HOMEOMORPHISMS OF ČECH–STONE REMAINDERS: 
THE ZERO-DIMENSIONAL CASE

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Abstract. We prove, using a weakening of the Proper Forcing Axiom, 
that any homemomorphism between Čech–Stone remainders of any two 
locally compact, zero-dimensional Polish spaces is induced by a homeo-
morphism between their cocompact subspaces.

1. Introduction

The Čech–Stone remainder (also known as corona) $\beta X \setminus X$ of a topological 
space $X$ will be denoted $X^*$. A continuous map $\varphi : X^* \rightarrow Y^*$ is called trivial 
if there is a continuous $e : X \rightarrow Y$ such that $\varphi = e^*$, where $e^* = \beta e \setminus e$ and $\beta e$ 
is the unique continuous extension of $e$ to $\beta X$. It follows that two remainders $X^*$ and $Y^*$ are homeomorphic via a trivial map if and only if there are 
cocompact subspaces of $X$ and $Y$ which are themselves homeomorphic. In 
this paper we prove the following (see §2 for the definitions).

Theorem 1.1. OCA and MA$_{\aleph_1}$ together imply that every homeomorphism 
between Čech–Stone remainders of locally compact, zero-dimensional, Polish 
spaces is trivial.

This proves a special case of the rigidity conjecture that forcing axioms 
imply all homeomorphisms between Čech–Stone remainders of locally compact, 
noncompact Polish spaces are trivial (see [10], [9], [3]). In contrast, the 
Continuum Hypothesis (CH), implies that Čech–Stone remainders of locally 
compact, noncompact, zero-dimensional Polish spaces are homeomorphic. 
This is a consequence of Parovičenko’s topological characterization of $\omega^*$ 
(see e.g., [25]). Stone duality between compact, zero-dimensional, Hausdorff 
spaces and Boolean algebras of their clopen sets provides a model-theoretic 
reformulation of this malleability phenomenon. For a locally compact, non-
compact Hausdorff space $X$ let $\mathcal{C}(X)$ denote the algebra the clopen subsets 
of $X$ and let $\mathcal{K}(X)$ denote its ideal of compact-open sets. If $X$ and $Y$ are 
in addition zero-dimensional, then continuous maps from $X^*$ to $Y^*$ func-
torially correspond to Boolean algebra homomorphisms from $\mathcal{C}(Y)/\mathcal{K}(Y)$ 
into $\mathcal{C}(X)/\mathcal{K}(X)$. All of these algebras are elementarily equivalent and 
(assuming CH) saturated, and therefore isomorphic (see [6] for the details 
and an extension to not necessarily zero-dimensional spaces).  

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\footnote{There is a deeper metamathematical explanation of the effect of CH; see [31].}
Back to rigidity, Theorem 1.1 belongs to a long line of results going back to Shelah’s groundbreaking construction of an oracle-cc forcing extension of the universe in which all autohomeomorphisms of $\omega^*$ are trivial ([22]). Shelah’s proof was recast in terms of forcing axioms PFA and OCA+MA$_{\aleph_1}$ in [23] and [27], respectively. The latter axiom also implies that homeomorphisms between Čech–Stone remainders between countable locally compact spaces, as well as their arbitrary powers, are trivial ([9, §4]) as well as strong negations of Parovičenko’s theorem ([5], [7]).

The interest in quotient rigidity results was rejuvenated by the discovery that the noncommutative analogue of ‘are all automorphisms of $\omega^*$ (or of $\mathcal{P}(\omega)/\text{fin}$) trivial?’ was a prominent open problem in the theory of operator algebras. Motivated by their work on analytic K-homology, Brown, Douglas, and Fillmore asked whether the Calkin algebra associated with the separable, infinite-dimensional, complex Hilbert space has outer automorphisms ([2]). Like its commutative analogue, this question cannot be resolved in ZFC, with CH and OCA implying the opposite answers ([21], [11]). Other rigidity results in the setting of C$^*$-algebras were proved for reduced products of the form $\prod_n A_n / \bigoplus_n A_n$ in case when all $A_n$ are matrix algebras ([16], [15]), separable UHF algebras ([19]) or unital separable nuclear C$^*$-algebras ([28], [20]).

