2. Lattice Methods

2.1 One-step binomial tree model
(Hull, Chap. 11, page 241)

Outline

1. No-Arbitrage Evaluation
2. Its relationship to “risk-neutral valuation”.

A Simple Binomial Model

A stock price is currently $20
After 3 months it will be either $22 or $18
A 3-month European call option on the stock with K=$21

Stock Price = $22
Option Price = $1

Stock Price = $18
Option Price = $0

Stock price = $20
Option Price =?

3 months
Consider the riskless portfolio: long $\Delta$ shares, short 1 call option.

Portfolio Value

$2\Delta - 1$

$18\Delta$

Portfolio is riskless when $2\Delta - 1 = 18\Delta$

i.e. $\Delta = 0.25$

Valuing the Portfolio and the Option
(Risk-Free Rate $r = 12\%$)

1. What is the value of the portfolio at the expiry of the option?
2. What is the value of the portfolio today?
3. What is the value of the option today?

General One-Step Binomial Tree
(No-arbitrage evaluation)

A derivative lasts for time $T$ and is dependent on a stock $S_0$.
General One-Step Binomial Tree
(No-arbitrage evaluation)

Consider the portfolio: long $\Delta$ shares, short 1 derivative

$$S_0\Delta - \nu$$

The portfolio is riskless when $S_0\Delta - \nu_S = S_0d\Delta - \nu_d$ or

$$\Delta = \frac{\nu_d - \nu_S}{S_0u - S_0d}$$

Value of the portfolio at time $T$:

$$S_0\Delta - \nu_S e^{-rT}$$

Value of the portfolio today:

$$S_0\Delta - \nu_S e^{-rT} = (S_0\Delta - \nu_S) e^{-rT}$$

Hence

$$\nu = S_0\Delta - (S_0\Delta - \nu_S) e^{-rT}$$

Another approach: risk-neutral evaluation

Substituting for $\lambda$, we obtain

$$\nu = \left[p\nu_u + (1-p)\nu_d\right] e^{-rT}$$

$$p = \frac{e^{rT} - d}{u - d}$$

Where

- The variables $p$ and $(1-p)$ can be interpreted as the risk-neutral probabilities of up and down movements.
- The value of a derivative can be considered as its expected payoff in a risk-neutral world discounted at the risk-free rate.
Comments and example

• Note: When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant.
• Use risk-neutral evaluation to calculate option price in the original example \( r = 12\% \), \( T = 3/12 \):

Original example revisited (risk-neutral evaluation)

\[
p = \frac{e^{rT} d - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523
\]
\[
S_0 v = 0.6523 S_0 v = 1
\]
\[
S_0 d = 0.3477 S_0 d = 0
\]

Therefore, the value of the option is

\[
v = e^{-0.22 \times 3/12} \left[ 0.6523 \times 1 + 0.3477 \times 0 \right] = 0.633
\]

Risk-neutral evaluation

• In a risk-neutral world, all investors are indifferent to risk, i.e., investors require no compensation for risk and the expected returns of all securities is the risk-free interest rate.
• The option price resulted from risk-neutral evaluation is correct not only in the risk-neutral world, but also in other world as well.
2. Lattice Methods

2.2 Binomial Methods
(Hull, Sec. 11.7, Sec. 17.1)

Outline

1. Extension to N-step binomial method
2. Determine the parameters
3. Binomial methods in option pricing

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time $\Delta t$ the stock price is assumed to move up by a proportional amount $u$ or to move down by a proportional amount $d$

Stock movements in time $\Delta t$
Math6911, S08, HM ZHU

Assumption 1

To model a continuous walk by a discrete walk in the following fashion:

\( u > 1 \)
\( d < 1 \)
\( p \): probability

\[
\begin{array}{c}
S \\
\downarrow 1-p \\
\downarrow \Delta t \\
S \\
\uparrow \Delta t \\
\uparrow p \cdot 2 \Delta t \\
\uparrow \Delta t \\
\uparrow u \cdot S \\
\end{array}
\]

The parameters \( u, d, \) and \( p \) in such a way that
mean and variance of the discrete random walk
coincide with those of the continuous random walk

