A proof of the Σ^2_1-absoluteness theorem

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This paper is a highly expanded version of Todorcevic's notes [**15**] and [**14**] which contain the proof of the following result.

THEOREM 1 (Woodin, [**16**]). *Assume there are class many measurable Woodin cardinals, and let* $\phi = (\exists x)\psi(x)$ *be a* Σ^2_1*-formula. If* ϕ *is true in some forcing extension, then it is true in every forcing extension that satisfies CH.*

Although this important result was proved more than twenty years ago, the first published proof, based as the original one on the stationary tower forcing, has appeared only recently in [**7**]. See [**18**] for a recent survey. The proof from [**15**] and [**14**] which we reproduce here, however, uses only elementary forcing arguments involving the Levy collapse and it could therefore be accessible to a wider audience.

None of the results are due to me and the sole purpose of this note is to make these proofs and methods available to a wider set theoretic community. Familiarity with forcing ([**6**]) and large cardinals ([**5**]) is assumed.

In §1 we review some folklore results, and in §2 we define a saturated ideal $\mathcal{I}_\delta$ in $V^{\mathrm{Coll}(\omega_1,<\delta)}$. In §3 basic facts about generic elementary embeddings are reviewed, and it is proved that large cardinals imply all definable sets of reals are Lebesgue measurable. Theorem 1 is proved in §4.

1. Preliminaries

NOTATION. Our set theoretic terminology is standard and the reader is assumed to be familiar with forcing and large cardinals. Undefined terms can be found e.g., in [**5**]. We assume each H_θ is equipped with a predicate for a well-ordering $<_\theta$ that provides Skolem functions. We moreover assume these orderings *cohere*: for $\theta < \lambda$ we have H_θ is a $<_\lambda$-initial segment of H_λ and $<_\lambda \restriction H_\theta = <_\theta$. If $X \subseteq H_\theta$ then $\mathrm{Hull}_\theta(X)$ is the Skolem hull of X inside H_θ.

If M and N are models of a large enough fragment of ZFC (typically elementary submodels of a large enough H_θ) and λ is a cardinal in N, then M is a λ-*extension* of

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Dedicated to all colleagues who boldly attempted reading early versions of this note. Each of these early versions contained a number of curiously looking sections starting with 'PROOF' and ending with '□' and otherwise not having much to do with what is normally called a proof. Needles to say, these sections were not present in Todorcevic's notes.

N ($N \prec_\lambda M$) if $M \succ N$ and $M \cap \lambda = N \cap \lambda$. It is a $<\lambda$-*extension* of N ($N \prec_{<\lambda} M$) if $M \succ_\kappa N$ for all $\kappa \in N \cap \lambda$, equivalently if $\min((M \setminus N) \cap \lambda) \geq \sup(N \cap \lambda)$. We say that M is a *strong* $<\delta$-*extension of* N, $N \prec^*_{<\delta} M$, if $N \cap V_\alpha = M \cap V_\alpha$ for all $\alpha \in N \cap \delta$. A *strong* λ-*extension*, $N \prec^*_\lambda M$, is defined analogously. With $\lambda^{<\lambda} = \lambda$ every λ-extension is a strong λ-extension.

An ideal $\mathcal{I}$ on ω_1 is *normal* if for every $A \in \mathcal{I}^+$, for every regressive $f\colon A \to \omega_1$ there exists $\xi \in \omega_1$ such that $f^{-1}(\xi) \in \mathcal{I}^+$. Let NS_{ω_1} denote the ideal of all nonstationary subsets of ω_1. If $\mathcal{X} \subseteq \mathcal{P}(\omega_1)$ includes NS_{ω_1} then the set of all diagonal unions $\bigcup_{\alpha<\omega_1}(A_\alpha \setminus (\alpha+1))$ of elements A_α ($\alpha < \omega_1$) of $\mathcal{X}$ is a normal ideal. We say that it is the *normal ideal generated by* $\mathcal{X}$.

Let $\mathcal{I}$ be a normal ideal on ω_1. Consider $\mathcal{I}^+ = \mathcal{P}(\omega_1) \setminus \mathcal{I}$ as a forcing notion ordered by the reverse inclusion. It is forcing equivalent to the quotient Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{I}$. All elementary submodels of some H_θ are assumed to be countable, unless otherwise specified.

Lemma 1.1. *Assume* $N \prec H_{(2^{\aleph_1})^+}$, $M \prec H_{(2^{2^{\aleph_1}})^+}$, *and* $M \cap H_{(2^{\aleph_1})^+} = N$. *If* $\alpha < 2^{\aleph_1}$ *is such that* $N \prec_{\omega_1} \mathrm{Hull}_{(2^{\aleph_1})^+}(N \cup \{\alpha\})$, *then* $M \prec_{\omega_1} \mathrm{Hull}_{(2^{2^{\aleph_1}})^+}(M \cup \{\alpha\})$.

Proof. It will suffice to prove that for every $\beta < \omega_1$ in $\mathrm{Hull}_{(2^{2^{\aleph_1}})^+}(M \cup \{\alpha\})$ we have $\beta \in \mathrm{Hull}_{(2^{\aleph_1})^+}(N \cup \{\alpha\})$. Pick such a β. Let $h\colon 2^{\aleph_1} \to \omega_1$ be a function in M such that $h(\alpha) = \beta$. Then $h \in H_{(2^{\aleph_1})^+}$, hence $h \in N$, and therefore $\beta = h(\alpha)$ is in $\mathrm{Hull}_{(2^{\aleph_1})^+}(N \cup \{\alpha\})$, as required. $\square$

Definition 1.2. Let $\mathcal{I}$ be a normal ideal on ω_1 and let $\mathfrak{A}$ be an antichain in $\mathcal{I}^+$. An $N \prec H_{(2^{\aleph_1})^+}$ *seals* $\mathfrak{A}$ if $\{\mathcal{I}, \mathfrak{A}\} \subseteq N$ and $N \cap \omega_1 \in \bigcup(N \cap \mathfrak{A})$.

If $\mathcal{I}$ is a normal ideal on ω_1 then in a forcing extension the ideal generated by all diagonal unions of elements of $\mathcal{I}$ is a normal ideal, and we still denote it by $\mathcal{I}$. If $\mathfrak{A}$ is an antichain in $\mathcal{I}^+$ then we say $\mathfrak{A}$ is *indestructible* if it is maximal in $\mathcal{I}^+$ in every $\aleph_1$-preserving forcing extension.

Lemma 1.3. *Let* $\mathcal{I}$ *be a normal ideal on* ω_1 *and let* $\mathfrak{A}$ *be an antichain in* $\mathcal{I}^+$. *Then conditions* (1) *and* (2) *below are equivalent and they imply* (3).

(1) *The set* $S_{\mathfrak{A}} = \{N \cap \omega_1 | N \prec H_{(2^{\aleph_1})^+}, \{\mathfrak{A}, \mathcal{I}\} \in N$ *and* N *does not seal* $\mathfrak{A}\}$ *belongs to* $\mathcal{I}$.
(2) *There is* $f\colon \omega_1 \to \mathfrak{A}$ *such that* $S = \omega_1 \setminus \bigcup_{\alpha<\omega_1}(f(\alpha) \setminus (\alpha+1))$ *belongs to* $\mathcal{I}$.
(3) $\mathfrak{A}$ *is maximal and moreover indestructible.*

Proof. Assume (1). Let N_α ($\alpha < \omega_1$) be a continuous chain of elementary submodels of $H_{(2^{\aleph_1})^+}$ such that $\{\mathcal{I}, \mathfrak{A}\} \in N_0$ and $N_\alpha \in N_{\alpha+1}$ for all α. For $\alpha \in \omega_1$ let $f(\alpha)$ be $a \in \mathfrak{A}$ such that $\alpha \in a$ if such a exists and $f(\alpha) = a_0$ for some fixed $a_0 \in \mathfrak{A}$ otherwise. Now let C be the club of all limit points in $C_0 = \{\alpha \mid N_\alpha \cap \omega_1 = \alpha\}$ and let $S = (\omega_1 \setminus C) \cup S_{\mathfrak{A}}$. For $\alpha \in C$ we have $N_\alpha = \bigcup_{\xi \in \alpha \cap C_0} N_\xi$; if moreover $\alpha \in C \setminus S$ then $\alpha \in \bigcup_{\xi \in \alpha \cap C_0} f(\xi)$, and (2) holds for f.

Assume (2). If $\mathfrak{A} \in N$ then $f \in N$ since we have definable Skolem functions. If moreover $N \cap \omega_1 \notin S$, then N seals $\mathfrak{A}$ so (1) follows.