A general rigidity conjecture for corona C$^*$-algebras was stated and partially verified in [3]. The model theory of coronas proved to be a bit more complex than that of Boolean algebras. While the reduced products are countably saturated ([14]), coronas possess only a modest degree of saturation ([12], [8], [30], [13]). In return, C$^*$-algebras provided a vantage point that resulted in the construction of nontrivial autohomeomorphisms of $X^*$ for every noncompact, locally compact, metrizable manifold using CH ([29]).

We note that Theorem 1.1 is not optimal. The first author’s proof that all zero-dimensional, locally compact, Polish spaces satisfy the weak extension principle ([9, Theorem 4.10.1]) will appear elsewhere. Dow refuted the related strong extension principle ([9, Question 4.11.4]) by constructing a nontrivial continuous map from $\omega^*$ into $\omega^*$ (i.e., one that does not have a continuous extension to a map from $\beta\omega$ into $\beta\omega$) in ZFC ([4]). An alternative proof of our main result from a stronger assumption (PFA) is given by [14, Theorem 4.3].

In section 2 we introduce some of the language required to prove Theorem 1.1. Section 3 treats embeddings of $\mathcal{P}(\omega)/\text{fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$, and we show that under OCA+MA$_{\aleph_1}$, every such embedding is trivial. Much of the proof follows the work in [27] and [26] with only minor modifications, so to avoid treading the same ground we only prove one of the ingredients going into this theorem. Section 4 completes the proof of Theorem 1.1 through an analysis of coherent families of continuous functions.

2The only previously known case was $X = \mathbb{R}$, see [17] and [14]
2. Notation

Our terminology is standard (see [18]). The assumption of Theorem 1.1 is a consequence of the Proper Forcing Axiom, PFA. OCA abbreviates the Open Coloring Axiom ([24]; not to be confused with the eponymous OCA of [1]), and MA_{\aleph_1} refers to Martin’s Axiom for \aleph_1 dense sets.

If E is a set, then |E|^2 will denote the set of unordered pairs from E. If \( M \subseteq [E]^2 \), then a set \( H \subseteq E \) is called \( M \)-homogeneous if \( |H|^2 \subseteq M \). The Open Coloring Axiom states: for every separable metric space \( E \) and every partition \( |E|^2 = M_0 \cup M_1 \) such that \( M_0 \) is open (here we identify \( |E|^2 \) with a symmetric subset of \( E \times E \) minus the diagonal), either

1. there is an uncountable \( M_0 \)-homogeneous set, or
2. there is a cover of \( E \) by countably-many \( M_1 \)-homogeneous sets.

We fix a zero-dimensional, locally compact and noncompact Polish space \( X \). Let \( \langle K_n \mid n < \omega \rangle \) be an increasing sequence of compact-open sets in \( X \), such that \( X = \bigcup K_n \). Then \( \mathcal{K}(X) \) is generated by \( \langle K_n \mid n < \omega \rangle \) since

\[
K \in \mathcal{K}(X) \iff \exists n \ K \subseteq K_n
\]

It is easy to see that \( \mathcal{C}(X) \) has size continuum, whereas \( \mathcal{H}(X) \) is countable. When \( A, B \in \mathcal{C}(X) \) are distinct, we write \( \delta(A, B) \) for the least \( n \) such that \( A \cap X_n \neq B \cap X_n \). If

\[
d(A, B) = \begin{cases} 2^{-\delta(A, B)} & A \neq B \\ 0 & A = B \end{cases}
\]

then \( d \) is a Polish metric on \( \mathcal{C}(X) \).

Let \( X_0 = K_0 \) and \( X_{n+1} = K_{n+1} \setminus K_n \). We will often identify \( \mathcal{C}(X) \) with \( \prod_n \mathcal{C}(X_n) \), and \( \mathcal{P}(\omega) \) with \( {< \omega}2 \). Under these identifications, \( \mathcal{K}(X) \) maps to \( \bigoplus_n \mathcal{C}(X_n) \) (the set of functions in \( \prod_n \mathcal{C}(X_n) \) which are nonempty on only finitely many coordinates) and fin to \( \leq \omega2 \). If \( Y \) and \( Z \) are zero-dimensional, locally compact Polish spaces, \( \varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(Z)/\mathcal{K}(Z) \) is a homomorphism, and \( U \in \mathcal{C}(Y) \), then we write \( \varphi\lceil U \) for the restriction \( \varphi\lceil \mathcal{C}(U)/\mathcal{K}(U) \).

