WOODIN'S PROOF THAT NS_{ω_1} SATURATED ALMOST IMPLIES CH FAILS

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This seminar note presents Woodin's proof of Theorem 1 as given in [3, §3.1] with some details added. It contains nothing new. Many of the lemmas—but not Theorem 13, and typically not Theorem 1 either, remain true when saturatedness is weakened to presaturatedness or when NS_{ω_1} is replaced with some normal ideal. Most of the results are given in [2] in a greater generality.

Let NS_{ω_1} be the ideal of nonstationary subsets of ω_1. It is *saturated* if every antichain in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ has size at most $\aleph_1$.

Theorem 1 (Woodin)**.** *Assume* NS_{ω_1} *is saturated and there exists a measurable cardinal. Then CH fails.*

Proof. Theorem follows immediately from lemmas 2, 3 and 4 below. □

In [3, §3.1] Woodin proved a stronger conclusion, that $\delta^1_2 = \omega_2$. The argument that in the presence of sharps the assumption of Lemma 4 implies $u_2 = \omega_2$ is given in [3]. The argument that, again in the presence of sharps, $u_2 = \delta^1_2$ is given in [4].

Terminology. Let ZFC^0 be a large enough fragment of ZFC that holds in $H(\theta)$ for a regular $\theta \geq \aleph_2$ (the precise definition will not be needed here; see [2, §1]). A *ctm* is a countable transitive model of ZFC^0. If N is a transitive model of ZFC^0 and $G \subseteq ({\mathrm{NS}_{\omega_1}}^+)^N$ is N-generic ultrafilter, then in $N[G]$ one defines the *generic ultrapower* as follows (see [1, p. 420] for or [2, §1] details). On the structure $({}^{\omega_1}N) = \{f \in N \models N \models f \text{ is a function and } dom(f) = \omega_1\}$ define relations $\sim_G$ and $\in_G$ via

$$f \sim_G g \leftrightarrow \{\xi < \omega_1^N \mid f(\xi) = g(\xi)\} \in G$$
$$f \in_G g \leftrightarrow \{\xi < \omega_1^N \mid f(\xi) \in g(\xi)\} \in G.$$

Then the generic ultrapower N^* is the set of all $\sim_G$-equivalence classes with respect to $\in_G$. The equivalence class of f is denoted by $[f]_G$. For $a \in N$ let f_a be the function mapping each $\xi \in \omega_1$ to a. The proof of Łos's theorem for ultraporoducts gives that the mapping $j\colon N \to N^*$ defined by $j(a) = [f_a]_G$ is an elementary embedding. In the case when N^* is well-founded we shall identify N^* with its transitive collapse and j with the corresponding embedding.

If N is a transitive model of ZFC^0 then a γ*-iteration* of N is $(N_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \gamma)$ where $j_{\xi\eta}\colon N_\xi \to N_\eta$ is a commuting family of elementary embeddings, $G_\eta \subseteq (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^{N_\eta}$ is a generic filter, $N_{\eta+1}$ is the transitive collapse of the generic ultrapower, and $j_{\eta\eta+1}$ is the corresponding generic ultrapower embedding , and for a limit η and $\xi < \eta$ $j_{\xi\eta}$ and N_η are the transitive collapse of the direct limit

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of $j_{\xi\zeta}$ and N_ζ for $\xi < \zeta < \eta$. (This is different from the standard definition which allows ill-founded models in iterations. This way we avoid having to define $N[G]$ for ill-founded N at the expense of creating other technical problems. What matters is that our definition of 'M is iterable' agrees with the standard one.) An iteration $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta < \alpha)$ is *well-founded* if its direct limit M_α is well-founded. A model M is γ-*iterable* if it has iterations of length γ and each of its γ-iterations is well-founded. Since ${\rm NS}_{\omega_1}{}^+$ collapses ω_1, each iteration has length at most ω_1 and we say M is *iterable* if it is ω_1-iterable. Note that a sufficient condition for the existence of a (possibly ill-founded) 1-iteration of a transitive model M of ${\rm ZFC}^0$ is that $\mathcal{P}(\mathcal{P}(\omega_1))^M$ is countable.

For $x \in \mathbb{R}$ the set

$$D_x = \{\alpha < \omega_1 \mid L_\alpha[x] \prec L_{\omega_1}[x]\}$$

is a club. If M is a ctm then it is coded by a real so D_M has the obvious meaning.