Now assume (2) holds and write $C' = \omega_1 \setminus S$. We will first prove that the range of f is a maximal antichain in $\mathcal{I}^+$. For $B \in \mathcal{I}^+$, for each $\alpha \in B \cap C'$ let $g(\alpha) = \min\{\xi | \alpha \in f(\xi)\}$. This function is well-defined and regressive. Since $\mathcal{I}$ is normal, there is ξ such that the set of $\alpha \in B$ for which $g(\alpha) = \xi$ is not in $\mathcal{I}$. But this means that $f(\xi) \cap B \in \mathcal{I}^+$. Since B was arbitrary, this proves $\mathfrak{A}$ is maximal. To prove $\mathfrak{A}$ is indestructible it remains to show that $\mathfrak{A}$ remains maximal in every $\aleph_1$-preserving forcing extension. But the condition (2) is preserved by $\aleph_1$-preserving forcing, so (3) follows. □

If $X \subseteq Y$ and $S \subseteq [Y]^\omega$ let $S \restriction X = \{A \cap X | A \in S\}$.

Lemma 1.4. *If $\theta \geq \aleph_3$ and $N \prec H_\theta$ is such that $N \cap \omega_1 = \nu$ is countable and $N \cap \omega_2$ is uncountable then for every $\alpha < \omega_1$ there is $f \colon \omega_1 \to \omega_1$ in N such that $f(\nu) = \alpha$.*

Proof. For each $\zeta \in [\omega_1, \omega_2)$ let $h_\zeta \colon \omega_1 \to \zeta$ be a $<_\theta$-minimal bijection. We claim $\operatorname{otp}(N \cap \omega_2) = \omega_1$. Otherwise, for some $\bar\eta \in N \cap \omega_2$ the set $N \cap \bar\eta$ is uncountable. Then $h^{-1}_{\bar\eta}(N \cap \bar\eta)$ is an uncountable subset of ω_1 included in N, and therefore $\omega_1 \subseteq N$. For $\eta \in [\omega_1, \omega_2)$ define $g_\eta \colon \omega_1 \to \omega_2$ by $g_\eta(\xi) = \operatorname{otp}(h_\eta{}''\xi)$. By elementarity, the sequence $\langle g_\eta : \eta \in [\omega_1, \omega_2)\rangle$ belongs to N. For $\beta \in N$ we have that $g_\beta \restriction \nu$ is a non-decreasing surjection from $N \cap \omega_1$ onto $N \cap \beta$, in particular $g_\beta(\nu) = \operatorname{otp}(N \cap \beta)$. Since $\operatorname{otp}(N \cap \omega_2) = \omega_1$, for each η we can pick $\beta(\eta) \in N \cap \omega_2$ so that $\operatorname{otp}(N \cap \beta(\eta)) = \eta$, and therefore $f_\eta = g_{\beta(\eta)}$ is in N and satisfies $f_\eta(\nu) = \eta$. □

2. A saturated ideal

In this section we prove that if δ is a Woodin cardinal then after the Levy collapse of δ to ω_2 there is a saturated ideal $\mathcal{I}_\delta$ on ω_1. The existence of a saturated ideal in a Levy collapse of a large cardinal is a result originally due to Foreman, Magidor and Shelah [**3**]. Their proof is based on Shelah's semi-proper iteration lemma and their famous result that Martin's Maximum implies NS_{ω_1} is a saturated ideal. In [**10**, §XVI] Shelah uses an iteration of semi-proper forcings up to a Woodin cardinal which does not add reals and forces the existence of a saturated ideal on ω_1 (this proof is also sketched in [**17**, Theorem 2.65]). The purpose of this section is to produce a naturally definable and easily manageable saturated ideal in the Levy collapse which will eventually be used in the proof of the Σ^2_1 absoluteness theorem. The construction of the saturated ideal will therefore not depend on the theory of revised countable support iteration of semi-proper forcing notion though we shall borrow several ideas from [**3**] and [**10**]. For a cardinal $\lambda > \omega_1$ let

$$V^\lambda = V^{\mathrm{Coll}(\omega_1, <\lambda)},$$

where $\mathrm{Coll}(\omega_1, <\lambda)$ is the Levy-collapse of all ordinals less than λ to ω_1. So $V^\lambda \models |\lambda| = \aleph_2 = 2^{\aleph_1}$ if λ is inaccessible.

Theorem 2.1. *For each measurable cardinal λ there is C_λ in $V^{(2^\lambda)^+}$ such that*

1. *Each C_λ is a stationary subset of ω_1,*
2. *$C_\lambda \setminus C_\kappa$ is nonstationary for all $\lambda < \kappa$,*
3. *If δ is Woodin then the normal ideal $\mathcal{I}_\delta$ in V^δ generated by $\{\omega_1 \setminus C_\lambda : \lambda < \delta\}$ is saturated.*

The sequence C_λ is canonical and it does not depend on δ. The rest of this section is devoted to the proof of Theorem 2.1. Let X be a set of size $\aleph_1$ and let $S \subseteq [X]^\omega$. We define the *projection of S to ω_1* as follows. Fix a continuous cofinal sequence Z_α ($\alpha < \omega_1$) in $[X]^\omega$ and let

$$\pi_{\omega_1}(S) = \{\alpha < \omega_1 | Z_\alpha \in S\}.$$

Clearly, $\pi_{\omega_1}(S)$ depends on the choice of the sequence Z_α $(\alpha < \omega_1)$ but the equivalence class of $\pi_{\omega_1}(S)$ in $\mathcal{P}(\omega_1)/\operatorname{NS}_{\omega_1}$ is uniquely defined.

2.1. The definition of $\mathcal{I}_\lambda$. Let E the class of all measurable cardinals. We will recursively define the following objects:

(4) $\mathcal{I}_\lambda$, a normal ideal on ω_1 in V^λ for $\lambda < \delta$,
(5) $\Delta_\lambda \subseteq [\lambda]^\omega$ in V^λ for $\lambda < \delta$, and
(6) $C_\lambda \subseteq \omega_1$, in $V^{(2^{2^\lambda})^+}$ for $\lambda \in \delta \cap E$.

The definition of these objects will not involve δ, but its appropriate large cardinal property will insure that $\mathcal{I}_\delta$ is saturated.

If $\lambda \leq \min E$ let $\mathcal{I}_\lambda = (\operatorname{NS}_{\omega_1})^{V^\lambda}$ and $\Delta_\lambda = [\lambda]^\omega$. Whenever $\mathcal{I}_\lambda$ and Δ_λ are defined, $\kappa \geq \lambda$ and $E \cap [\lambda, \kappa) = \emptyset$ we let $\mathcal{I}_\kappa = \mathcal{I}_\lambda$ and $\delta_\kappa = \delta_\lambda$. Pick another $\lambda \in E$ and assume $\mathcal{I}_\lambda$ and Δ_κ, C_κ $(\kappa \in \lambda \cap E)$ have been defined.

Work in V^λ. Let (taking $\bigcap \emptyset = \omega_1$)

(7) $\Delta_\lambda = \{A \in [\lambda]^\omega | A \cap \omega_1 \in \bigcap_{\xi \in A \cap E} C_\xi\}$.

A word on the purpose of Δ_λ. If $M \prec H_\theta$ is countable then $M \cap \omega_1$ belongs to all clubs $C \subseteq \omega_1$ in M. Similarly, $\{M \prec H_\theta \mid M \cap \lambda \in \Delta_\lambda\}$ is the set of all models that are 'well-positioned' with respect to all C_ξ, $\xi \in E \cap \lambda$. See also (iii)$_\lambda$ of Lemma 2.5.

In order to define C_λ, let

$$\begin{aligned}\mathfrak{M}_\lambda = \{\mathfrak{A} : \mathfrak{A} \text{ is a maximal antichain in } \mathcal{I}_\lambda^+ \text{ and}\\ (\forall M \prec H_{(2^{2^{\aleph_1}})^+})(\{\mathcal{I}_\lambda, \mathfrak{A}\} \subseteq M \wedge M \cap \lambda \in \Delta_\lambda\\ \Rightarrow (\exists A \in \mathfrak{A}) M \cap \omega_1 \in A\\ \wedge M \prec_{\omega_1} N = \operatorname{Hull}_{(2^{2^{\aleph_1}})^+}(M \cup \{A\}) \wedge N \cap \lambda \in \Delta_\lambda)\}.\end{aligned}$$

This is the collection of all antichains that are ready to be frozen at the stage λ. Recall that $M \prec H_\theta$ seals an antichain $\mathfrak{A}$ if $M \cap \omega_1 \in \bigcup(M \cap \mathfrak{A})$. Still in V^λ, consider the following set:

(8) $\tilde{C}_\lambda = \{M \prec H_{(2^{\aleph_1})^+} | M \cap \lambda \in \Delta_\lambda \text{ and } M \text{ seals every } \mathfrak{A} \in \mathfrak{M}_\lambda \cap M\}$.