When working with the quotient \( \mathcal{G}(X)/\mathcal{K}(X) \) we will write \([A]\) for the equivalence class of some \( A \in \mathcal{C}(X) \).

3. Embeddings of \( \mathcal{P}(\omega)/\text{fin} \) into \( \mathcal{C}(X)/\mathcal{K}(X) \)

Let \( e : X \to \omega \) be a continuous map. If \( e^{-1}(n) \) is compact for every \( n \), then we say \( e \) is compact-to-one. If \( e \) is compact-to-one, then the map \( a \mapsto e^{-1}(a) \), from \( \mathcal{P}(\omega) \) to \( \mathcal{C}(X) \), induces a homomorphism \( \varphi_e : \mathcal{P}(\omega)/\text{fin} \to \mathcal{C}(X)/\mathcal{K}(X) \). Moreover, \( \varphi_e \) is injective if and only if \( e \) is finite on compact sets. We call a homomorphism \( \varphi : \mathcal{P}(\omega)/\text{fin} \to \mathcal{C}(X)/\mathcal{K}(X) \) trivial if it is of the form \( \varphi_e \) for some compact-to-one, continuous \( e \).

In this section we prove

**Theorem 3.1.** Assume OCA+MA_{\aleph_1}, and suppose

\[
\varphi : \mathcal{P}(\omega)/\text{fin} \to \mathcal{C}(X)/\mathcal{K}(X)
\]
is an injective homomorphism. Then \( \varphi \) is trivial.

Towards the proof of Theorem 3.1, we fix an injective homomorphism 
\( \varphi : P(\omega)/\text{fin} \to C(X)/\mathcal{H}(X) \) and we define 
\[
I = \{ a \subseteq \omega \mid \varphi|a \text{ is trivial} \}
\]
Note that \( I \) is an ideal on \( \omega \).

A family \( \mathcal{A} \subseteq P(\omega) \) is called almost disjoint if for all distinct \( a, b \in \mathcal{A} \), \( a \cap b = {}^* \emptyset \). Such a family \( \mathcal{A} \) is called treelike if there is some tree \( T \) on \( \omega \) and a bijection \( t : \omega \to \omega \omega \) under which each \( a \in \mathcal{A} \) corresponds to a branch through \( T \), and vice-versa. The following lemma is proven in [27].

**Lemma 3.2.** Assume \( MA_{\aleph_1} \). Then for every uncountable almost-disjoint family \( \mathcal{A} \) of subsets of \( \omega \) we may find an uncountable \( \mathcal{B} \subseteq \mathcal{A} \) and partitions \( b = b_0 \cup b_1 \) for \( b \in \mathcal{B} \) such that each family \( \mathcal{B}_i = \{ b \mid b \in \mathcal{B} \} \) is treelike.

The following three lemmas do not directly follow from the work in [27], but their proofs are nearly the same, modulo some minor modifications. Recall that an ideal \( I \subseteq P(\omega) \) is a P-ideal if for each countable sequence \( A_n \in I \ (n < \omega) \) there is an \( A \in I \) such that for all \( n < \omega \), \( A_n \subseteq {}^* A \).

**Lemma 3.3.** Assume \( OCA + MA_{\aleph_1} \). If \( I \) is a dense P-ideal then \( \varphi \) is trivial.

**Lemma 3.4.** Assume \( b > \aleph_1 \). If \( I \) is not a dense P-ideal, then there is an uncountable almost-disjoint family \( \mathcal{A} \subseteq P(\omega) \) which is disjoint from \( I \).

**Lemma 3.5.** Assume \( OCA \). Let \( \mathcal{A} \) be an uncountable, tree-like, almost-disjoint family of subsets of \( \omega \). Then \( I \setminus \mathcal{A} \) is countable.

Theorem 3.1 now follows from a straightforward combination of Lemmas 3.2, 3.3, 3.4 and 3.5. To illustrate the kind of modifications necessary in translating from [27], we will give a proof of Lemma 3.3.