Lemma 2. *For every iterable M, every $C \in M$ such that $M \models$'$C \subseteq \omega_1$ is a club,' and every ω_1-iteration $j\colon M \to M_{\omega_1}$ we have $D_M \subseteq j(C)$.*

Lemma 3. *If for every club $C \subseteq \omega_1$ there is a real x such that $D_x \subseteq C$ then CH fails.*

Lemma 4. *Assume ${\rm NS}_{\omega_1}$ is saturated and there is a measurable cardinal. Then for every $A \subseteq \omega_1$ there is an iterable M and $a \in M$ such that for some iteration $j\colon M \to M_{\omega_1}$ we have $j(a) = A$.*

Proof of Lemma 3. Since every ω_1-sequence of clubs can be diagonalized by a club, there are more than $\aleph_1$ reals. $\square$

Proof of Lemma 2. First note that for every iteration $(M_\alpha, G_\alpha, j_{\alpha\beta}, \alpha < \beta \leq \omega_1)$ and every club $C \in M_0$ we have $\omega_1^{M_\alpha} \in j_{0\alpha}(C)$ for all α.

Fix M and $C \in M$ as in the statement of Lemma. Also let α be a countable ordinal. The set

$$X = \{x \mid \ x \text{ codes } \omega_1^{M_\alpha} \text{ for the final model of some } \alpha\text{-iteration of } M\}$$

is analytic. This is because $x \in X$ if and only if there is a sequence $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \alpha)$ such that $M_0 = M$, each $G_\xi \subseteq {\rm NS}_{\omega_1}{}^{M_\xi}$ is M_ξ-generic, $j_{\xi\eta}$ are generic ultrapower maps, and x codes $\omega_1^{M_\alpha}$. Since M is iterable, X is included in the complete coanalytic set $\{x \mid x \text{ codes a countable well-ordering}\}$. By the Boundedness Lemma $g_M(\alpha) = \sup\{\omega_1^N \mid N \in X\}$ is a countable ordinal. For each η closed under g_{M_0} and every iteration $(M_\alpha, G_\alpha, j_{\alpha\beta}, \alpha < \beta \leq \omega_1)$ we have $j_{0\eta}(\omega_1^{M_0}) = \eta$, thus $\eta \in j_{0\omega_1}(C)$ for every club C in M_0.

We have $g_M \in L[M]$, and also $g_M \restriction \alpha \in L_{\omega_1}[M]$ since all the reals of $L[M]$ are in $L_{\omega_1}[M]$. Thus every $\eta \in D_M$ is closed under g_M and therefore belongs to $j(C)$. $\square$

Proof of Lemma 4. Let $<_w$ be a fixed well-ordering of $H(\aleph_2)$.

Lemma 5. *Assume ${\rm NS}_{\omega_1}$ is saturated, $M \prec (H(\aleph_2), <_w)$ is countable, and N is the Skolem hull in $(H(\aleph_2), <_w)$ of $M \cup \{M \cap \omega_1\}$. Then the transitive collapse of N is a 1-iteration of the transitive collapse of M via $G = \{A \in \mathcal{P}(\omega_1)^M \mid M \cap \omega_1 \in A\}$.*

Proof. We first prove G is M-generic. Let $\mathcal{A} \subseteq {\rm NS}_{\omega_1}{}^+$ be a maximal antichain in M, enumerated as A_α $(\alpha < \omega_1)$. Then $C = \bigcup_{\alpha<\omega_1}(A_\alpha \setminus (\alpha+1))$ includes a club, and therefore $\nu = M \cap \omega_1 \in C$. This implies ν belongs to $\bigcup_{\alpha<\nu} A_\alpha$, which is $M \cap \mathcal{A}$.