Move to $V^{(2^\lambda)^+}$. Since $V^\lambda \models (2^{\aleph_1})^+ = \lambda^+$, the set $\bigcup(\tilde{C}_\lambda)^{V^\lambda}$ has size $\aleph_1$. Let

(9) $C_\lambda = \pi_{\omega_1}(\tilde{C}_\lambda)$,

where the name $\langle \dot{Z}_\alpha : \alpha < \omega_1\rangle$ for the sequence $\langle Z_\alpha : \alpha < \omega_1\rangle$ used in the definition of π_{ω_1} is the $<_\theta$-minimal name for such a club for θ large enough (the choice of this sequence is actually irrelevant; see the remark after the definition of π_{ω_1}).

Finally, if $\mathcal{I}_\kappa, \Delta_\kappa$, and C_κ have been defined for all relevant κ, let $\mathcal{I}_\lambda$ be the normal ideal generated by $\{\omega_1 \setminus C_\kappa | \kappa \in \lambda \cap E\} \cup \operatorname{NS}_{\omega_1}$. In the following lemma and elsewhere, H_θ is as computed in the model we are working in; we shall write e.g., $(H_\theta)^V$ if this is the intended meaning.

Lemma 2.2. *In V^λ for $\lambda \in E$ and every large enough θ: Every $M \prec H_\theta$ such that $M \cap \lambda \in \Delta_\lambda$ has an ω_1-extension $N \prec H_\theta$ such that $N \cap H_{(2^{\aleph_1})^+} \in \tilde{C}_\lambda$.*

Proof. By Lemma 1.1 it suffices to consider the case when $\theta = (2^{\aleph_1})^+$. Build $M_i \prec H_{(2^{\aleph_1})^+}$ $(i \in \omega)$ such that $M_0 = M$ and for all i

(a) $M_i \prec_{\omega_1} M_{i+1}$,
(b) for every $\mathfrak{A} \in M_i \cap \mathfrak{M}_\lambda$ there is $j > i$ such that M_j seals $\mathfrak{A}$, and
(c) $M_i \cap \lambda \in \Delta_\lambda$.

This sequence is constructed by straightforward bookkeeping. Now let $N = \bigcup_i M_i$. Then N seals every $\mathfrak{A} \in N \cap \mathfrak{M}_\lambda$. Also, every $\xi \in N \cap \lambda \cap E$ belongs to some M_i, implying $N \cap \omega_1 = M_i \cap \omega_1 \in C_\xi$ and $N \cap \lambda = \Delta_\lambda$. Hence $N \in \tilde{C}_\lambda$. □

Consider $\mathrm{Coll}(\omega_1, <\kappa)$ as a subordering of $\mathrm{Coll}(\omega_1, <\lambda)$ for $\kappa < \lambda$ so that the phrase 'p and q are compatible' is meaningful for p and q coming from any of these posets. The following is one of the key technical devices in the proof of Theorem 2.1.

Lemma 2.3. *In V: Assume $\kappa < \lambda \le \delta$, $\kappa \notin E$, $\lambda \in E \cup \lim(E)$, $q \in \mathrm{Coll}(\omega_1, < \lambda)$ $\{\kappa, \lambda, q\} \subseteq M \prec H_{(2^\delta)^+}$, $p \in \mathrm{Coll}(\omega_1, <\kappa)$ is M-generic, $p \Vdash_\kappa M \cap \kappa \in \dot{\Delta}_\kappa$, and p is compatible with q.*

Then there is a strong κ-extension N of M and N-generic $r \in \mathrm{Coll}(\omega_1, <\lambda)$ extending p and q such that $r \Vdash_\lambda N \cap \lambda \in \dot{\Delta}_\lambda$.

Proof. By induction. Assume λ, κ are as above and that the statement is proved for all appropriate pairs κ', λ' such that $\kappa' < \lambda' < \lambda$ or $\kappa' < \kappa < \lambda' = \lambda$, and fix M, p, q as in the statement.

In the case when $\lambda = \min(E \setminus \kappa)$ we have $\Delta_\lambda = \{A \in [\lambda]^\omega \mid A \cap \kappa \in \Delta_\kappa\}$ and therefore we can let $N = M$ and freely pick an N-generic r extending p and q in $\mathrm{Coll}(\omega_1, <\lambda)$.

Now assume E is cofinal in λ. Recursively find the following objects.

(10) $\kappa = \kappa_0 < \kappa_1 < \kappa_2 < \dots$, inaccessibles in $\lambda \setminus E$,

(11) $M = N_0 \prec^*_{\kappa_0} N_1 \prec^*_{\kappa_1} N_2 \prec^*_{\kappa_2} \dots$, countable elementary submodels of $H_{(2^\delta)^+}$,

(12) $p = p_0 > p_1 > p_2 > \dots$, so that $p_i \in \mathrm{Coll}(\omega_1, <\kappa_i)$ is N_i-generic and $p_i \Vdash_{\kappa_i} N_i \cap \kappa_i \in \dot{\Delta}_{\kappa_i}$, and

(13) $q = q_0 > q_1 > q_2 > \dots$, so that $q_i \in N_i \cap \mathrm{Coll}(\omega_1, <\lambda)$.

Furthermore assure p_i and q_i are compatible for all i, and that with $N = \bigcup_i N_i$ the condition $\bigcup_i q_i$ is N-generic; the latter only requires a simple bookkeeping device.

The above objects are constructed by simultaneous recursion. Assume appropriate p_i, q_i, N_i, κ_i have been chosen for $i \le n$. If $\mathrm{cf}(\lambda) > \omega$, pick $\kappa_{i+1} \in N_i$ such that $q_i \in \mathrm{Coll}(\omega_1, <\kappa_{i+1})$. If $\mathrm{cf}(\lambda) = \omega$ then $\kappa_{i+1} \in N_i$ are chosen to assure $\sup_i \kappa_i = \lambda$. Either way, choose $\kappa_{i+1} \in \lambda \setminus E$ inaccessible. Using the inductive hypothesis, find a strong κ_i-extension N_{i+1} of N_i and p_{i+1} extending both $q_i \restriction \kappa_i$ and p_i that is N_{i+1}-generic and satisfies $p_{i+1} \Vdash_{\kappa_{i+1}} N_{i+1} \cap \kappa_{i+1} \in \dot{\Delta}_{\kappa_{i+1}}$. It only remains to find q_{i+1}. Find an N_{i+1}-generic filter $\mathrm{Coll}(\omega_1, <\lambda)$ containing p_{i+1} and q_i, and pick $q_{i+1} \in N_{i+1} \cap G$ extending q_i and meeting the $i+1$-st dense open set. This assures q_{i+1} and p_{i+1} are compatible.

After the construction is completed, let $N = \bigcup_i N_i$ and $r = \bigcup_i p_i$. Then r extends all p_i and all q_i, hence it is N-generic for $\mathrm{Coll}(\omega_1, <\lambda)$. Also, (11) implies $M \prec^*_\kappa N$. Since $p_i \Vdash N_i \cap \omega_1 \in \bigcap_{\xi \in E \cap N_i \cap \kappa_i} C_\xi$ for each i, we have $r \Vdash N \cap \omega_1 \in \bigcap_{\xi \in E \cap N \cap \lambda} C_\xi$, and therefore $r \Vdash N \cap \lambda \in \dot{\Delta}_\lambda$.

Easy implications aside, it remains to handle the case when $\lambda' = \min(E \setminus \kappa) < \lambda$ and E has no elements between λ' and λ. The rest of the proof deals with this case. By the induction hypothesis there is a strong κ-extension N' of M and N'-generic p_1 extending p and $q \restriction \lambda'$ in $\mathrm{Coll}(\omega_1, <\lambda')$ such that $p_1 \Vdash_{\lambda'} N' \cap \lambda' \in \dot{\Delta}_{\lambda'}$.