**Proof of Lemma 3.3.** For each \( a \in I \), we fix \( Z_a \in C(X) \) and a continuous, compact-to-one map \( e_a : Z_a \to a \) such that \( \varphi([a]) = [Z_a] \) and for all \( b \subseteq a \), \( \varphi([b]) = [e_a^{-1}(b)] \). We define \( f_a : \omega \to \mathcal{H}(X) \) by 
\[
f_a(n) = e_a^{-1}(\{n\})
\]
Define a partition \( [I]^2 = M_0 \cup M_1 \) by placing \( \{a, b\} \in M_0 \) if and only if there is some \( n \in a \cap b \) such that \( f_a(n) \neq f_b(n) \). Then \( M_0 \) is open when \( I \) is given the topology obtained by identifying \( a \in I \) with \( (a, f_a) \in P(\omega) \times {}^* \mathcal{H}(X) \).

**Claim 3.6.** There is no uncountable, \( M_0 \)-homogeneous subset \( H \) of \( I \).

**Proof.** Assume \( H \) is such a set, and that \( |H| = \aleph_1 \). Since \( I \) is a P-ideal, there is a set \( \bar{H} \subseteq I \) such that for every \( a \in H \) there is some \( b \in \bar{H} \) with \( a \subseteq {}^* b \), and moreover \( \bar{H} \) is a chain of order-type \( \omega_1 \) with respect to \( \subseteq {}^* \). By OCA, there is an uncountable subset of \( \bar{H} \) which is homogeneous for one of the two colors \( M_0 \) and \( M_1 \); hence, by passing to this subset, we may assume \( \bar{H} \) is either \( M_0 \) or \( M_1 \) homogeneous.
Say $\bar{H}$ is $M_1$-homogeneous. Put $\bar{a} = \bigcup \bar{H}$, and $\bar{f} = \bigcup_{a \in \bar{H}} f_a$. Then $\bar{f} : \bar{a} \to \mathcal{C}(X)$, and for all $a \in H$ we have $a \subseteq \bar{a}$ and $f_a \upharpoonright (a \cap \bar{a}) = f_a \upharpoonright (a \cap \bar{a})$. Choose $n$ so that for uncountably many $a \in H$, we have $a \setminus n \subseteq \bar{a}$, and $f_a \upharpoonright a \setminus n = f_a \upharpoonright a \setminus n$. Then if $a, b \in H$ are such, and $f_a[n] = f_b[n]$, we have \{a, b\} $\in M_1$, a contradiction.

So $H$ is $M_0$-homogeneous. Define a poset $\mathbb{P}$ as follows. Put $p \in \mathbb{P}$ if and only if $p = (A_p, m_p, H_p)$ where $m_p < \omega$, $A_p \in \mathcal{C}(K_{m_p})$, and $H_p \in [\bar{H}]^{<\omega}$, and for all distinct $a, b \in H_p$, there is an $n \in a \cap b$ such that

$$-(f_a(n) \cap A_p = \emptyset \iff f_b(n) \cap A_p = \emptyset)$$

That is, one of $f_a(n)$, $f_b(n)$ is disjoint from $A_p$, and the other isn’t. Put $p \leq q$ if and only if $m_p \geq m_q$, $A_p \cap K_{m_q} = A_q$, and $H_p \supseteq H_q$.

First we must show that $\mathbb{P}$ is ccc. Suppose $\mathcal{X}$ is an uncountable subset of $\mathbb{P}$. We may assume without loss of generality that for some fixed $m$ and $A \in \mathcal{C}(K_m)$, and for all $p \in \mathcal{X}$, $m_p = m$ and $A_p = A$, and moreover that $H_p$ is the same size for all $p \in \mathcal{X}$. Let $a_p$ be the minimal element of $H_p$ under $\subseteq^*$, for each $p \in \mathcal{X}$. Find $n_p$ so that for all $a \in H_p$,

$$f_{a_p}[\upharpoonright (a_p \setminus n_p)] \subseteq f_a \quad e_{a_p}^\prime K_m \subseteq n_p$$

