 441)

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The S.D of change in the value of the portfolio in 1-day is $5,000,000 \times 0.01 = $50,000
- The 10-day 99% VaR is $50,000 \times 2.33 \times \sqrt{10} = $368,405
• Now consider a portfolio consisting of both Microsoft and AT&T
• Suppose that the correlation between the returns is 0.3
• X: change in the value of the position in Microsoft over 1-day period
• Y: change in the value of the position in AT&T over 1-day period

S.D. of Portfolio

• A standard result in statistics states that
  \[ \sigma_{X,Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho \sigma_X \sigma_Y} \]
• In this case \( \sigma_X = 200,000 \) and \( \sigma_Y = 50,000 \) and \( \rho = 0.3 \). The standard deviation of the change in the portfolio value in one day is therefore 220,227

VaR for Portfolio

• The 10-day 99% VaR for the portfolio is
  \[ 220,227 \times \sqrt{10} \times 2.33 = \$1,622.657 \]
• The benefits of diversification are
  \[ (1,473,621+368,405)–1,622,657=\$219,369 \]
• What is the incremental effect of the AT&T holding on VaR?
The Linear Model

We assume
• The daily change in the value of a portfolio is linearly related to the daily returns from market variables
• The returns from the market variables are normally distributed

The General Linear Model continued
(equations 18.1 and 18.2)

\[ \Delta P = \sum \alpha_i \Delta x_i \]
\[ \sigma^2 = \sum \sum \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij} \]
\[ \sigma_p^2 = \sum \alpha_i^2 \sigma_i^2 + 2 \sum \sum \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij} \]

where \( \Delta x_i \): the return on asset \( i \) in 1 day
\( \alpha_i \): amount being invested on asset \( i \)
\( \sigma_i \): the volatility of variable \( i \)
\( \sigma_{p}^2 \): the portfolio's standard deviation

The Linear Model and Options

Consider a portfolio of options dependent on a single stock price, \( S \). Define

\[ \delta = \frac{\Delta P}{\Delta S} \]

and

\[ \Delta x = \frac{\Delta S}{S} \]
Linear Model and Options
continued (equations 18.3 and 18.4)

- As an approximation
  \[ \Delta P = \delta \Delta S = S \delta \Delta x \]
- Similarly when there are many underlying market variables
  \[ \Delta P = \sum_i S_i \delta_i \Delta x_i \]
  where \( \delta_i \) is the delta of the portfolio with respect to the \( i \)th asset.

Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

\[ \Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2 \]

this becomes

\[ \Delta P = S \delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2 \]

Quadratic Model continued

With many market variables we get an expression of the form

\[ \Delta P = \sum_i S_i \delta_i \Delta x_i + \sum_i \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j \]

where

\[ \delta_i = \frac{\partial P}{\partial S_i}, \quad \gamma_{ij} = \frac{\partial^2 P}{\partial S_i \partial S_j} \]

This is not as easy to work with as the linear model.
Comparison of Approaches

• Historical simulation lets historical data determine distributions, but is computationally slower
• Monte Carlo simulation can handle any distribution and is easy to incorporate back and stress tests. It allows to achieve any accuracy (if the time is not an issue). It is computationally intensive.
• Model building approach assumes normal distributions for market variables. It tends to give poor results for low delta portfolios

Stress Testing

• This involves testing how well a portfolio performs under some of the most extreme market moves seen in the last 10 to 20 years

Back-Testing

• Tests how well VaR estimates would have performed in the past
• We could ask the question: How often was the actual 10-day loss greater than the 99%/10 day VaR?