Go to $V^{\lambda'}$ below p_1. We have $N'[G_{\lambda'}] \prec (H_{(2^\delta)^+})^V[G_{\lambda'}]$ — which is $H_{(2^\delta)^+}$ as computed in $V^{\lambda'}$. By Lemma 2.2 there is an ω_1-extension $N'' \prec H_{(2^\delta)^+}$ of $N'[G_{\lambda'}]$

such that $N''\cap\lambda' \in \Delta_{\lambda'}$ and moreover $N''\cap H_{(2^{\aleph_1})^+} \in \tilde{C}_{\lambda'}$. In $V^{\lambda'}$ we have $|\kappa| = \aleph_1$ and CH hence $N'[G_{\lambda'}] \prec^*_\kappa N''$. Find an N''-generic $p_2/G_{\lambda'}$ in $\mathrm{Coll}(\omega_1, <\lambda)/G_{\lambda'}$ extending q. Then p_2 forces $N''\cap\omega_1 \in C_{\lambda'}$ and therefore $N''\cap\lambda \in \dot{\Delta}_\lambda$. We may assure p_2 is strong enough to decide N such that $N'' = N[G_{\lambda'}]$. Since $N''\cap\lambda = N\cap\lambda$, an $r \in \mathrm{Coll}(\omega_1, <\lambda)$ extending $p_1 * p_2$ and q forces $N\cap\lambda \in \dot{\Delta}_\lambda$. Now, $N'[G_{\lambda'}] \cap H_{(2^\delta)^+} = N'$ and $N''\cap H_{(2^\delta)^+} = N$, hence $N' \prec^*_\kappa N$ and $M \prec^*_\kappa N$. □

Lemma 2.4. *For every $\lambda \in E$ we have the following:*

(14) *Δ_λ and C_λ are stationary.*
(15) *$C_\xi \setminus C_\eta \in \mathrm{NS}_{\omega_1}$, for all $\xi > \eta$.*
(16) *The ideal $\mathcal{I}_\lambda$ is normal and proper.*

Proof. (14) We prove Δ_κ is stationary by induction. If $\kappa = \min(E)$ then Δ_κ is a club. Now assume Δ_λ is stationary for all $\lambda < \kappa$ in E. Then by Lemma 2.3 and using Skolem functions every club in $[\kappa]^\omega$ intersects Δ_κ. The stationarity of Δ_κ and Lemma 2.2 imply $\tilde{C}_\kappa$ and C_κ are stationary. To prove (15), in V^ξ let $M^\eta = (H_{(2^{2^\eta})^+})^{V^\eta}$ and $M^\xi = (H_{(2^{2^\xi})^+})^{V^\xi}$. The sequences used to define $C_\eta = \pi_{\omega_1}(\tilde{C}_\eta)$ and $C_\xi = \pi_{\omega_1}(\tilde{C}_\xi)$ satisfy $Z^\eta_\alpha = Z^\xi_\alpha \cap M^\eta$ for club many $\alpha < \omega_1$. So it will suffice to prove that for every $N \in \tilde{C}_\xi$ we have $N \cap M^\eta \in \tilde{C}_\eta$, and this is immediate from the definitions. (16) Normality follows immediately from the definition and the properness follows from stationarity of all C_λ. □

Lemma 2.5. *In V^λ, for $A \subseteq \omega_1$ the following are equivalent.*

(i)$_\lambda$ *$A \in \mathcal{I}_\lambda$.*
(ii)$_\lambda$ *$A \cap C_\eta \in \mathrm{NS}_{\omega_1}$ for some $\eta \in \lambda\cap E$.*
(iii)$_\lambda$ *For all $M \prec H_{(2^{\aleph_1})^+}$ such that $\{\mathcal{I}_\lambda, A\} \subseteq M$ and $M\cap\lambda \in \Delta_\lambda$ we have $M\cap\omega_1 \notin A$.*
(iv)$_\lambda$ *$\Vdash_{(2^\lambda)^+} \pi_{\omega_1}(\Delta_\lambda) \cap A \in \mathrm{NS}_{\omega_1}$.*

Also, for $A \in V^\lambda$ we have $A \in \mathcal{I}^+_\lambda$ if and only if $A \in \mathcal{I}^+_\kappa$ for all $\kappa \geq \lambda$.

Proof. It is obvious that (i)$_\lambda$ is equivalent to (ii)$_\lambda$.

We prove that (ii)$_\lambda$ implies (iii)$_\lambda$. Assume (ii)$_\lambda$ and M, A, λ are as in (iii)$_\lambda$. By elementarity there is $\eta < \lambda$ in M such that (ii)$_\lambda$ holds. Then $M\cap\omega_1 \in C_\eta$, and since $A\cap C_\eta \in M$ is nonstationary, $M\cap\omega_1 \notin A$.

Now assume (ii)$_\lambda$ fails; then $A\cap C_\eta$ is stationary for each $\eta \in \lambda\cap E$. Hence there is an $M \prec H_{(2^{2^\lambda})^+}$ containing A and η such that $M\cap\omega_1 \in A\cap C_\eta$, hence (iii)$_\lambda$ fails.

Condition (iv)$_\lambda$ implies (iii)$_\lambda$ since we have definable Skolem functions.

By Lemma 2.3, if (iii)$_\lambda$ fails, then (iii)$_\kappa$ fails for all $\kappa \geq \lambda$ in E, hence by the equivalence of (i)$_\lambda$ and (iii)$_\lambda$ we have $A \in \mathcal{I}^+_\lambda$ in V^λ implies $A \in \mathcal{I}^+_\kappa$ in V^κ for all $\kappa > \lambda$. □

It remains to prove that every maximal antichain in $\mathcal{I}^+_\delta$ reflects to some $\mathfrak{M}_\lambda$.

Definition 2.6. For a maximal antichain $\mathfrak{A}$ in $\mathcal{I}^+_\delta$ define $f_{\mathfrak{A}}, g_{\mathfrak{A}} \colon \delta \to \delta$ as follows.

$g_{\mathfrak{A}}(\xi) = \min\{\eta |$for every $\mathrm{Coll}(\omega_1, <\xi)$-name B for an $\mathcal{I}_\xi$-positive subset of ω_1 there is a $\mathrm{Coll}(\omega_1, <\eta)$-name A for a member of $\mathfrak{A}$ such that $\Vdash_\eta A\cap B \notin \mathcal{I}_\eta\}$.

$f_{\mathfrak{A}}(\xi) = \beth_\omega(\min(E \setminus (g_{\mathfrak{A}}(\beth_\omega(\xi))))$.

Note that $g_{\mathfrak{A}}$—and therefore $f_{\mathfrak{A}}$—is well-defined by Lemma (2.5). Also, if $\kappa < \delta$ is closed under either of these functions, then $\mathrm{Coll}(\omega_1, < \kappa)$ forces $\mathfrak{A} \restriction \kappa$ is a maximal antichain in $\mathcal{I}_\kappa$.

DEFINITION 2.7. Assume $\mathfrak{A}$ is a maximal antichain in $\mathcal{I}_\delta$ and $\kappa \in \delta \cap E$. We say that $\mathfrak{A}$ is *sealed at* κ if in V^κ every $M \prec H_{(2^{2^{\aleph_1}})^+}$ such that $\mathfrak{A} \in M$ and $M \cap \kappa \in \Delta_\kappa$ seals $\mathfrak{A}$ (i.e., $M \cap \omega_1 \in \bigcup(\mathfrak{A} \cap M)$).

LEMMA 2.8. *If δ is such that every antichain $\mathfrak{A}$ in $\mathcal{I}^+_\delta$ is sealed at some $\kappa < \delta$, then $\mathcal{I}_\delta$ is saturated.*

PROOF. Assume $\mathfrak{A}$ is sealed at κ. Then $\mathfrak{A} \in \mathfrak{M}_\kappa$ and every $N \in \tilde{C}_\kappa$ such that $\mathfrak{A} \in N$ seals $\mathfrak{A}$.

In $V^{(2^{2^\kappa})^+}$: Assume $N \prec H_{(2^{2^{\aleph_1}})^+}$ is large enough so that both $\tilde{C}_\kappa$ and the sequence $\{Z_\alpha | \alpha < \omega_1\}$ used to define $C_\kappa = \pi_{\omega_1}(\tilde{C}_\kappa)$ belong to N. Then $N \cap (H_{(2^{\aleph_1})^+})^{V^\kappa} = Z_\nu$ with $\nu = N \cap \omega_1$, therefore $N \cap \omega_1 \in C_\kappa$ implies $N \cap (H_{(2^{\aleph_1})^+})^{V^\kappa} \in \tilde{C}_\kappa$. By the assumption, this implies that N seals $\mathfrak{A}$. Since for club many models N we have $N \cap \omega_1 \in C_\kappa$ implies N seals $\mathfrak{A}$, Lemma 1.3 implies $\mathfrak{A}$ is maximal and indestructible. Since $|\mathfrak{A}|$ is collapsed to $\aleph_1$ and $\mathfrak{A}$ was arbitrary, $\mathcal{I}_\delta$ is saturated. □

The following lemma is the only use of the Woodin cardinal strength (Definition 2.10) in the proof of Theorem 2.1 (cf. [**7**, Theorem 2.5.9]).