We may assume that for some fixed $n$, we have $n_p = n$ for all $p \in \mathcal{X}$. Find $p, q \in \mathcal{X}$ with $f_{a_q}[n] = f_{a_q}[n]$. Since $\{a_p, a_q\} \in M_0$, there is some $k \in a_p \cap a_q$ such that $f_{a_p}(k) \neq f_{a_q}(k)$. Then $k \geq n$, and so $f_{a_q}(k) \cap K_m = f_{a_q}(k) \cap K_m = \emptyset$. At least one of $f_{a_p}(k) \setminus f_{a_q}(k)$ and $f_{a_q}(k) \setminus f_{a_q}(k)$ must be nonempty; whichever one it is, call it $B$. Put $A_r = A \cup B$ and $H_r = H_p \cup H_q$, and choose $m_r$ large enough that $A_r \subseteq K_{m_r}$. Then $r = (A_r, m_r, H_r) \in \mathbb{P}$, and $r \leq p, q$.

By MA$_{\aleph_1}$, there is a set $A \in \mathcal{C}(X)$ and an uncountable $H^* \subseteq \bar{H}$ such that for all distinct $a, b \in H^*$,

$$\exists n \in a \cap b \quad -(f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

Fix $x \subseteq \omega$ such that $F(x) = A$. Then for all $a \in H^*$, $e_a^{-1}(x \cap a)\Delta (A \cap F(a))$ is compact; hence there are $k_a$ and $m_a$ such that $e_a^{-1}(x \cap k_a) = (A \cap F(a)) \setminus K_{m_a}$ and $e_a^{-1}(a \setminus k_a) = F(a) \setminus K_{m_a}$.

Then, for all $n \in a \setminus k_a$, $n \in x$ implies $f_a(n) \not\subseteq A$, and $n \notin x$ implies $f_a(n) \cap A = \emptyset$. Fix distinct $a, b \in H^*$ with $k_a = k_b = k$, and $f_a[k] = f_b[k]$. Then

$$\forall n \in a \cap b \quad (f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

This contradicts the choice of $A$. \hfill \Box

By OCA, there is a cover of $\mathcal{I}$ by countably many sets $\mathcal{I}_n$, each of which is $M_1$-homogeneous. Since $\mathcal{I}$ is a P-ideal, at least one of the $\mathcal{I}_n$’s must be cofinal in $\mathcal{I}$ with respect to $\subseteq^*$. Choose such an $\mathcal{I}_n$, and let $f = \bigcup \{f_a \mid a \in \mathcal{I}_n\}$. Then $f$ is a function from some subset of $\omega$ to $\mathcal{C}(X)$.

Setting $e(x) = n$ if and only if $x \in f(n)$, we get a function $e : X \to \omega$,
and since $\mathcal{I}$ is dense and $\mathcal{I}_n$ cofinal in $\mathcal{I}$, $a \mapsto e^{-1}(a)$ witnesses that $\varphi$ is trivial. \hfill $\square$

4. COHERENT FAMILIES OF CONTINUOUS FUNCTIONS

**Theorem 4.1.** Assume $OCA+MA_{\aleph_1}$. Let $X$ and $Y$ be zero-dimensional, locally compact Polish spaces, and let $\varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$ be an isomorphism. Then there are compact-open $K \subseteq X$ and $L \subseteq Y$, and a homeomorphism $e : X \setminus K \to Y \setminus L$, such that for all $A \in \mathcal{C}(Y \setminus L)$, $\varphi([A]) = [e^{-1}(A)]$.

By Stone duality, a homeomorphism $\varphi : X^* \to Y^*$ induces an isomorphism $\hat{\varphi} : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$, and any map $e$ as in the conclusion to Theorem 4.1 will in this case be a witness to the triviality of $\varphi$. Hence Theorem 4.1 implies Theorem 1.1. Before proving Theorem 4.1 we note a corollary involving definable isomorphisms.

**Corollary 4.2.** Suppose $X$ and $Y$ are zero-dimensional, locally compact, Polish spaces, and $\varphi : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$ is an isomorphism such that the set

$$\Gamma = \{(A,B) \in \mathcal{C}(Y) \times \mathcal{C}(X) \mid \varphi([A]) = [B]\}$$

is Borel. Then $\varphi$ is trivial.