Assumption 1 (cont)

Assumption 2: Risk-Neutral Valuation

We may assume investors are risk-neutral (even though most are not)

In terms of option pricing (or other derivatives), we can assume:

- The expected return from all traded securities is the risk-free interest rate
- Future cash flows can be valued by discounting their expected values at the risk-free interest rate

The tree is designed to represent the behavior of a stock price in risk-neutral world
**Binomial Method: Assumption 2**

Under the assumption of risk-neutral world, the value $V^m$ of the put option at $m \Delta t$ is given by the expected value $\mathbb{E}[V^{m+1}]$ of the option at $(m+1)\Delta t$ discounted by the risk-free interest rate $r$.

$$V^m = E[e^{-r\Delta t} V^{m+1}]$$

**Tree parameters $p$, $u$, and $d$ for non-dividend paying stock**

The parameters $u$, $d$, and $p$ in such a way that mean and variance of the discrete random walk coincide with those of the continuous random walk in risk-neutral world, i.e.,

$$E[S^{m+1} | S^m] = E_S[S^{m+1} | S^m]$$

and

$$\text{var}[S^{m+1} | S^m] = \text{var}_S[S^{m+1} | S^m]$$

The parameters $p$, $u$, and $d$ must give correct values for the mean and variance of stock price changes during $\Delta t$.

**Tree parameters $u$, $d$, and $p$**

$$E[S^{m+1} | S^m] = E_E[S^{m+1} | S^m]$$

$$\Rightarrow e^{-r\Delta t} S^m = [pu + (1-p)d] S^m$$

$$\Rightarrow e^{-r\Delta t} = pu + (1-p)d \quad (1)$$
Tree parameters $u$, $d$, and $p$

\[
\text{var}[S^n] = \text{var}[S^d] \Rightarrow \\
(\Delta t)^2 \phi^u(\phi^u - 1) = \left[ pu^2 + (1-p)d^2 - \phi^u \right] (\Delta t)^2 \\
\phi^{2u+2d} = pu^2 + (1-p)d^2 
\] (2)

In order to determine these three unknowns uniquely, we need to add another condition (somewhat arbitrary).

Note: $\sigma$ follows a lognormal distribution. It can be shown that

1. The expected value of $S^n$
2. The variance of $S^n$.

One further condition: $u = \frac{1}{d}$

One of the popular choices is $u = \frac{1}{d}$.

It generates a symmetric tree w.r.t. the initial value $S_0$:

\[
u e^{\sigma \Delta t\sqrt{\frac{1}{2}}} \quad , \quad d = e^{\sigma \Delta t\sqrt{\frac{1}{2}}} \quad , \quad p = \frac{e^{\sigma \Delta t} - d}{u - d} 
\]

[reference: Cox, Ross and Rubinstein (1979)]

Often, ignoring the terms in $\Delta t$ and higher power of $\Delta t$ gives

\[
u e^{\sigma \Delta t} \quad , \quad d = e^{\sigma \Delta t} \quad , \quad p = \frac{e^{\sigma \Delta t} - d}{u - d}
\]

Note: If $\Delta t$ is too large, $p$ or $(1-p) < 0$. The binomial method fails.

Symmetric Tree: $u = \frac{1}{d}$
Another further condition: \( p = \frac{1}{2} \)

One of the popular choices is \( p = \frac{1}{2} \).

It generates an asymmetric tree, oriented in the direction of the drift:

\[
\begin{align*}
    u &= e^{\sigma \Delta t} \left( 1 + \sqrt{e^{2 \sigma \Delta t} - 1} \right), \\
    d &= e^{\sigma \Delta t} \left( 1 - \sqrt{e^{2 \sigma \Delta t} - 1} \right), \\
    p &= \frac{1}{2}
\end{align*}
\]

[Kwok, 1998; Wilmott et al, 1995]

Note: If \( \Delta t \) is too large, \( d < 0 \). The binomial method fails.