Now we check N is the generic ultrapower. Since $<_w$ provides definable Skolem functions, $N = \{f(\nu) \mid f\colon \omega_1 \to M, f \in M\}$. Let $X = \{f \mid f\colon \omega_1 \to M, f \in M\}$. Then the generic ultrapower, M^*, is equal to $(\{[f]_G \mid f \in X\}, \epsilon)$, where $[f]_G =_G [g]_G$ if and only if $A = \{\xi \mid f(\xi) = g(\xi)\} \in G$ and $[f]_G \epsilon_G [g]_G$ if and only if $B = \{\xi \mid f(\xi) \in g(\xi)\} \in G$. But by elementarity $A \in G$ if and only if $f(\nu) = g(\nu)$ and $B \in G$ if and only if $f(\nu) \in g(\nu)$. Therefore the map $[f]_G \mapsto f(\nu)$ is an isomorphism from M^* onto N. □

The proof of Lemma 5 included verification of the following fact worth recording.

Lemma 6. *If* NS_{ω_1} *is saturated in* M *and* $G \subseteq (\mathrm{NS}_{\omega_1}{}^+)^M$ *is* M*-generic then the generic ultrapower is equal to* $\{(jf)(\omega_1^M) \mid f\colon \omega_1 \to M, f \in M\}$. □

Lemma 7. *Assume* NS_{ω_1} *is saturated. Then for every countable* $M \prec H(\aleph_2)$ *there is an* ω_1*-iteration of its transitive collapse* $\bar{M}$ *such that* $j(\bar{A}) = A$ *for all* $A \subseteq \omega_1$ *in* M *(where* $\bar{A}$ *denotes the image of* A *under the collapsing map for* M*).*

Proof. Define M_ξ ($\xi \leq \omega_1$) recursively by $M_0 = M$, $M_{\alpha+1} = \mathrm{Hull}_{H(\aleph_2)}(M_\alpha, \{M_\alpha \cap \omega_1\})$ and $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ when $\beta \leq \omega_1$ is a limit ordinal. Then by Lemma 5 the sequence of transitive collapses $\bar{M}_\alpha$ of these models defines an ω_1-iteration of $\bar{M}$, and $j_{0\omega_1}(\bar{A}) = A$ for every $A \in M$ that is a subset of ω_1. □

Lemma 8. *Assume* M *and* N *are transitive models of* ZFC^0 *such that* $\mathcal{P}(\omega_1)^M = \mathcal{P}(\omega_1)^N$ *and* NS_{ω_1} *is saturated in* N*. Then the following hold.*

1. *Every* N*-generic* $G \subseteq (\mathrm{NS}_{\omega_1}{}^+)^N$ *is* M*-generic.*
2. *For every* $\alpha \leq \omega_1$ *and every iteration* $(N_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \alpha)$ *of* $N = N_0$ *the sequence* $(M_\xi, G_\xi, k_{\xi\eta}, \xi \leq \eta \leq \alpha)$ *with* $M = M_0$ *and* $k_{\xi,\xi+1}\colon M_\xi \to M_{\xi+1}$ *induced by* G_ξ*, is an iteration of* M*.*
3. *If moreover* $N \subseteq M$*, then* $N_\xi \subseteq M_\xi$ *for all* $\xi \leq \alpha$ *and* $j_{\xi\eta} = k_{\xi\eta} \restriction N_\xi$ *for all* $\xi \leq \eta \leq \omega_1$*.*

Proof. (1) Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ in M. By saturatedness it can be coded by a subset of ω_1, and it is therefore a maximal antichain of $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ in N.

We prove (2) by induction on α. The case $\alpha = 1$ is (1). Assume $\alpha = \beta + 1$ and the statement is true for β. If $(N_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \beta + 1)$ is an iteration, then N_β and M_β have the same $\mathcal{P}(\omega_1)$. Assume $A \in \mathcal{P}(\omega_1)^{N_{\beta+1}}$ and let $f\colon \omega_1 \to \mathcal{P}(\omega_1)$ be a function in N_β such that $[f]_{G_\beta} = A$. Work in N_β. For each $\xi < \omega_1$ there is a maximal antichain $\mathcal{A}_\xi$ in $\mathrm{NS}_{\omega_1}{}^+$ whose elements decide $f(\xi)$. Each of these antichains, as well as the whole structure, is of size $\aleph_1$. Therefore $f \in M_\beta$. Since f is computed with respect to the same G_β in M, we have $A = [f]_{G_\beta} \in \mathcal{P}(\omega_1)^M$. This proves $\mathcal{P}(\omega_1)^{N_\alpha} \subseteq \mathcal{P}(\omega_1)^{M_\alpha}$, and the proof of the other inclusion is analogous. By (1) the inductive hypothesis holds for α.