**Proof of Corollary 4.2.** The fact that $\varphi$ is an isomorphism between $\mathcal{C}(Y)/\mathcal{K}(Y)$ and $\mathcal{C}(X)/\mathcal{K}(X)$ can be written as a $\Pi^1_2$ statement using $\Gamma$; hence by Schoenfield absoluteness, if $V^P$ is a forcing extension satisfying $OCA+MA_{\aleph_1}$ (see [24]), then in $V^P$ the map $\hat{\varphi} : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$, defined from the reinterpretation of $\Gamma$ in $V^P$, is also an isomorphism. By Theorem 4.1, then, we have in $V^P$ that

$$\exists e \in C(X,Y) \forall A \in \mathcal{C}(Y) \hat{\varphi}([A]) = [e^{-1}(A)]$$

where $C(X,Y)$ denotes the space of continuous maps from $X$ to $Y$. This can be written as a $\Sigma^1_2$ statement and so by Schoenfield absoluteness again, it must be true in $V$ with $\varphi$ replacing $\hat{\varphi}$. \hfill $\square$

Before the proof of Theorem 4.1 we set down some more notation. Fix $X, Y$ and $\varphi$ as in the statement of the theorem. Let $L_n$ be an increasing sequence of compact subsets of $Y$, with union $Y$, and let $Y_{n+1} = L_{n+1} \setminus L_n$ and $Y_0 = L_0$. Let $\mathcal{B}$ be a countable base for $Y$ consisting of compact-open sets, such that

- for all $U \in \mathcal{B}$, the set of $V \in \mathcal{B}$ with $V \supseteq U$ is finite and linearly ordered by $\subseteq$, and
- for all $U \in \mathcal{B}$ and all $n < \omega$, either $U \subseteq Y_n$ or $U \cap Y_n = \emptyset$.

It follows that for all $U, V \in \mathcal{B}$, either $U \cap V = \emptyset$, $U \subseteq V$, or $V \subseteq U$. Let $\mathcal{P}$ be the poset of all partitions of $Y$ into elements of $\mathcal{B}$, ordered by refinement;

$$P \prec Q \iff \forall U \in P \exists V \in Q \ U \subseteq V$$
We also use $\prec$ to denote eventual refinement:

$$P \prec Q \iff \forall U \in P \exists V \in Q \ U \subseteq V$$

When $P \prec Q$ we let $\Gamma(P, Q)$ be the least $n$ such that every $U \in P$ disjoint from $L_n$ is contained in some element of $Q$.

For a given $P \in \mathcal{P}$, let $s_P : Y \to P$ be the unique function satisfying $x \in s_P(x)$ for all $x \in Y$; similarly, when $P, Q \in \mathcal{P}$ and $P \prec Q$ we let $s_{PQ} : P \to Q$ be the unique function satisfying $U \subseteq s_{PQ}(U)$ for all $U \in P$.

These maps induce embeddings $\sigma_P : \mathcal{P}(P)/\text{fin} \to \mathcal{C}(Y)/\mathcal{K}(Y)$ and $\sigma_{PQ} : \mathcal{P}(Q)/\text{fin} \to \mathcal{P}(P)/\text{fin}$ in the usual way.

Finally, we need to prove a uniqueness result for maps $e : Z \to \omega$ inducing the same map $\mathcal{P}(\omega)/\text{fin} \to \mathcal{C}(\mathcal{Z})/\mathcal{K}(\mathcal{Z})$.

**Lemma 4.3.** Suppose $Z \in \mathcal{C}(X)$ and $e, f : Z \to \omega$ are continuous, compact-to-one maps, such that $e^{-1}(a) \Delta f^{-1}(a)$ is compact for every $a \subseteq \omega$. Then $\{x \in Z \mid e(x) \neq f(x)\}$ is compact.

