7. Greek Letters, Value-at-Risk

7.1 Stop and loss hedging

(Hull’s book, Chapter 15)
Outline
(Hull, Chap 15.1-15.3; Brandimarte, Chap 8.1.2)

• Naked and cover position
• Stop-Loss Strategy
• Simulating the stop-loss strategy
Example

• A bank has sold for $7.00 a European call option on one share of a nondividend paying stock
  \( S_0 = $50, \quad K = $50, \quad r = 5\%, \quad \sigma = 40\%, \quad T = 5 \text{ months}, \quad \mu = 10\% \)

• The Black-Scholes value of the option is $5.6150
• How does the bank hedge its risk?
Naked & Covered Positions

- Naked position: Take no action
- Covered position: Buy 1 share today
- Either strategy leaves the bank exposed to significant risk
Stop-Loss Strategy

This involves:

• Buying 1 share as soon as price rises above $50
• Selling 1 share as soon as price falls below $50
Stop-Loss Strategy

- This simple hedging strategy is to ensure that at time T, the bank owns the stock if the option closes in the money and does not own it if the option closes out of money.
- We can evaluate its performance in discrete time by Monte Carlo simulation.
Stop-Loss Strategy

<table>
<thead>
<tr>
<th>B-S Price ($)</th>
<th># of time steps</th>
<th>10</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5.6150</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stop-Loss cost</td>
<td>5.5780</td>
<td>5.6295</td>
<td>5.6437</td>
<td></td>
</tr>
<tr>
<td>Elapsed time (loops, sec)</td>
<td>2.814</td>
<td>3.235</td>
<td>3.595</td>
<td></td>
</tr>
</tbody>
</table>

This deceptively simple hedging strategy does not work well.
7. Greek Letters, Value-at-Risk

7.2 Greek Letters

(Hull’s book, Chapter 15)
Outline

- Delta, Delta hedging
- Theta
- Gamma
- Relationship between delta, Theta and Gamma
- Vega
Delta (See Figure 15.2, page 345)

\[ \Delta = \frac{\partial \Pi}{\partial S} \]

- Delta (\( \Delta \)) is the rate of change of the value of the portfolio with respect to the underlying asset price.
Delta

- An important parameter in the pricing and hedging of options
- It is # of the units of the stock we should hold for each option shorted to create a riskless hedge. Such a construction of riskless hedging is called “delta hedging”.
- $\Delta$ of a call option is positive whereas $\Delta$ of a put option is negative
Delta Hedging

- This involves maintaining a delta neutral portfolio
- The hedge position must be frequently rebalanced
- Delta hedging a written option involves a “buy high, sell low” trading rule
## Delta hedging

<table>
<thead>
<tr>
<th></th>
<th># of time steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>B-S Price ($)= $5.6150</td>
<td></td>
</tr>
<tr>
<td>Stop-Loss cost</td>
<td>5.5780</td>
</tr>
<tr>
<td>Cost of delta-hedging</td>
<td>5.6165</td>
</tr>
</tbody>
</table>
Theta \( \Theta = \frac{\partial \Pi}{\partial t} \)

- Theta \( (\Theta) \) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time.
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of the option declines.
Gamma

\[ \Gamma = \frac{\partial^2 \Pi}{\partial S^2} \]

- Gamma (\( \Gamma \)) is the rate of change of delta (\( \Delta \)) with respect to the price of the underlying asset.
- Gamma neutrality protects against large change in the price of the underlying asset between hedge re-balancing.
Gamma Addresses Delta Hedging Errors Caused By Curvature
(Figure 15.7, page 355)
Relationship Between Delta, Gamma, and Theta

For a portfolio of derivatives on a stock

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$
Vega \( \nu = \frac{\partial \Pi}{\partial \sigma} \)

- Vega (\( \nu \)) is the rate of change of the value of a derivatives portfolio with respect to volatility (volatility change over time)
- Vega changes when there are large price movements in the underlying asset and vega falls as the option gets closer to maturity
- Vega neutrality protects for volatility changes
Hedging in Practice

• Traders usually ensure that their portfolios are delta-neutral at least once a day.
• Traders monitor gamma and vega. If they get too large, either corrective action is taken or trading is curtailed.
7. Greek Letters, Value-at-Risk

7.3 Value-at-Risk
(Hull’s, Chapter 18)
Outline
(Hull, Chap 18)

• What is Value at Risk (VaR)?
• Historical simulations
• Monte Carlo simulations
• Model based approach
  – Variance-covariance method
• Comparisons of methods
• Testing
The Question Being Asked in VaR

“What loss level is such that we are $X\%$ confident it will not be exceeded in next $N$ business days?”