Asymmetric Tree: \( p = \frac{1}{2} \)

2. Lattice Methods
The Idea of Binomial Methods

1. Build a tree of possible values of asset prices and their probabilities, given an initial asset price
2. Once we know the value of the option at the final nodes, work backward through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Advantages:
easily deal with possibility of early exercise and with dividend payments

Binomial Method for valuing non-dividend-paying options

Step 1. Build up a tree of possible asset prices for all time points and their probabilities, given the initial price $S_0^n$

$$S_n^m = u^m d^{n-m} S_0^n$$ for $n = 0, 1, \ldots, m$

$S_n^m$: the n-th possible value of $S$ at m-th time-step

Build A Complete Tree of Asset Prices

Tree structure used the relationship $u = 1/d$
Step 2: Valuing an European option:

Step 2. Use this tree to calculate the possible value of option at expiry. Then work back down the tree to calculate the price of the option.

2.1. Value the option at expiry, i.e., time-step $M\Delta t$

Put: $V_n^0 = \max\left(K - S_n^0, 0\right)$, $n = 0, 1, \ldots, M$

Call:

2.2. Find the expected value of the option at a time step prior to expiry

$$V_n^m = E\left[e^{-r\Delta t}V_{n+1}^m\right] = e^{-r\Delta t}\left[pV_{n+1}^m + (1-p)V_{n+1}^m\right]$$

and so on, back to time-step $0$.

Example 1: European 2-year Put Option

$S_0 = 50; K = 52; r = 5\%; \sigma = 30\%$

$T = 2$ years; $\Delta t = 1$ year

The parameters imply:

$u = e^{\sigma\sqrt{\Delta t}} = 1.3499$

$d = \frac{1}{u} = 0.7408$

$a = e^{0} = 1.0513$

$p = \frac{1.0513 - 0.7408}{1.3499 - 0.7408} = 0.5097$

$1-p = 0.4903$

Example 1: European 2-year Put Option

The parameters imply:

$u = e^{\sigma\sqrt{\Delta t}} = 1.3499$

$d = \frac{1}{u} = 0.7408$

$a = e^{0} = 1.0513$

$p = \frac{1.0513 - 0.7408}{1.3499 - 0.7408} = 0.5097$

$1-p = 0.4903$
Example 2: European Put Option

\[ S_0 = 50; \quad K = 50; \quad r = 10\%; \quad \sigma = 40\%; \]
\[ T = 5 \text{ months} = 0.4167; \]
\[ \Delta t = 1 \text{ month} = 0.0833 \]

The parameters imply
\[ a = e^{\Delta t r} = 1.1224; \]
\[ d = e^{-\Delta t r} = 0.8909; \]
\[ u = e^{\sigma \sqrt{\Delta t}} = 1.0084; \]
\[ p = \frac{u-d}{u+d} = 0.5076 \]
\[ 1-p = 0.4924 \]
Step 2: Valuing an American put option

At time step $k$ and at asset price $S^i$, there are two possibilities:

1. Exercise the option which yields a profit

$$\text{payoff} \left( S^i \right) = \max \left( K - S^i, 0 \right)$$

2. Retain the option, the value of the option is then

$$V^i = E \left[ e^{-r \Delta t} V^i_{i+1} \right] = e^{-r \Delta t} \left[ p V^i_{i+1} + (1-p) V^i_{i-1} \right]$$

Therefore, the value of the option is the maximum of two possibilities, i.e.,

$$V^i = \max \left( \text{payoff} \left( S^i \right), e^{-r \Delta t} \left[ p V^i_{i+1} + (1-p) V^i_{i-1} \right] \right)$$

Example 1: American 2-year Put Option

$S_0 = 50$; $K = 52$; $r = 5\%$, $\sigma = 30\%$, $T = 2$ years, $\Delta t = 1$ year

<table>
<thead>
<tr>
<th>$1$ year</th>
<th>$1$ year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$50$</td>
<td>$50$</td>
</tr>
<tr>
<td>$91.11$</td>
<td>$67.49$</td>
</tr>
<tr>
<td>$27.44$</td>
<td>$24.56$</td>
</tr>
</tbody>
</table>

Example 2: American put option of the same stock: $S_0 = 50$; $K = 50$; $r = 10\%$, $\sigma = 40\%$; $T = 5$ months; $\Delta t = 1$ month