**Proof.** Suppose not; then for some infinite set $I \subseteq \omega$ and all $n \in I$, there is a point $x_n \in Z \cap X_n$ such that $e(x_n) \neq f(x_n)$. Since $e$ and $f$ are compact-to-one, we may assume also that $m \neq n$ implies $e(x_m) \neq e(x_n)$ and $f(x_m) \neq f(x_n)$. Now define a coloring $F : [I]^2 \to 3$ by

$$F(\{m < n\}) = \begin{cases} 0 & e(x_m) \neq f(x_n) \land e(x_n) \neq f(x_m) \\ 1 & e(x_m) = e(x_n) \land f(x_m) \neq f(x_n) \\ 2 & e(x_m) \neq f(x_n) \land f(x_m) = f(x_n) \end{cases}$$

By Ramsey’s theorem, there is an infinite set $a \subseteq I$ which is homogeneous for this coloring. Suppose first that $a$ is 1-homogeneous, and let $m < n < k$ be members of $a$. Then

$$e(x_m) = f(x_n) \quad \text{and} \quad e(x_m) = f(x_k) \quad \text{and} \quad e(x_n) = f(x_k)$$

which implies $e(x_n) = f(x_n)$, a contradiction. Similarly, $a$ cannot be 2-homogeneous.

Now suppose $a$ is 0-homogeneous. Let $a = a_0 \cup a_1$ be a partition of $a$ into two infinite sets, and put $W_i = \{x_n \mid n \in a_i\}$ and $W = \{x_n \mid n \in a\} = W_0 \cup W_1$. From the homogeneity of $a$, it follows that $e^W \cap f^W = \emptyset$, and hence (as $e$ and $f$ are injective on $W$)

$$W \cap e^{-1}((e''W_0) \cup (f''W_1)) = W_0 \quad \text{and} \quad W \cap f^{-1}((e''W_0) \cup (f''W_1)) = W_1$$

So, if $b = e''W_0 \cup f''W_1$, we have $W \subseteq e^{-1}(b) \Delta f^{-1}(b)$. But $W$ is not compact, so this is a contradiction. $\square$

**Proof of Theorem 4.1.** For each $P \in \mathcal{P}$, let $\varphi_P = \varphi \circ \sigma_P$. Then $\varphi_P$ is an embedding of $\mathcal{P}(P)/\text{fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$. By Theorem 4.1, there is a continuous map $e_P : X \to P$ such that $a \mapsto e_P^{-1}(a)$ lifts $\varphi_P$. Note that if
$P, Q \in \mathbb{P}$ and $P \prec^* Q$, then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P}(P)/\text{fin} & \xrightarrow{\varphi_P} & \mathcal{C}(X)/\mathcal{X}(X) \\
\sigma_{PQ} & & \downarrow \varphi_Q \\
\mathcal{P}(Q)/\text{fin} & & \\
\end{array}
\]

So by Lemma 4.3, the set \( \{ x \in X \mid s_{PQ}(e_P(x)) \neq e_Q(x) \} \) is compact.

Now let \( [\mathbb{P}]^2 = M_0 \cup M_1 \) be the partition defined by

\[
\{ P, Q \} \in M_0 \iff \exists x \in X \quad s_{P \vee Q}(e_P(x)) \neq s_{Q \vee Q}(e_Q(x))
\]

Here \( P \vee Q \) is the finest partition coarser than both \( P \) and \( Q \). If we define \( f_P : \mathcal{B} \to \mathcal{C}(X) \) by

\[
f_P(U) = \{ x \in X \mid e_P(x) \subseteq U \}
\]

then we have

\[
\{ P, Q \} \in M_0 \iff \exists U \in \mathcal{B} \quad f_P(U) \neq f_Q(U)
\]

and it follows that \( M_0 \) is open in the topology on \( \mathbb{P} \) obtained by identifying \( P \) with \( f_P \).

**Claim 4.4.** There is no uncountable, \( M_0 \)-homogeneous subset of \( \mathbb{P} \).

**Proof.** Suppose \( H \) is such, and has size \( \aleph_1 \). Using \( MA_{\aleph_1} \) with a simple modification of Hechler forcing, we see that there is some \( \bar{P} \in \mathbb{P} \) such that \( P \prec^* \bar{P} \) for all \( P \in H \). By thinning out \( H \) and refining a finite subset of \( \bar{P} \), we may assume that \( P \succ \bar{P} \) for all \( P \in H \), and moreover that there is an \( \bar{n} \) such that for all \( P \in H \),

\[
\{ x \in X \mid s_{P \vee e_{\bar{P}}}(e_P(x)) \neq e_P(x) \} \subseteq K_{\bar{n}}
\]