![Probability Distribution of Portfolio Value](image_url)
VaR and Regulatory Capital
(Hull’s Business Snapshot 18.1, page 436)

• We are X\% certain that we will not lose more than V dollars in the next N days
• VaR is the loss level V that will not be exceeded with a specified probability
• Regulators use VaR in determining the capital a bank is required to keep to reflect the market risks it is bearing
• The market-risk capital is $k$ times the 10-day 99\% VaR where $k$ is at least 3.0
Advantages of VaR

• It captures an important aspect of risk in a single number
• It is easy to understand
• It asks the simple question: “How bad can things get?”
Time Horizon $N$

- Instead of calculating the $N$-day, $X\%$ VaR directly, analysts usually calculate a 1-day $X\%$ VaR and assume

$$N \text{-day VaR} = \sqrt{N} \times 1\text{-day VaR}$$

- This is exactly true when portfolio changes on successive days come from independent identically distributed normal distributions
Methods of Estimating VaR

- Historical simulations
- Monte Carlo simulations
- Model building approaches
Historical Simulation
(Hull, See Tables 18.1 and 18.2, page 438-439)

• Create a database of the daily movements in all market variables.
• The first simulation trial assumes that the percentage changes in all market variables are as on the first day
• The second simulation trial assumes that the percentage changes in all market variables are as on the second day
• and so on
Historical Simulation continued

• Suppose we use \( m \) days of historical data
• Let \( v_i \) be the value of a variable on day \( i \)
• There are \( m-1 \) simulation trials
• The \( i \)th trial assumes that the value of the market variable tomorrow (i.e., on day \( m+1 \)) is

\[
v_{m+1} = v_m \frac{v_i}{v_{i-1}}
\]
Example A
(Hull, page 438)

<table>
<thead>
<tr>
<th>Day</th>
<th>Market Variable 1</th>
<th>Market Variable n</th>
<th>Portfolio value ($ millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.33</td>
<td>65.37</td>
<td></td>
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<tr>
<td>1</td>
<td>20.78</td>
<td>64.91</td>
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<tr>
<td>2</td>
<td>21.44</td>
<td>65.02</td>
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<td></td>
<td>20.97</td>
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<td></td>
</tr>
<tr>
<td>500</td>
<td>25.85</td>
<td>62.10</td>
<td>23.50</td>
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<td></td>
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</tbody>
</table>
### Example A
*(Hull, page 439)*

Scenarios generated for tomorrow (Day 501)

<table>
<thead>
<tr>
<th>Scenario Number</th>
<th>Market Variable 1</th>
<th>Market Variable n</th>
<th>Portfolio value ($ millions)</th>
<th>Change in value ($ millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.42</td>
<td>61.66</td>
<td>23.71</td>
<td>0.21</td>
</tr>
<tr>
<td>2</td>
<td>26.67</td>
<td>62.21</td>
<td>23.12</td>
<td>-0.38</td>
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</tr>
<tr>
<td>499</td>
<td>25.88</td>
<td>61.87</td>
<td>23.63</td>
<td>0.13</td>
</tr>
<tr>
<td>500</td>
<td>25.95</td>
<td>62.21</td>
<td>22.87</td>
<td>-0.63</td>
</tr>
</tbody>
</table>

Scenarios generated for tomorrow (Day 501)
Example B

- A portfolio consists of 9 different stocks
- The number of each stock in the portfolio is 905, 569, 632, 234, 549, 932, 335, 656, 392, respectively.
- We have 3 year historical data of each stock
- Estimate 1-Day 95% VaR using historical simulations
Example B

- 787 simulations; VaR = 9.7388e+004 (the 39th)
To calculate VaR using M.C. simulation we
• Value portfolio today
• Sample once from the multivariate distributions of the $\Delta x_i$
• Use the $\Delta x_i$ to determine market variables at end of one day
• Revalue the portfolio at the end of day
Monte Carlo Simulation

- Calculate $\Delta P$
- Repeat many times to build up a probability distribution for $\Delta P$
- VaR is the appropriate fractile of the distribution times square root of $N$
- For example, with 1,000 trials the 1 percentile is the 10th worst case.
Example B

- A portfolio consists of 9 different stocks.
- The number of each stock in the portfolio is 905 569 632 234 549 932 335 656 392, respectively.
- We have 3 year historical data of each stock.
- Estimate 1-Day 95% VaR using historical simulations.
Example B

- 1000 simulations; VaR = 1.2656e+005 (the 50th)
The Model-Building Approach

• Another alternative approach is to make assumptions about the probability distributions of return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically.