LEMMA 2.9. *Assume $\mathfrak{A}$ is a maximal antichain in $\mathcal{I}^+_\delta$ such that for some $\kappa < \delta$ closed under $f_{\mathfrak{A}}$ there is an elementary embedding $j \colon V \to M$ such that* $\mathrm{cr}(j) = \kappa$ *and $V_{(jf_{\mathfrak{A}})(\kappa)} \subseteq M$. Then $\mathfrak{A}$ is sealed at* $\min(E \setminus (\kappa + 1))$.

PROOF. Fix a $j \colon V \to M$ such that $\mathrm{cr}(j) = \kappa$ and $V_{(jf_{\mathfrak{A}})(\kappa)} \subseteq M$. Since $\mathrm{Coll}(\omega_1, < \kappa)$ is included in V_κ and a regular subordering of $\mathrm{Coll}(\omega_1, < j(\kappa))$, by the Kunen–Paris method (see [**5**, p. 226]) j extends to an elementary embedding (also denoted by j) $j \colon V^\kappa \to M^{j(\kappa)}$, where $M^{j(\kappa)}$ is $M^{\mathrm{Coll}(\omega_1, < j(\kappa))}$. Let $\mathfrak{A}' = \mathfrak{A} \restriction \kappa$.

In V^κ. Assume for a moment that for some club C in $[H_{(2^{\aleph_1})^+}]^\omega$ the family $\nabla(\mathfrak{A}') = \{N \prec H_{(2^{\aleph_1})^+} | \{\mathfrak{A}', \mathcal{I}_\kappa\} \subseteq N$ and some $M \succ_{\omega_1} N$ seals $\mathfrak{A}'\}$ includes

$$C \cap \{M \prec H_{(2^{\aleph_1})^+} | \mathfrak{A}' \in M, M \cap \kappa \in \Delta_\kappa\}.$$

Then by using Skolem functions we get $\mathfrak{A}' \in \mathfrak{M}_\kappa$, hence every $M \in \tilde{C}_\kappa$ seals $\mathfrak{A}'$ and therefore it seals $\mathfrak{A}$ as well. Hence if $\lambda = \min(E \setminus (\kappa + 1))$ then in V^λ only nonstationary many $M \prec H_{(2^{2^\lambda})^+}$ such that $\{\mathfrak{A}, \kappa\} \in M$ and $M \cap \lambda \in \Delta_\lambda$ do not seal $\mathfrak{A}$. Hence the conclusion follows.

Now assume there is no such club C and therefore the set

$$S = \{N \prec H_{(2^{\aleph_1})^+} | \mathfrak{A}' \in N, N \cap \kappa \in \Delta_\kappa \text{ but there does not exist } N' \succ_{\omega_1} N \text{ such that } N' \cap \kappa \in \Delta_\kappa \text{ and } N' \text{ seals } \mathfrak{A}'\}$$

is stationary in $\{N : N \cap \kappa \in \Delta_\kappa\}$.

In $V^{(2^\kappa)^+}$. The set $X = \bigcup S$ is of size $\aleph_1$ and S is a stationary subset of X. Hence $B = \pi_{\omega_1}(S)$ is a stationary subset of C_κ. Since $\mathcal{I}_\kappa$ is generated by C_κ over NS_{ω_1}, B is $\mathcal{I}_\kappa$-positive. Since $V_{\beth_\omega(\kappa)} \subseteq M$, these sets belong to $M^{j(\kappa)}$.

In $M^{j(\kappa)}$. Moreover, since $\mathfrak{A}'$ is a maximal antichain in $\mathcal{I}^+_\kappa$, we can find $A \in j(\mathfrak{A}')$ such that $A \cap B$ is positive. Also, by the definition of $g_{\mathfrak{A}}$ and $f_{\mathfrak{A}}$ a name for A can be found in $V_{(jg_{\mathfrak{A}})(\beth_\omega(\kappa))}$. Recall that $V_{(jg_{\mathfrak{A}})(\beth_\omega(\kappa))} \subseteq M$.

In V. Let $\lambda = \min(E \setminus (g_{\mathfrak{A}}(\beth_\omega(\kappa))))$ and work below a condition q_0 deciding all of the above. Let $\langle \dot{Z}_\alpha | \alpha < \omega_1 \rangle$ be a name of the sequence used to define $B = \pi_{\omega_1}(S)$. Pick $N_0 \prec H_{(2^\delta)^+}$ such that $\{\dot{\mathfrak{A}}, A, j \restriction V_\lambda, \langle \dot{Z}_\alpha | \alpha < \omega_1 \rangle\} \subseteq N_0$ and a $\mathrm{Coll}(\omega_1, <\lambda), N_0$-generic condition $q \leq q_0$ that forces $N_0 \cap \kappa \in \Delta_\kappa$ and $N_0 \cap \omega_1 \in A \cap B$. This is possible because $B = \pi_{\omega_1}(S)$ for $S \subseteq \Delta_\kappa$. Since $\langle \dot{Z}_\alpha | \alpha < \omega_1 \rangle \in N_0$ and q forces $N_0 \cap \omega_1 \in B$, $q \restriction \kappa$ forces that in V^κ we have $N_0 \cap H_{(2^{\aleph_1})^+} \in S$. By Lemma 2.3 find a κ-extension $N \prec H_{(2^\delta)^+}$ of N_0 and a $\mathrm{Coll}(\omega_1, <\lambda)$-generic condition $p \leq q$ such that p forces $N \cap \kappa \in \Delta_\kappa$ and $N \cap \omega_1 \in C_\kappa$, hence $N \in \tilde{C}_\kappa$. Since $\beth_\omega(\lambda) \leq (jf_{\mathfrak{A}})(\kappa)$, p and N belong to M.

In $M^{j(\kappa)}$. By Lemma 2.3, find $N' \prec H_{(2^\delta)^+}$ such that N' is a $<\kappa$-extension of N and an $N', \mathrm{Coll}(\omega_1, < j(\kappa))$-generic $p' \leq p$ such that $p' \Vdash N' \cap j(\kappa) \in \Delta_{j(\kappa)}$. Note that N' seals $j(\mathfrak{A}')$ (via A). Let $N_0' = j''(N_0 \cap H_{(2^{2^\kappa})^+})$ (recall $j \restriction V_\lambda \in N \subseteq N'$). Then $N_0' = j(N_0 \cap H_{(2^{2^\kappa})^+}) \prec H_{(2^{2^{j(\kappa)}})^+}$, also $N_0' \in j(S)$, and N' is an ω_1-extension of N_0' that seals $j(\mathfrak{A}')$.

In V^κ. By elementarity, there is a member of S whose ω_1-extension seals $\mathfrak{A}'$; a contradiction. This concludes the proof. □

Definition 2.10. (See [**8, 7**]) A cardinal δ is a *Woodin cardinal* if

(W) for every $f\colon \delta \to \delta$ there is a $\gamma < \delta$ such that $f''\gamma \subseteq \gamma$ and an elementary embedding $j\colon V \to M$ for some transitive inner model M such that $\mathrm{cr}(j) = \gamma$ and $V_{(j(f))(\gamma)} \subseteq M$.

Proof of Theorem 2.1. It remains to prove that if δ is a Woodin cardinal then $\mathcal{I}_\delta$ is a normal saturated ideal on ω_1 in V^δ. By Lemma 2.4, $\mathcal{I}_\delta$ is proper and normal. Fix a name $\mathfrak{A}$ for a maximal antichain in $\mathcal{I}_\delta$, and let $f_{\mathfrak{A}}$ be as in Definition 2.6. Since δ is Woodin there is $\gamma < \delta$ such that the assumptions of Lemma 2.8 are satisfied. By Lemma 2.8 and Lemma 2.9, the size of $\mathfrak{A}$ is at most $\aleph_1$. Since $\mathfrak{A}$ was arbitrary, the ideal $\mathcal{I}_\delta$ is saturated. □

The ideas presented above essentially give the proof of the following result due to Woodin (see also [**4**]). An ideal $\mathcal{I}$ on ω_1 is *presaturated* if for every sequence $\mathfrak{A}_i$ ($i \in \omega$) of maximal antichains in $\mathcal{I}^+$ there is $D \in \mathcal{I}^+$ such that $\{A \in \mathfrak{A}_i \mid A \cap D \in \mathcal{I}^+\}$ has size at most $\aleph_1$ for each $i \in \omega$. A normal presaturated ideal gives rise to a generic ultrapower containing all the reals, via a minor modification of the proof of Lemma 3.1 below.

