Many thanks to the organizers!
Many thanks to all the organizers!

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Lecture 3 recap:
Another look at the types

Definition (Type as a character)
Fix $n$, $B \leq C$, and $\bar{c} \in C^n$. The *complete type* of $\bar{c}$ in $C$ over $B$ is

$$\text{type}_C(\bar{c}/B) : \mathbb{F}^n(B) \ni \varphi \mapsto \varphi^C(\bar{c}).$$

Lemma

1. $\text{type}_C(\bar{c}/B)$ is a character on $\mathbb{F}^n(B)$.
2. $\text{type}_C(\langle \rangle / \mathbb{C}) = \text{Th}(C)$.
3. $\text{type}_C(\langle \rangle / B)$ is the theory of $C$ in $\mathbb{F}^0(B)$. 
Type as a set of conditions

With $B \leq C$, a $k$-condition over $C$ in $\mathbb{F}^k(B)$

$$\varphi(\bar{x}) = r$$

where $\varphi \in \mathbb{F}^k(B)$ and $r \in \mathbb{R}$.

$k$-type over $B$ is a set of $k$-conditions over $B$.

The type of $\bar{c}$ in $C$ over $B$:

$$\{ \varphi(\bar{x}) = \varphi^C(\bar{c}) | \varphi \in \mathbb{F}^k(B) \}.$$  

Definition

1. $\bar{c} \in C^k_1$ realizes $t(\bar{x})$ if $\varphi^C(\bar{c}) = r$ for every $\varphi(\bar{x}) = r$ in $t(\bar{x})$.
2. $t(\bar{x})$ is consistent if every finite subset is approximately realized.
Definition

$C$ is countably saturated if for every separable $B \leq C$, every consistent type over $B$ is realized in $C$.

(Note: Type being consistent depends on the way $B$ sits inside $C$.) Thanks to Jamie Gabe for suggesting the following.

Proposition

TFAE

1. $C$ is countably saturated.

2. For every $k$ and separable $B \leq C$,

$$\{\text{type}_C(\bar{c}/B) | \bar{c} \in C_1^k\}$$

is weak*-closed in $\mathbb{F}^k(B)^*$.
Theorem

For every $B$, $B_U$ and $B_\infty$ are countably saturated.
The Continuum Hypothesis

CH, the Continuum Hypothesis:
Every set of cardinality $2^\aleph_0$ (e.g., $A_\mathcal{U}$, $A_\infty$, $\ell_2(\mathbb{N})$, $B(\ell_2(\mathbb{N}))$, ...) has a well-ordering all of whose proper initial segments are countable.

CH is independent from ZFC (Gödel, Cohen).

Proposition (Platek)

A classification result for separable $C^*$–algebras proved using CH can be proved in ZFC alone.

I will assume the CH throughout today’s lecture.
Recall
\( \mathbb{F}^0(\mathbb{C}) \): the \( \mathbb{R} \)-algebra of all sentences.
\( \text{Th}(A) \): the evaluation character on \( \mathbb{F}^0(\mathbb{C}) \), defined by \( \varphi \mapsto \varphi^A \).

**Definition**

\( A \) and \( B \) are *elementarily equivalent*, \( A \equiv B \), if \( \text{Th}(A) = \text{Th}(B) \).

**Example**

1. Łoś: \( A \equiv A_U \).
2. Ghasemi: If \( \mathcal{F} \) is a filter such that the restriction of \( \mathcal{F} \) to any \( \mathcal{F} \)-positive set is not an ultrafilter, then \( A_\infty \equiv A_\mathcal{F} \).
3. There are elementarily equivalent, but not isomorphic, separable, simple AF algebras.
4. Ditto for Kirchberg algebras.
The density character of $C$, $\chi(C)$, is the smallest cardinality of a dense subset. (Thus $\chi(C) = \aleph_0$ iff $C$ is separable.)

**Definition**
We say that $C$ is *saturated* if every consistent type $t$ over $B \leq C$ such that $\chi(B) < \chi(C)$ is realized in $C$.

**Lemma (CH)**
$C_\mathcal{U}$ and $C_\infty$ are saturated for every $C$ such that $\chi(C) \leq 2^{\aleph_0}$.

**Proof.**
CH implies
$|X| < 2^{\aleph_0}$ iff $X$ is countable. 

Theorem (Keisler, I think)

If $C$ and $D$ are saturated and $\chi(C) = \chi(D)$, then $C \equiv D$ iff $C \cong D$. 
Corollary

*CH implies that for all $A, \mathcal{U}, \mathcal{V}$ we have*

$$A_\mathcal{U} \cong A_\mathcal{V} \cong (A_\mathcal{U})_\mathcal{V} \cong ((A_\mathcal{U})_\mathcal{U})_\mathcal{V} \ldots$$

$$A_\infty \cong (A_\infty)_\infty \cong (A_\mathcal{U})_\infty \cong (A_\infty)_\mathcal{U} \cong ((A_\infty)_\mathcal{U})_\mathcal{V} \infty \ldots$$

Many of these are independent from ZFC. (Shelah, F., Dow–Hart, McKenney–Vignati, Vignati...)
Theorem

$CH$ implies $A_U \cap A' \cong A_V \cap A'$.