• This is known as the model building approach or the variance-covariance approach.
Daily Volatilities

• In option pricing we measure volatility “per year”
• In VaR calculations we measure volatility “per day”

\[ \sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}} \approx 6\% \sigma_{\text{year}} \]
Daily Volatility continued

- Strictly speaking we should define $\sigma_{\text{day}}$ as the standard deviation of the continuously compounded return in one day.
- In practice we assume that it is the standard deviation of the percentage change in one day.
We have a position worth $10 million in Microsoft shares
The volatility of Microsoft is 2% per day (about 32% per year)
We use $N=10$ and $X=99$
Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- The standard deviation of the change in the portfolio in 1 day is $200,000
- We assume that the change in the value of the portfolio is normally distributed
- Since $N(-2.33)=0.01$, the 1-day 99% VaR is $2.33 \times 200,000 = $466,000
The 10-day 99% VaR

\[ 466,000 \sqrt{10} = \$1,473,621 \]
AT&T Example (page 441)

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The S.D of change in the value of the portfolio in 1-day is
  \[ 5,000,000 \times 0.01 = 50,000 \]
- The 10-day 99% VaR is
  \[ 50,000 \times 2.33 \times \sqrt{10} = 368,405 \]
Now consider a portfolio consisting of both Microsoft and AT&T
Suppose that the correlation between the returns is 0.3
X: change in the value of the position in Microsoft over 1-day period
Y: change in the value of the position in AT&T over 1-day period
S.D. of Portfolio

• A standard result in statistics states that

\[ \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} \]

• In this case \( \sigma_X = 200,000 \) and \( \sigma_Y = 50,000 \) and \( r = 0.3 \). The standard deviation of the change in the portfolio value in one day is therefore 220,227
VaR for Portfolio

• The 10-day 99% VaR for the portfolio is
  \[220,227 \times \sqrt{10} \times 2.33 = \$1,622,657\]

• The benefits of diversification are
  \[1,473,621 + 368,405 - 1,622,657 = \$219,369\]

• What is the incremental effect of the AT&T holding on VaR?
The Linear Model

We assume

• The daily change in the value of a portfolio is linearly related to the daily returns from market variables

• The returns from the market variables are normally distributed
The General Linear Model continued
(equations 18.1 and 18.2)

\[ \Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i \]

\[ \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij} \]

\[ \sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij} \]

where \( \Delta x_i \): the return on asset \( i \) in 1 day
\( \alpha_i \): amount being invested on asset \( i \)
\( \sigma_i \): the volatility of variable \( i \)
\( \sigma_P \): the portfolio's standard deviation
Consider a portfolio of options dependent on a single stock price, $S$. Define

\[ \delta = \frac{\Delta P}{\Delta S} \]

and

\[ \Delta x = \frac{\Delta S}{S} \]
• As an approximation

\[ \Delta P = \delta \Delta S = S\delta \Delta x \]

• Similarly when there are many underlying market variables

\[ \Delta P = \sum_i S_i \delta_i \Delta x_i \]

where \( \delta_i \) is the delta of the portfolio with respect to the \( i \)th asset
Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

$$\Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2$$

this becomes

$$\Delta P = S\delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2$$
Quadratic Model continued

With many market variables we get an expression of the form

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where

$$\delta_i = \frac{\partial P}{\partial S_i} \quad \gamma_{ij} = \frac{\partial^2 P}{\partial S_i S_j}$$

This is not as easy to work with as the linear model
Comparison of Approaches

- Historical simulation lets historical data determine distributions, but is computationally slower.
- Monte Carlo simulation can handle any distribution and is easy to incorporate back and stress tests. It allows to achieve any accuracy (if the time is not an issue). It is computationally intensive.
- Model building approach assumes normal distributions for market variables. It tends to give poor results for low delta portfolios.
Stress Testing

• This involves testing how well a portfolio performs under some of the most extreme market moves seen in the last 10 to 20 years
Back-Testing

- Tests how well VaR estimates would have performed in the past
- We could ask the question: How often was the actual 10-day loss greater than the 99%/10 day VaR?