Theorem 2.11. *Assume* δ *is a Woodin cardinal. Then* $\mathrm{Coll}(\omega_1, <\delta)$ *forces that* NS_{ω_1} *is presaturated.*

Proof. Fix names $\mathfrak{A}_i$ ($i \in \omega$) for maximal anticahins in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. Let $g, f\colon \delta \to \delta$ be defined by

$g(\xi)$ $= \min\{\eta |$for every $\mathrm{Coll}(\omega_1, <\xi)$-name B for a stationary subset of ω_1 and every $i < \omega$ there is a $\mathrm{Coll}(\omega_1, <\eta)$-name A for a member of $\mathfrak{A}_i$ such that $\Vdash_\eta A \cap B \notin \mathrm{NS}_{\omega_1}\}$.

$f_{\mathfrak{A}}(\xi)$ $= \beth_\omega(\min(E \setminus (g_{\mathfrak{A}}(\beth_\omega(\xi)))))$.

Find $\kappa < \delta$ and an elementary embedding $j\colon V \to M$ with κ as the critical point and such that $V_{(jf)(\kappa)} \subseteq M$. Lift j to $j\colon V^\kappa \to M^{j(\kappa)}$.

Assume for a moment that in V^κ relative to $\{M \prec H_{(2^{\aleph_1})^+} \mid M \cap \kappa \in \Delta_\kappa\}$ club many models seal each $\mathfrak{A}_i$. By the argument of Lemma 2.2, the set

$$\tilde{D} = \{M \prec H_{(2^{\aleph_1})^+} \mid (\mathfrak{A}_i)_i \in M, M \cap \kappa \in \Delta_\kappa \text{ and } M \text{ seals each } \mathfrak{A}_i\}$$

is stationary, and $D = \pi_{\omega_1}(\tilde{D})$ is a stationary subset of ω_1 as required.

Otherwise, for some $i \in \omega$ the set $\{M \prec H_{(2^{\aleph_1})^+} \mid \mathfrak{A}_i \in M, M \cap \kappa \in \Delta_\kappa$, and no $N \succ_{\omega_1} M$ such that $N \cap \kappa \in \Delta_\kappa$ seals $\mathfrak{A}_i\}$ is stationary. An argument identical to that in the proof of Lemma 2.9 leads to a contradiction. □

3. The generic ultrapower

If $\mathcal{I}$ is an ideal on ω_1 and $G \subseteq \mathcal{I}^+$ is a generic ultrafilter, then in $V[G]$ we can define the *generic ultrapower* V^{ω_1}/G. Let us include a proof of the following well-known lemma.

LEMMA 3.1. *If $\mathcal{I}$ is a normal saturated ideal on ω_1 and $\mathcal{U} \subseteq \mathcal{I}$ is a generic ultrafilter, then the generic ultrapower is closed under ω-sequences. In particular it is well-founded and it contains all reals from $V[\mathcal{U}]$.*

PROOF. Write M for V^{ω_1}/G. We first prove a useful fact about the representation of the names in M. For every ordinal $\alpha \in M$ there is a name $\dot{f}$ for a function in V^{ω_1} such that $[\dot{f}]_G = \alpha$. Find a maximal antichain A^ξ $(\xi < \omega_1)$ and $f^\xi \in V^{\omega_1}$ such that $A^\xi \Vdash \dot{f} = \check{f}^\xi$. The size of this antichain is $\aleph_1$ by the saturatedness of $\mathcal{I}$. By replacing A^ξ with $A^\xi \setminus \bigcup_{\eta<\xi} A^\eta$ we can assume the sets in the antichain are pairwise disjoint, and then we can define f^* with domain $\bigcup_{\xi<\omega_1} A^\xi$ such that $f^* \restriction A^\xi = f^\xi$ for all ξ. Therefore $\Vdash [\dot{f}] = [\check{f}^*]$. In order to prove M is closed under ω-sequences fix a sequence of ordinals α_n $(n \in \omega)$ in M. By the above, for each α_n there is a ground-model $f_n \in V^{\omega_1}$ such that $[\check{f}_n]_G = \alpha_n$. Then $g\colon \omega_1 \to V^\omega$ defined by $g(\alpha) = \langle f_n(\alpha) : n \in \omega\rangle$ is a name for the sequence $\langle \alpha_n : n \in \omega\rangle$. □

LEMMA 3.2. *If δ is a Woodin cardinal and ϕ is a Σ^2_1-statement that holds in V, then ϕ holds in $V^{\mathrm{Coll}(\omega_1,<\delta)*\dot{\mathcal{I}}^+_\delta}$.*

PROOF. Let $G \subseteq \mathrm{Coll}(\omega_1, <\delta)$ be a generic filter. Since $\mathrm{Coll}(\omega_1, <\delta)$ does not add new reals $V[G] \models \phi$. In $V[G]$ let $\dot{\mathcal{U}} \subseteq \dot{\mathcal{I}}^+_\delta$ be the generic filter and $j\colon V[G] \to M$ be the generic embedding. By elementarity, $M \models \phi$. By Lemma 3.1, $L(\mathbb{R})^{V[G][\dot{\mathcal{U}}]} = L(\mathbb{R})^M$, thus by the upwards absoluteness of existential formulas $V[G][\mathcal{U}] \models \phi$. □

The following lemma is true for an arbitrary Σ^m_n formula ϕ (see [**7**, Theorem 2.5.10]), but we shall need only the special case stated below.

LEMMA 3.3. *Assume δ is a Woodin cardinal, ϕ is a Σ^2_1-formula and $\mathcal{P}, \mathcal{Q}$ are forcing notions of size $< \delta$. If $\mathcal{P}$ forces ϕ, then $\mathcal{Q}$ forces that there is a forcing notion $\mathcal{P}'$ that forces ϕ.*

PROOF. By Lemma 3.2, $\mathcal{R} = \mathcal{P} * \mathrm{Coll}(\dot{\omega_1}, <\delta) * \dot{\mathcal{I}}^+_\delta$ forces ϕ. Since $\mathcal{R}$ collapses $2^{|\mathcal{Q}|}$ to ω, $\mathcal{Q}$ is a regular subordering of the regular open algebra of $\mathcal{R}$, and therefore $\mathcal{Q}$ forces that $\dot{\mathcal{P}}' = \mathcal{R}/\mathcal{Q}$ forces ϕ. □

It would be a shame not to mention the following result of Shelah–Woodin, [**11**].

THEOREM 3.4. *If there is a weakly compact Woodin cardinal κ, then $L(\mathbb{R})$ is elementarily equivalent to $L(\mathbb{R})$ of some Solovay's model, $V^{\mathrm{Coll}(\omega,<\kappa)}$. In particular, all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable, have Property of Baire, perfect-set property and Ramsey property.*

Proof. Let $G \subseteq \mathrm{Coll}(\omega_1, < \kappa)$ be a generic filter. Note that $L(\mathbb{R}) = L(\mathbb{R})^{V[G]}$. By Woodinness of κ and Theorem 2.1, in $V[G]$ there is a normal saturated ideal $\mathcal{I}_\kappa$ on ω_1. Let $H \subseteq \mathcal{I}_\kappa^+$ be a $V[G]$-generic ultrafilter. By Lemma 3.1, the generic ultrapower $(V[G])^{\omega_1}/H$ contains all the reals of $V[G][H]$. Since $L(\mathbb{R})^V = L(\mathbb{R})^{V[G]}$, the restriction of the generic elementary embedding $j\colon V[G] \to (V[G])^{\omega_1}/H$ to $L(\mathbb{R})^V$ is an elementary embedding into $L(\mathbb{R})^{V[G][H]}$. It is well-known (see [**11**]) that the weak compactness of κ in $V[G][H]$ there is a forcing notion $\mathcal{Q}$ that adds a $\mathrm{Coll}(\omega, < \kappa)$-generic over V filter, call it K, such that $\mathbb{R}^{V[G][H]} = \mathbb{R}^{V[K]}$.

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Coll}(\omega_1,<\kappa)} & V[G] \\ {\scriptstyle \mathrm{Coll}(\omega,<\kappa)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(\omega_1)/\mathcal{I}_\kappa} \\ V[K] & \xrightarrow{\text{same } L(\mathbb{R})} & V[G][H] \end{array}$$

Hence $L(\mathbb{R})^{V[G][H]} = L(\mathbb{R})^{V[K]} = L(\mathbb{R})^{\mathrm{Coll}(\omega,<\kappa)}$, and $L(\mathbb{R})^V$ is elementarily embeddable into this model.

$$\begin{array}{ccc} L(\mathbb{R}) & \overset{j}{\searrow} & \\ \downarrow & & \\ L(\mathbb{R})^{V[K]} & = & L(\mathbb{R})^{V[G][H]} \end{array}$$

The proof is concluded by applying Solovay's and Mathias's results on regularity properties of sets of reals in $L(\mathbb{R})^{\mathrm{Coll}(\omega,<\kappa)}$ ([**12**], [**9**]). □

The large cardinal assumption in Theorem 3.4 is not optimal. As shown in [**11**], by a 'switching quantifiers' argument the conclusion of Theorem 3.4 follows from the assumption that there are infinitely many Woodin cardinals.