The case when α is a limit ordinal is immediate since we are taking direct limits.

Clause (3) is proved by induction. For the case $\alpha = 1$ we need to check that for every $f \in {}^{\omega_1}N$, $g \in {}^{\omega_1}M$ and $B \in \mathrm{NS}_{\omega_1}{}^+$ such that $B \Vdash_{\mathrm{NS}_{\omega_1}} [g]_{G_0} \in_{G_0} [f]_{G_0}$ there is a $g' \in {}^{\omega_1}N$ such that $B \Vdash [g']_{G_0} =_{G_0} [g]_{G_0}$. Let $A = \{\xi \in B \mid g(\xi) \in f(\xi)\}$. Then $\omega_1 \setminus A \in \mathrm{NS}_{\omega_1}$, hence g' defined by $g'(\xi) = g(\xi)$ if $\xi \in A$ and $g'(\xi) = 0$ for $\xi \notin A$ is in N as required.

The proof of the successor case is identical, and the limit case is automatic since we are taking direct limits. □

The *critical sequence* of an iteration $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \gamma)$ is $j_{0\alpha}(\omega_1^{M_0})$ $(\alpha < \gamma)$. An elementary embedding $j\colon M \to M^*$ can be naturally extended to include definable subsets of M in its domain. If $X = \{x \mid M \models \phi(x,a)\}$ for a parameter $a \in M$ let $j(X) = \{x \mid M^* \models \phi(x, j(a))\}$.

Lemma 9. *Assume M is a transitive model of ZFC^0 such that NS_{ω_1} is saturated in M and X is a definable subset M.*

(1) *If $G \subseteq {\mathrm{NS}_{\omega_1}}^+$ is M-generic, $j\colon M \to M^*$ is the generic ultrapower map then for every ${\mathrm{NS}_{\omega_1}}^+$-name $\dot{x}$ for an element of $j(X)$ there is $f\colon \omega_1 \to X$ in M such that $\Vdash [f]_G = \dot{x}$. (Equivalently, $(jf)(\omega_1^M) = \mathrm{int}_G(\dot{x})$.)*

(2) *If $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \gamma)$ is an iteration then for every $x \in j_{0\gamma}(X)$ then there is $n \in \omega$, an n-tuple $\vec{\xi}$ in the critical sequence and $f\colon \omega_1^n \to X$ in M_0 such that $a = (j_{0\gamma} f)(\vec{\xi})$.*

(3) *In every iteration $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \gamma)$ $j_{0\gamma}{}''\,\mathrm{Ord}^{M_0}$ is cofinal in Ord^{M_γ}.*

Proof. (1) Let $\mathcal{A}$ be a maximal antichain such that each $A \in \mathcal{A}$ decides $f_A \in M$ such that $f_A\colon A \to \mathrm{Ord}$ and $A \Vdash [f_A]_G = \dot{x}$. Since $\mathcal{A}$ is of size at most $\aleph_1$, we can assume its elements are pairwise disjoint and therefore $f' = \bigcup_{A \in \mathcal{A}} f_A \restriction A$ is a function whose domain includes a club such that $[f']_G = \dot{x}$. The set $\{\xi \mid f'(\xi) \notin X\}$ is nonstationary and therefore we can define $f\colon \omega_1 \to X$ in M that coincides with f' on a club.

(2) Induction on γ. Fix an iteration $(M_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \gamma)$ and write $\xi_\alpha = j_{0\alpha}(\omega_1^M)$. For the successor case, $\gamma = \beta + 1$, by Lemma 6 let $f'\colon \omega_1 \to j_{0\beta}(X)$ in M_β be such that $(j_{\beta\beta+1} f')(\xi_\beta) = x$. By the inductive assumption there is $f\colon \omega_1^n \to {}^{\omega_1} X$ in M_0 and $\vec{\xi}'$ in the critical sequence such that $(j_{0\beta} f)(\vec{\xi}') = f'$. Then with $\vec{\xi} = \vec{\xi}' {}^\frown \xi_\beta$ we have $(j_{0\gamma} f)(\vec{\xi}) = x$. If γ is limit, there is nothing to prove since we are taking direct limits.