<table>
<thead>
<tr>
<th>$1$ month</th>
<th>$1$ month</th>
</tr>
</thead>
<tbody>
<tr>
<td>$50.00$</td>
<td>$50.34$</td>
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<tr>
<td>$44.55$</td>
<td>$44.55$</td>
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<tr>
<td>$39.69$</td>
<td>$39.69$</td>
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<td>$35.36$</td>
<td>$35.36$</td>
</tr>
<tr>
<td>$31.50$</td>
<td>$31.50$</td>
</tr>
<tr>
<td>$28.07$</td>
<td>$28.07$</td>
</tr>
<tr>
<td>$24.56$</td>
<td>$24.56$</td>
</tr>
<tr>
<td>$21.93$</td>
<td>$21.93$</td>
</tr>
</tbody>
</table>
Step 2: Valuing an American call option

At time step $k$ and at asset price $S^k$, there are two possibilities:
1. Exercise the option which yields a profit
   \[ \text{payoff} (S^k) = ? \]
2. Retain the option, the value of the option is then
   \[ V_k^e = ? \]
Therefore, the value of the option is the maximum of two possibilities, i.e.,
\[ V_k^e = ? \]

Exercise 2: American call option of the same stock:
\( S_0 = 50; \ K = 50; \ r = 10\%; \ \sigma = 40\%; \ T = 5 \text{ months}; \ \Delta t = 1 \text{ month} \)

\[ \begin{array}{c|c|c}
\text{Price} & 89.07 & 89.07 \\
\text{Option} & 79.35 & 79.35 \\
\text{Price} & 70.70 & 70.70 \\
\text{Option} & 62.99 & 62.99 \\
\text{Price} & 56.12 & 56.12 \\
\text{Option} & 50.00 & 50.00 \\
\end{array} \]

Convergence of the Price of the American Put on Non-Dividend-Paying Option Example
2. Lattice Methods

2.4 Dealing with Options on Dividend-paying Stocks

With Continuous Dividend Yields

Assume there is a constant dividend yield \( D_t \) paid on the underlying. Then the expected return of the underlying is at the rate \( (r - D_t) \).

To accommodate the constant dividend yield in the tree model, replace \( r \) by \( r - D_t \) in the parameters \( u, d, \) and \( p \) in the tree construction of stock prices. For example, in the case \( u = 1/d \), it becomes:

\[
\frac{e^{r t} u - D_t}{d - D_t} \quad u = \frac{e^{r t} - D_t}{u - D_t}
\]

For the case when \( p = \frac{1}{2} \), it becomes:

\[
\frac{e^{r t} u - D_t}{1 + \sqrt{e^{2 r t} - 1}} \quad d = \frac{e^{r t} - D_t}{1 - \sqrt{e^{2 r t} - 1}}
\]

With Continuous Dividend Yields

2. Use this tree to calculate the possible value of option at expiry. Then work back down the tree to calculate the present value of the option at previous time points is obtained using

\[
V^n = E \left[ e^{-r t} p^n \right]
\]
2. Use this tree to calculate the possible value of option at expiry. Then work back down the tree to calculate the present value of the option at previous time points is obtained using

\[ V^n = E \left[ \sum_{i=1}^{n} e^{-r\Delta t} \right] \]

With Continuous Dividend Yields

A better procedure:
- Draw the tree for the stock price less the present value of the dividends
- Create a new tree for the stock price by adding the present value of the dividends at each node

This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes model is used

With Dollar Dividend

Assume that there is one ex-dividend date, \( \tau \), during the life of the option.
- Construct a tree for the uncertain component \( S' \), i.e.,

\[ S' = \ \begin{cases} \hat{S} - De^{-(r+\sigma^2)} & \text{if } i\Delta t < \tau \\ \hat{S} & \text{if } i\Delta t > \tau \end{cases} \]

Using the volatility \( \sigma' \) of \( S' \), a tree can be constructed in usual way to model \( S' \).
- To model \( S \), we simply add back the present value of future dividend, i.e.,

\[ S^n = \ \begin{cases} \hat{S}^n d^+ + De^{-r(i\Delta t)} & \text{if } i\Delta t < \tau \\ \hat{S}^n d^- & \text{if } i\Delta t > \tau \end{cases} \]
Example 1:
American put option on a stock:
$S_0 = 52$;
$K = 50$;
$D = 52.06$
$r = 10\%$;
$\sigma = 40\%$;
$T = 5$ months;
$t = 3.5$ months
$\Delta t = 1$ month