Now fix \( P, Q \in H \) such that \( e_P|K_{\bar{n}} = e_Q|K_{\bar{n}} \). Then \( s_{P \vee Q} \circ e_P = s_{Q \vee Q} \circ e_Q \), contradicting the fact that \( \{ P, Q \} \in M_0 \). \( \square \)

By OCA, there is a countable cover of \( \mathbb{P} \) by \( M_1 \)-homogeneous sets; since \( \mathbb{P} \) is countably directed under \( \succ^* \), it follows that one of them, say \( Q \), is cofinal in \( \mathbb{P} \). It follows moreover that for some \( n \), we have

\[
\forall P \in \mathbb{P} \exists Q \in Q \quad \Gamma(Q, P) \leq n
\]

That is, \( Q \) is cofinal in \( \mathbb{P} \) under \( \succ^n \) defined by

\[
P \prec^n Q \iff \forall U \in P \quad (U \cap L_n = \emptyset \implies \exists V \in Q \ U \subseteq V)
\]

**Claim 4.5.** There is a compact set \( K \subseteq X \) and a unique continuous map \( e : X \setminus K \to Y \) satisfying

\[
\forall x \in X \setminus K \quad e(x) \in \bigcap_{P \in Q} e_P(x)
\]
Proof. Fix \( x \in X \). If \( P, Q \in \mathbb{Q} \), then by \( M_1 \)-homogeneity of \( \mathbb{Q} \) we have
\[
s_{P \lor Q}(e_P(x)) = s_{Q \lor Q}(e_Q(x))
\]
Then, the unique member of \( P \lor Q \) containing \( e_P(x) \) is the same as the unique member of \( P \lor Q \) containing \( e_Q(x) \). It follows that \( e_P(x) \cap e_Q(x) \neq \emptyset \), and so either \( e_P(x) \subseteq e_Q(x) \) or vice-versa. Then the collection \( \{e_P(x) \mid P \in \mathbb{Q}\} \) is a chain, and hence by compactness has nonempty intersection.

Now let
\[
K = \{x \in X \mid \forall P \in \mathbb{Q} \ e_P(x) \subseteq L_n \} \subseteq \bigcap_{P \in \mathbb{Q}} e^{-1}_P(P \cap \mathcal{C}(L_n))
\]
Then \( K \) is contained in a compact set. If \( x \in X \setminus K \) and \( P \in \mathbb{Q} \), then \( e_P(x) \) is disjoint from \( L_n \). Then for any \( x \in X \setminus K \) and \( \epsilon > 0 \), there is some \( P \in \mathbb{Q} \) such that \( e_P(x) \) has diameter less than \( \epsilon \) (since \( \mathbb{Q} \) is cofinal in \( \mathbb{P} \) under \( \succ^n \)). Thus \( e \), as defined above, is unique.

To see that \( e \) is continuous, note that for any open \( U \subseteq X \),
\[
x \in e^{-1}(U) \iff \exists P \in \mathbb{Q} \ e_P(x) \subseteq U
\]
\[\square\]

Claim 4.6. The map \( U \mapsto e^{-1}(U) \) lifts \( \varphi \).

Proof. Fix \( P \in \mathbb{Q} \), and let \( U \in P \). Then clearly, for all \( x \in X \setminus K \), \( e_P(x) = U \) if and only if \( e(x) \in U \). Since there are only finitely many \( U \in P \) such that one of \( e^{-1}_P(\{U\}) \) or \( e^{-1}(U) \) meets \( K \), it follows that
\[
\forall \infty U \in P \ e^{-1}_P(\{U\}) = e^{-1}(U)
\]
Then \( U \mapsto e^{-1}(U) \) lifts \( \varphi_P \).

Now fix \( A \in \mathcal{C}(Y) \). Then there is some \( P \in \mathbb{P} \) such that \( A \) can be written as a union of a subset of \( P \). Find \( Q \in \mathbb{Q} \) with \( Q \prec^* P \); then, up to a compact set, \( A \) can be written as a union of some subset \( a \) of \( Q \). Hence,
\[
\varphi[A] = \varphi_Q[a] = [e^{-1}(A)]
\]
\[\square\]

References