(3) Letting $X = \mathrm{Ord}^M$ and applying (2), for an ordinal α in M_γ fix $f \in M_0$ such that $(jf)(\vec{\xi}) = \alpha$. If $\beta = \sup \mathrm{range}(f)$ then clearly $j(\beta) \geq \alpha$. □

Lemma 10. *If M is a ctm such that NS_{ω_1} is saturated in M then M is n-iterable for all $n \in \omega$.*

Proof. It suffices to check the well-known case when $n = 1$, and for this we only need to check the following.

> If $G \subseteq {\mathrm{NS}_{\omega_1}}^+$ is generic and $j\colon M \to M^*$ is a generic ultrapower embedding then every ω-sequence of elements of M^* that belongs to $M[G]$ belongs to M^*.

It suffices to prove the assertion for sequences of ordinals. Assume $\dot{\alpha}_n$ $(n < \omega)$ is a sequence of ${\mathrm{NS}_{\omega_1}}^+$-names for ordinals in M. By Lemma 9 there are functions $f_n \in M$ such that $\Vdash [\check{f}_n]_G = \dot{\alpha}_n$, and therefore the function $g\colon \omega_1 \to {}^\omega\,\mathrm{Ord}$ defined by $g(\xi)(n) = f_n(\xi)$ is such that $\Vdash [\check{g}]_G = (\dot{\alpha}_n \mid n < \omega)$.

Since M^* is closed under ω-sequences in $M[G]$, it is well-founded in $M[G]$. But $M[G]$ is well-founded, and therefore so is M^*. □

For $X \subseteq M$ we write $j''X = \{j(x) \mid x \in X\}$. The proof of Lemma 10 clearly gives that for every $X \subseteq M$ of size $\aleph_1$ we have $j''X \in M^*$, but note that M^* is not closed under ω_1-sequences.

In the following ZFC^* is a finite fragment of ZFC large enough to imply 'there is no largest cardinal.'

Lemma 11. *If M is a transitive model of ZFC*, $\mathcal{P}(\omega_1)^M$ is countable, and NS_{ω_1} is saturated in M, then M is α-iterable for every $\alpha \in M$.*

Proof. Since $\mathcal{P}(\omega_1)^M$ is countable we can construct iterations of M of an arbitrary countable length, and $\mathcal{P}(\omega_1)^{M_\xi}$ will be countable for every $\xi < \omega_1$. Assume the assertion fails and fix an iteration $(M_\xi, G_\xi, j_{\xi\eta}, \xi \le \eta \le \alpha)$ such that $M_0 = M$ and M_α is ill-founded. We can assume α is the minimal ordinal for which such an iteration exists. By (3) of Lemma 9, $j_{0\alpha}{}'' \operatorname{Ord}^{M_0}$ is cofinal in $\operatorname{Ord}^{M_\alpha}$, and therefore there is $\beta \in \operatorname{Ord}^{M_0}$ such that $j_{0\alpha}(\beta)$ is ill-founded. For each $\kappa \in M$ such that $M \models \mathrm{cf}(\kappa) \ge \omega_2$ we have $(V_\kappa)^M \models \mathrm{ZFC}^0$ hence by Lemma 8 each iteration of M induces an iteration of $(V_\kappa)^M$. By Lemma 8(3), this iteration is ill-founded.

Then α is a limit ordinal by Lemma 10, and it is clear that β is also a limit ordinal. We cannot define an iteration of $M \cap V_\kappa$ in M because its ω_1 is uncountable, hence let $G \subseteq \mathrm{Coll}(\omega, |V_\kappa|^+)^M$ be M-generic. For a ctm N and $\alpha < \omega_1$ 'there exists an ill-founded α-iteration of N' is a $\mathbf{\Sigma}^1_1$ statement, and therefore absolute between transitive models of ZFC^0. Therefore for $N_0 = V_\kappa^M$ the following holds in $M[G]$:

P_{α,β,N_0}: There is an ill-founded α-iteration $(N_\xi, G_\xi, j_{\xi\eta}, \xi \le \eta < \alpha)$ such that $j_{0\alpha}(\beta)$ is ill-founded.