Proof.

Find isomorphism $\Phi: A_U \rightarrow A_V$ so that $\Phi \upharpoonright A = \text{id}_A$. 

This is independent from ZFC (F.–Hart–Sherman, F.–Shelah).
Löwenheim–Skolem–Blackadar

**Theorem**

If $C$ is nonseparable and $B \leq C$ is separable, then there is $B \leq A \leq C$.

**Proof:**

**Reduction 1, the Tarski–Vaught test:**

$A \leq C$ iff

$$\inf_{x \in A_1} \varphi^C(\bar{a}, x) \leq \inf_{x \in C_1} \varphi^C(\bar{a}, x).$$

(1)

**Reduction 2**

It suffices to assure (1) for a dense set of $\bar{a} \in A^n$ and a dense subset of $\varphi \in F^n(B)$.

Build $B = A_0 \leq A_1 \leq A_2 \leq \ldots$ and let $A = \bigcup_n A_n$. 

\[\square\]
Corollary (CH)

If $C$ is countably saturated and $\chi(C) = \aleph_1$, then $C = \lim_{\alpha < \aleph_1} A_\alpha$ so that for all $\alpha$

1. $A_\alpha \leq C$, and
2. $(A_\alpha)_U \cong C$ (the isomorphism fixes $A_\alpha$).

Corollary (Keisler–Shelah)

$A \equiv B$ iff $A_U = B_U$ for some $U$. 
Corollary (CH)

If $A$ is separable, then $A_\infty \cong B_\mathcal{U}$ for some separable $B$.

Can we construct $B$ from $A$?

We can construct $\text{Th}_A(B)$ (the theory of $B$ in $\mathcal{F}(A)$) from $\text{Th}(A)$,
Lemma (Eagle–Vignati)

Let $K$ denote the Cantor space.

1. If $A$ is separable, then $A \equiv C(K)$ if and only if $A \approx C(K)$.
2. If $A \leq \ell_\infty/c_0$, $A$ contains the unit, and $A \approx C(K)$, then $A \leq \ell_\infty/c_0$.

(In model-theoretic jargon: Th($C(K)$) is $\aleph_0$-categorical,\(^1\) model-complete, and it admits elimination of quantifiers.)

Proof relies on analyzing the joint spectra of tuples of normal operators.

Corollary (CH)

$C(K)_\infty \approx C(K)_U \approx \ell_\infty/c_0$.

\(^1\)They actually say $\omega$-categorical, but let’s not go there.
Lemma (F.)

Suppose $A$ is unital. Then $A \otimes C(K) \prec A_\infty$, for any $C(K) \hookrightarrow \ell_\infty/c_0$.

\[
\begin{array}{ccc}
C(K) & \xrightarrow{1_A \otimes \text{id}_{C(K)}} & A \otimes C(K) \\
\downarrow & & \downarrow \\
\ell_\infty/c_0 & \rightarrow & A_\infty
\end{array}
\]

(The vertical arrows are elementary embeddings.)

Corollary (CH)

$A_\infty \cong (A \otimes C(K))_\mathcal{U}$ for some (any) $\mathcal{U}$.

(The isomorphism is equal to the identity on $A$.)
Theorem (F.)

Suppose $A$ is separable and unital. Then there are $\Theta_\mathcal{U}$, $\Phi_\mathcal{U}$ such that the diagram commutes.
Proof.
Fix a point $0 \in K$. Consider the structure

$$(A \otimes C(K), A, \Theta, \Phi),$$

where $\Theta(a) = a \otimes 1_{C(K)}$, $\Phi(a \otimes f) = f(0)a$. Then $\Phi \circ \Theta = \text{id}_A$ and $\Theta \circ \Phi$ is a conditional expectation onto $A \otimes 1_{C(K)}$. By Łoś (in the language $\mathbb{F}(A)$ (with added constants for $\Theta, \Phi$, and the distance function to $A$), $((A \otimes C(K))_U, A_U, \Theta_U, \Phi_U)$ satisfies this. Since we worked in $\mathbb{F}(A)$, the embeddings of $A$ into $A_U$ and $A_\infty$ commute. \qed
The following was a question (conjecture?) of Chris Schafhauser

**Theorem**

If $F : \mathcal{C}^*\text{-algebras} \to \mathcal{C}$ is a functor, then for unital, separable $A$ and $B$ and $\varphi : F(A) \to F(B)$, TFAE

1. $\varphi$ is realized by a $^*$-homomorphism $\Phi : A \to B_{\mathcal{U}}$ for some (any) nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

2. $\varphi$ is realized by a $^*$-homomorphism $\Phi : A \to B_\infty$.

$(1) \Rightarrow (2)$, together with the Kirchberg–Phillips–Gabe reindexing, implies that $\varphi$ is realized by $\Psi : A \to B$. 