4. Σ^2_1-absoluteness

This section contains the proof of Theorem 1 (Woodin, [**16**]).

Theorem 4.1. *Assume there are class many measurable Woodin cardinals, and let $\phi = (\exists x)\psi(x)$ be a Σ^2_1-formula. If ϕ is true in some forcing extension, then it is true in every forcing extension that satisfies CH.*

By using a result of Steel (see [**13**, Corollary 4.6(1)] or [**7**, Theorem 1.7.2]) this result can be extended to formulas ϕ that involve arbitrary universally Baire sets of reals as predicates. A variation not depending on CH is given in [**2**, Theorem 4.1]. We use the terminology and notation of §2. Let δ be a Woodin cardinal and let

$$W = \{\kappa < \delta : \ \kappa \text{ is Woodin}\}$$
$$E = \{\kappa < \delta : \ \kappa \text{ is measurable and not Woodin}\}.$$

Assume δ satisfies the following condition

(WW) For every $N_0 \prec H_{(2^{2^\delta})^+}$ such that $\{E, W\} \subseteq N_0$ there is a strong $<\delta$-extension N of N_0 such that $N \prec H_{(2^{2^\delta})^+}$ and $\min((N \setminus N_0) \cap \delta) \in W$.

An easy closure argument shows that (WW) is equivalent to its version in which moreover $\min((N \setminus N_0) \cap \delta)$ can be chosen to be an arbitrarily large cardinal below δ.

Lemma 4.2. *Assume δ is a measurable Woodin cardinal. Then (WW) holds.*

PROOF. If $j\colon V \to M$ is an elementary embedding with $\mathrm{cr}(j) = \delta$ obtained from a normal ultrafilter $\mathcal{U}$ on δ, then δ is a Woodin cardinal in M. This is because the elementary embeddings witnessing Woodinness of δ can be coded by extenders that belong to V_δ; see [**8**, Lemma 4.3]. Therefore the set of Woodin cardinals below δ belongs to $\mathcal{U}$. Now let N_0 be as in (WW), and let f_i ($i \in \omega$) be all regressive functions from W into δ that belong to N_0. By the normality and σ-completeness of $\mathcal{U}$, there is a set $A \in \mathcal{U}$ such that $f_i(\xi) \in N_0$ for all $i \in \omega$ and all $\xi \in A$. Pick $\kappa \in (A \cap W) \setminus \sup(N_0 \cap \delta)$. Then $N = \mathrm{Hull}_{(2^{2^\delta})^+}(N_0 \cup \{\kappa\})$ is as required. $\square$

LEMMA 4.3. *Assume $\mathcal{P} \subseteq V_\delta$ has δ-cc and $\mathcal{P} \in M \prec_{<\delta} N \prec H_{(2^\delta)^+}$. Then every $A \subseteq \mathcal{P} \cap M$ in N belongs to M and in N we have that $\mathcal{P} \cap M$ is a regular subordering of $\mathcal{P} \cap N$.*

PROOF. Let α be minimal such that $A \in V_\alpha$. Then $\alpha \le \sup(M \cap \delta)$ and therefore $\alpha \in M$. But $M \cap V_\alpha = N \cap V_\alpha$. Now assume $A \subseteq \mathcal{P} \cap M$ is a maximal antichain that belongs to N. By the first part $A \in M$, and by the elementarity A is maximal in $\mathcal{P} \cap N$. $\square$

LEMMA 4.4. *In V^δ, if $M \prec H_{(2^{2^{\aleph_1}})^+}$ and $M \cap \delta \in \Delta_\delta$, then there is $N \prec H_{(2^{2^{\aleph_1}})^+}$ that strongly (and properly) $< \delta$-extends M and is such that $N \cap \delta \in \Delta_\delta$. Moreover, $\min((N \setminus M) \cap \delta) \in W$ can be chosen to be an arbitrarily large member of W.*

PROOF. Pick $M_0 \prec H_{(2^{2^\delta})^+}$ and an $(M_0, \mathrm{Coll}(\omega_1, < \delta))$-generic $p_0 \in G_\delta$ that forces $M_0[G_\delta] = M$. By (WW) we can find $\kappa \in W$ such that $\kappa > \sup((M_0 \cap \delta) \cup \mathrm{supp}(p))$ and $M_1 = \mathrm{Hull}_{(2^{2^\delta})^+}(M_0 \cup \{\kappa\}) \succ_\kappa M_0$. Then $p_0 \in \mathrm{Coll}(\omega_1, < \kappa)$ and $p_0 \Vdash_\kappa M_1 \cap \kappa \in \dot{\Delta}_\kappa$ since $M_1 \cap \kappa = M_0 \cap \kappa$. Since $W \cap E = \emptyset$, we have $\kappa \notin E$ and by the case when $\lambda = \delta$ of Lemma 2.3 we can find a κ-extension M_2 of M_1 and a $\mathrm{Coll}(\omega_1, < \delta), M_2$-generic $p \le p_0$ such that $p \Vdash M_2 \cap \delta \in \dot{\Delta}_\delta$. Then $N = M_2[G_\delta]$ is as required. $\square$

LEMMA 4.5. *Assume δ is Woodin and satisfies (WW). For every $\theta \ge (2^{2^\delta})^+$ and countable $N_0 \prec H_\theta$ there is a continuous $\prec^*_{<\delta}$-chain N_ξ ($\xi \le \omega_1$) such that each $\min(N_{\xi+1} \setminus N_\xi)$ is Woodin and there is an N_{ω_1}-generic $G \subseteq N_{\omega_1} \cap \mathrm{Coll}(\omega_1, < \delta)$.*

PROOF. We may assume $\theta = (2^{2^\delta})^+$. Working in V recursively construct the following for $\xi < \omega_1$:

(17) $N_\xi \prec H_{(2^{2^\delta})^+}$ ($\xi < \omega_1$),
(18) $\kappa_\xi < \lambda_\xi < \kappa_{\xi+1} < \delta$ and
(19) $p_\xi \in \mathrm{Coll}(\omega_1, < \kappa_\xi)$

so that for all $\eta < \xi < \omega_1$:

(20) $N_\xi \cap \omega_1 = \nu$, for some fixed $\nu < \omega_1$,
(21) $N_\eta \prec^*_{<\delta} N_\xi$,
(22) $p_\eta \ge p_\xi$,
(23) $\lambda_\xi = \min(N_{\xi+1} \setminus N_\xi) \in W$,
(24) $p_\xi \Vdash N_\xi \cap \delta \in \Delta_\delta$,
(25) p_ξ is $\mathrm{Coll}(\omega_1, < \delta), N_\xi$-generic and $\mathrm{supp}(p_\xi) \subseteq \kappa_\xi$,
(26) for every $\mathrm{Coll}(\omega_1, < \delta)$-name τ for a subset of ω_1 that belongs to N_ξ the condition p_ξ decides whether $\nu \in \tau$.

Let us describe the construction. Pick $N_0 \prec H_{(2^{2^\delta})^+}$, let $\lambda_0 = 0$, let p_0 be $\mathrm{Coll}(\omega_1, < \delta)$, N_0-generic and let κ_0 be such that $\mathrm{supp}(p_0) \subseteq \kappa_0$. Assume $N_\eta, \kappa_\eta, \lambda_\eta, p_\eta$ $(\eta < \xi)$ have been defined. If ξ is a limit ordinal, let $N_\xi = \bigcup_{\eta<\xi} N_\eta$, $p_\xi = \bigcup_{\eta<\xi} p_\eta$, $\kappa_\xi = \bigcup_{\eta<\xi} \kappa_\xi$. If $\xi = \eta + 1$ for some η let $G \subseteq \mathrm{Coll}(\omega_1, < \delta)$ be generic such that $p_\eta \in G$. In $V[G]$ use Lemma 4.4 to pick $N^0_\xi \prec H_{(2^{2^{\aleph_1}})^+}$ such that $N^0_\xi \cap \delta \in \Delta_\delta$ and $\lambda_\eta = \min((N^0_\xi \setminus N_\eta[G]) \cap \delta) \in W \setminus (\kappa_\xi + 1)$. Pick $p_\xi \leq p_\eta$ in G that decides $N^0_\xi = N_\xi[G]$ for some ground-model $N_\xi \prec H_{(2^{2^\delta})^+}$ and satisfies (26). Then p_ξ forces $N_\xi[G] \prec H_{(2^{2^\delta})^+}[G]$ so it is N_ξ-generic. Let $\kappa_\xi > \lambda_\eta$ be such that $\mathrm{supp}(p_\xi) \subseteq \kappa_\xi$. This describes the construction. Let $N_{\omega_1} = \bigcup_{\xi<\omega_1} N_\xi$. Then $G = \bigcup_{\xi<\omega_1} p_\xi$ is $\mathrm{Coll}(\omega_1, < \delta)$-generic over N_{ω_1}. □