Let $(\alpha_0, \beta_0, \kappa_0)$ be the lexicographically minimal triple such that $P_{\alpha_0,\beta_0,(V_{\kappa_0})^M}$ holds in $M[G]$ and $\mathrm{cf}(\kappa)^M > \omega_1$. Now let $N_0 = M \cap V_{\kappa_0}$, fix a witnessing iteration $(N_\xi, G_\xi, k_{\xi\eta}, \xi \le \eta \le \alpha_0)$ in $M[G]$. Let ζ_n $(n \in \omega)$ be a decreasing sequence of ordinals in N_{α_0} below $j_{0\alpha_0}(\beta_0)$. By (3) of Lemma 8 this sequence is in N_{α_0}. Since $j_{0\alpha_0}(\beta_0)$ is a direct limit, we can find $\alpha_1 < \alpha_0$ and $\beta_1 < j_{0\alpha_1}(\beta_0)$ such that $j_{\alpha_1\alpha_0}(\beta_1) = \zeta_2$, hence it is ill-founded. By Lemma 8 and the minimality of α_0, in the induced iteration $(N_\xi, G_\xi, k_{\xi\eta}, \xi \le \eta \le \alpha_0)$ all N_ξ $(\xi < \alpha_0)$ are well-founded. By the elementarity, the triple $j_{0\alpha_1}(\alpha_0, \beta_0, \kappa)$ in has the same minimality property as $(\alpha_0, \beta_0, \kappa)$. But the tail $(N_\xi, G_\xi, j_{\xi\eta}, \alpha_1 \le \xi \le \eta \le \alpha_0)$ of the iteration is an iteration of length $\alpha_2 \le j_{0\alpha_1}(\alpha_0)$ and it witnesses that $(\alpha_2, \beta_1, j_{0\alpha_1}(\kappa))$ have the same property. This is a contradiction since $\beta_1 < j(\beta_0)$. □

The following is standard, and our only use of the assumption that there exists a measurable cardinal is its conclusion for the case $\mathfrak{A} = H(\aleph_2)$.

Lemma 12. *Assume κ is a measurable cardinal and $\mathfrak{A} \in V_\kappa$ is transitive. Then for every $\theta > \kappa$ the set*

$$\{M \prec \mathfrak{A} \mid M \text{ is countable and there is an uncountable } N \prec H(\theta) \text{ such that } N \cap \mathfrak{A} = M\}$$

contains a club.

Proof. Let $<_w$ be a fixed well-ordering of V_κ and let $M \prec (V_\kappa, <_w)$ be countable and such that $\mathfrak{A} \in M$. Let f_n $(n \in \mathbb{N})$ be the enumeration of all Skolem functions for V_κ with parameters in M. Find $A \subseteq \kappa$ of full measure such that each f_n with range in M is constant on A. Then $M_1 = \mathrm{Hull}(M \cup \{\alpha\})$, for $\alpha = \min(A)$, is an end-extension of M. Iterating this process find an ω_1-elementary chain of end-extensions and let N be its union. □

Proof of Lemma 4. Assume NS_{ω_1} is saturated and let $A \subseteq \omega_1$. With $\theta = \kappa^+$, by Lemma 12 we can find $N \prec H(\theta)$ such that $A \in N$, $M = N \cap H(\aleph_2)$ is countable and N is of size $\aleph_1$. Let $\bar{M}$ and $\bar{N}$ be the transitive collapses of these two models. By Lemma 7, $N_0 = \bar{N}$ has an ω_1-iteration $(N_\xi, G_\xi, k_{\xi\eta}, \xi \leq \eta \leq)$ such that $A = j_{0\omega_1}(a)$ for some $a \in N_0$. We only need to show that $\bar{N}$ is iterable. Note that $\mathcal{P}(\omega_1)^{\bar{M}} = \mathcal{P}(\omega_1)^{\bar{N}}$. We claim $\bar{M}$ is iterable. Assume not and fix an ill-founded iteration $(\bar{M}_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \alpha)$ of minimal length; then $\alpha < \omega_1$. By Lemma 8, these generics define an iteration $(\bar{N}_\xi, G_\xi, j_{\xi\eta}, \xi \leq \eta \leq \alpha)$ such that $\bar{N}_\xi \subseteq \bar{M}_\xi$. The latter iteration is well-founded by Lemma 8, a contradiction. □

Note that we have actually proved the following.

Theorem 13. *Assume that for every $A \subseteq \omega_1$ there is an iterable M and $a \in M$ such that for some iteration $j\colon M \to M_{\omega_1}$ we have $j(a) = A$. Then CH fails.* □

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