Example 1:
Tree model for $S'$

Example 1:
American put option on a stock:
$S_0 = 52$;
$K = 50$;
$D = 52.06$
$r = 10\%$;
$\sigma = 40\%$;
$T = 5$ months;
$t = 3.5$ months
$\Delta t = 1$ month
2. Lattice Methods

2.5 Further Comments

### Delta ($\Delta$)

- An important parameter in the pricing and hedging of options
- It is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- It is the number of units of the stock we should hold for each option shorted to create a riskless hedge. Such a construction of riskless hedging is called "delta hedging".
- $\Delta$ of a call option is positive whereas $\Delta$ of a put option is negative

### Valuing Delta's

<table>
<thead>
<tr>
<th>Value at $\Delta t$: $\Delta = \frac{2.0257 - 0}{22 - 18} = 0.5064$</th>
<th>Value at $2\Delta t$: $\Delta = \frac{3.2 - 0}{24.2 - 19.8} = 0.7273$</th>
</tr>
</thead>
<tbody>
<tr>
<td>If upward movement over the first time step, $\Delta = \frac{3.2 - 0}{24.2 - 19.8} = 0.7273$</td>
<td>Otherwise, $\Delta = \frac{0 - 0}{19.8 - 16.2} = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.0257</td>
<td>19.8</td>
<td>18</td>
<td>0.0</td>
<td>16.2</td>
</tr>
<tr>
<td>1.2823</td>
<td>3.2</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Delta values calculated at different points in the lattice.
Delta

• The value of $\Delta$ varies over time and from node to node
• To maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically

Trees for Options on Indices, Currencies and Futures Contracts
(Hull, Sec. 11.9, Sec. 17.2)

As with Black-Scholes:
– For options on stock indices, replace the continuous dividend yield $D_0$ with the dividend yield on the index
– For options on a foreign currency, $D_0$ equals the foreign risk-free rate $r_f$
– For options on futures contracts $D_0 = r$

Time Dependent Interest Rate and Dividend Yield (page 409)

• Making interest rate $r$ or dividend yield $D$ a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time as well
• Discounting factor becomes a function of time as well
• Changing \( \sigma \) at each time step does affect the geometry of the tree. (The probabilities on the tree become functions of time)

• Or we can make \( \sigma \) a function of time by making the lengths of the time steps inversely proportional to the variance rate.

**Trinomial Tree**
(see Technical Note 9, www.rotman.utoronto.ca/~hull)

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} \\
    d &= 1/u \\
    p_u &= \frac{\Delta t}{12 \sigma^2} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6} \\
    p_m &= \frac{2}{3} \\
    p_d &= -\frac{\Delta t}{12 \sigma^2} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6}
\end{align*}
\]

\( p_u, p_m, p_d \)

**Explicit FDM = Trinomial Tree**

\[
\begin{align*}
    V_{i+1,j+1} &= V_{i,j} + \Delta t \left( r V_{i,j} + \frac{\sigma^2}{2} \right) + \frac{1}{2} \sigma^2 \Delta t \\
    V_{i+1,j} &= V_{i,j} + \Delta t \left( r V_{i,j} + \frac{\sigma^2}{2} \right) - \frac{1}{2} \sigma^2 \Delta t \\
    V_{i+1,j-1} &= V_{i,j} + \Delta t \left( r V_{i,j} + \frac{\sigma^2}{2} \right) - \frac{1}{2} \sigma^2 \Delta t
\end{align*}
\]

Explicit Method
Further Reading

1. D. Leisen and M. Reimer. Applied Mathematical Finance, 1996 (developed a general convergence rate theory)

Adaptive Mesh Model

- This is a way of grafting a high resolution tree on to a low resolution tree
- We need high resolution in the region of the tree close to the strike price and option maturity
- Numerically efficient over a binomial or trinomial tree