LEMMA 4.6. *Assume δ, N_{ω_1} and G are as in Lemma 4.5 and let $x \mapsto \bar{x}$ be the Mostowski collapse of N_{ω_1}.*

(27) *$\bar{C} = \{\bar{\lambda}_\xi | \xi < \omega_1\}$ is a club in $\omega_1 = \bar{\delta}$.*
(28) *The set $H = \{A \subseteq \omega_1 | A \in N_{\omega_1}[G]$ and $\nu \in A\}$ is $\dot{\mathcal{I}}^+_\delta$-generic over $N_{\omega_1}[G]$.*
(29) *Assume $\langle r_\xi | \xi < \omega_1\rangle$ is an ω_1-sequence of reals belonging to N_0. Then $r_\xi \in N_{\omega_1}[G][H]$ for all $\xi < \omega_1$.*

PROOF. (27) follows from (23).

(28) Fix a name $\dot{\mathfrak{A}}$ for a maximal antichain in $\dot{\mathcal{I}}^+_\delta$ that belongs to N_{ω_1}. If ξ is the minimal ordinal such that $\dot{\mathfrak{A}} \in N_\xi$, then $\lambda_{\xi+1}$ is a Woodin cardinal and $\dot{\mathfrak{A}}$ is a $\mathrm{Coll}(\omega_1, < \lambda_{\xi+1})$-name for a maximal antichain in $\mathcal{I}^+_{\lambda_{\xi+1}}$. Therefore $\mathrm{Coll}(\omega_1, < \lambda_{\xi+1})$ forces that $C_\eta \setminus \nabla\dot{\mathfrak{A}}$ is nonstationary for all $\eta \in N_\xi$, and hence $\nabla\, \mathrm{int}_G(\dot{\mathfrak{A}}) \ni N_\xi \cap \omega_1$, and some $A \in \mathrm{int}_G(\dot{\mathfrak{A}})$ belongs to H.

(29) For an $f\colon \omega_1 \to \omega_1$ in $N_{\omega_1}[G]$ define a name τ_f for a real by $\tau_f(n) = i$ if $\{\xi < \omega_1 : r_{f(\xi)}(n) = i\} \in H$ for each $i \in \omega$. Then by the definition of H we have $\mathrm{int}_H(\tau_f) = r_{f(\nu)}$. By Lemma 1.4 for each $\alpha < \omega_1$ there is an $f \in N_{\omega_1}[G]$ such that $f(\nu) = \alpha$, hence $N_{\omega_1}[G][H]$ includes $\langle r_\xi : \xi < \omega_1\rangle$. □

PROOF OF THEOREM 1. Let $\mathcal{P}$ be a forcing notion that forces ϕ. By Lemma 3.3, it will suffice to prove ϕ holds in V assuming CH holds in V. Let $\langle r_\xi | \xi < \omega_1\rangle$ be an enumeration of all reals in V. Let δ be the minimal Woodin cardinal above $|\mathcal{P}|$ satisfying (WW). By Lemma 3.2, $V^{\mathcal{P}*\mathrm{Coll}(\dot{\omega}_1,<\delta)*\dot{\mathcal{I}}^+_\delta} \models (\exists x)\psi(x)$. Let $\dot{S}$ be a name for a set of reals such that $V^{\mathcal{P}*\mathrm{Coll}(\dot{\omega}_1,<\delta)*\dot{\mathcal{I}}^+_\delta} \models \psi(\dot{S})$. Take $N_{\omega_1} \prec H_{(2^{2^\delta})^+}$ such that $\{\mathcal{P}, \dot{S}, \langle r_\xi | \xi < \omega_1\rangle\} \subseteq N_{\omega_1}$ and N_{ω_1} and G satisfy the conclusion of Lemma 4.5. With H as in (28), (29) implies that the transitive collapse $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$ contains all reals. For $\kappa < \delta$ let $\mathcal{Q}_\kappa = \mathrm{Coll}(\omega_1, < \kappa) * \dot{\mathcal{I}}^+_\kappa$ and in $V^{\mathcal{P}}$ let $\dot{\mathcal{Q}}_\kappa = \mathrm{Coll}(\dot{\omega}_1, < \kappa) * \dot{\mathcal{I}}^+_\kappa$.

CLAIM 1. *For an $\alpha \in \bar{C}$, an $\bar{N_{\omega_1}}$-generic $G_\alpha \subseteq \bar{\mathcal{P}} * \bar{\mathcal{Q}}_\alpha$ in $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$ and a real r in $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$ there is $\beta > \alpha$ in $\bar{C}$ and $\bar{N_{\omega_1}}$-generic $G_\beta \subseteq \bar{\mathcal{P}} * \bar{\mathcal{Q}}_\beta$ such that $G_\beta \cap \bar{\mathcal{P}} * \bar{\mathcal{Q}}_\alpha = G_\alpha$ and $r \in \bar{N_{\omega_1}}[G_\beta]$.*

PROOF. Work in $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$. Since $\bar{\mathcal{Q}}_{\bar{\delta}}$ is $\bar{\delta}$-cc we can find a regular subordering $\mathcal{P}_r$ of $\bar{\mathcal{Q}}_{\bar{\delta}}$ of size less than $\bar{\delta}$ such that r is added by $\mathcal{P}_r$. Pick $\beta \in \bar{C} \setminus (\alpha+1)$ large enough so that $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\beta)/(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\alpha)$ collapses $2^{|\mathcal{P}_r|}$ to ω. Then $\mathcal{P}_r$ is a regular subordering of $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\beta)/(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\alpha)$. If $\alpha = \bar{\lambda}_\xi$ and $\beta = \bar{\lambda}_\eta$ then $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\xi}) \cap \bar{N_{\omega_1}} = (\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\delta}}) \cap \bar{N}_\xi$ and $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\eta}) \cap \bar{N_{\omega_1}} = (\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\delta}}) \cap \bar{N}_\eta$. By Lemma 4.3 applied to $\mathcal{P} * \mathcal{Q}_\delta$, N_{λ_ξ} and N_{λ_η}

we have that $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\alpha) \cap \bar{N_{\omega_1}}$ is a regular subordering of $(\bar{\mathcal{P}} * \bar{\mathcal{Q}}_\beta) \cap \bar{N_{\omega_1}}$. Since this remains true in $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$ and $\bar{N}_{\lambda_\eta}$ is countable we can find G_β as required. □

Working in V, find a club $D \subseteq \bar{C}$ and G_ξ $(\xi \in D \cup \{\omega_1\})$ such that for each $\eta < \xi \in D$ we have:

(30) $G_\xi \subseteq \bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\xi}$ that is $\bar{N_{\omega_1}}, \bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\xi}$-generic,
(31) $G_\xi \in \bar{N_{\omega_1}}[\bar{G}][\bar{H}]$,
(32) $G_\xi \cap \bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\eta} = G_\eta$,
(33) $G_{\omega_1} = \bigcup_{\xi \in D} G_\xi$ is $\bar{N_{\omega_1}}, \bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\omega_1}$-generic,
(34) $\mathbb{R} \subseteq \bar{N_{\omega_1}}[G_{\omega_1}]$.

The construction of G_ξ for a successor ξ and assuring (34) uses Claim 1. If ξ is a limit in D, let $G_\xi = \bigcup_{\eta \in D \cap \xi} G_\eta$. Since $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$ contains all reals and $\langle G_\eta : \eta < \xi \rangle$ can be coded by a real, G_ξ belongs to $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$. Also, since $\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\xi}$ is a direct limit of $\bar{\mathcal{P}} * \bar{\mathcal{Q}}_{\bar{\lambda}_\eta}$ for $\eta < \xi$ in $\bar{N_{\omega_1}}[\bar{G}][\bar{H}]$, G_ξ is generic. Then (33) implies that $\bar{N_{\omega_1}}[G_{\omega_1}]$ satisfies $\phi = \psi(\mathrm{int}_{\bar{G}_\delta}(\dot{S}))$. Since by (34) and Claim 29 this model contains all reals, ϕ is true in V. □

Remark. The idea of Todorcevic's proof of Theorem 4.1 can also be used in other places in the literature where forcing with stationary tower is used. For example, one can use it to reprove results from [**2**], [**1**] and [**19**, §5.1.3].

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