C* ALGEBRAS AND THEIR REPRESENTATIONS

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The original version of this note was based on two talks given by Efren Ruiz at the Toronto Set Theory seminar in November 2005. This very tentative note and my Luminy talk form an attempt to record the ideas from these talks and draw more attention of set theorists to Naimark's problem (Problem 5.1 below). An excellent invitation to C*-algebras for set-theorists can be found in [5]. All mistakes are due to me; I would like to thank Efren Ruiz for reducing their number.

1. Basics: definitions, examples, representation theorems

Definition 1.1. A Banach algebra A with an involution * is a C^*-algebra if for all a, b in A and $\lambda \in \mathbb{C}$ we have

(1) $(a+b)^* = a^* + b^*$,
(2) $(\lambda a)^* = \bar{\lambda} a^*$,
(3) $a^{**} = a$,
(4) $(ab)^* = b^* a^*$,
(5) $\|a^* a\| = \|a\|^2$.

A C*-algebra is *unital* if it has the multiplicative identity.

Lemma 1.2. *Every C*-algebra A is contained in a unital C*-algebra $A \oplus \mathbb{C}$, with naturally defined operations.*

Proof. On $A \times \mathbb{C}$ define the operations as follows: $(a, \lambda)(b, \xi) = (ab + \lambda b + \xi a, \lambda \xi)$, $(a, \lambda)^* = (a^*, \bar{\lambda})$ and $\|(a, \lambda)\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$. For verification see [3, Proposition I.1.3]. □

Example 1.3. (1) If X is a Hausdorff locally compact space take $C_0(X)$, the space of all continuous complex functions f on X vanishing at infinity with respect to the sup norm and $f^* = \bar{f}$.

(2) $B(H)$, the algebra of all bounded operators on a Hilbert space H with respect to the operator norm $\|T\| = \sup\{\|T(\eta)\| \mid \eta \in H, \|\eta\| = 1\}$. The adjoint is defined implicitly via $(T\eta, \xi) = (\eta, T^*\xi)$ for all ξ, η in H. Note that for a fixed T and ξ this formula uniquely defines a functional, which has to be of the form $(\cdot, \zeta)$ for some vector $\zeta \in H$.

(3) Let H and $B(H)$ be as in (2). If $X \subseteq B(H)$ then there is the minimal C*-subalgebra of $B(H)$ that contains X. This is the norm-closure of the closure of X under $+, \cdot$ and *. C*-algebras of this form are called *concrete* C*-algebras.

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(4) $K(H)$ (also denoted by $C(H)$ or $B_0(H)$, in assorted fonts), the algebra of all compact operators on a Hilbert space H. (An operator a on H is *compact* if the closure of the a-image of the unit ball is compact.) It is a subalgebra of $B(H)$, and even a (closed, two-sided and *self-adjoint*: $a \in K(H)$ implies $a^* \in K(H)$) *ideal* in $B(H)$.

The fact that every C*-algebra is isomorphic to a closed subalgebra of some $B(H)$ (Corollary 3.7) justifies the following terminology. An $a \in A$ is *self-adjoint* if $a^* = a$. A self-adjoint idempotent is a *projection.*

2. Commutative C^*-algebras

A C*-algebra is *commutative* if its multiplication operation is commutative. For a C^*-algebra B let

$$\hat{B} = \{\phi \colon B \to \mathbb{C} \mid \phi \text{ is a nonzero homomorphism}\}.$$

If X is locally compact and $x \in X$, then $\phi_x \colon C_0(X) \to \mathbb{C}$ given by $\phi_x(b) = b(x)$ is a homomorphism. All homomorphisms of $C_0(X)$ are of this form.

Lemma 2.1. *If B is a commutative C^*-algebra, then the space $\hat{B}$ is locally compact in its weak*-topology. Also, B is isomorphic to $C_0(\hat{B})$ via the map $B \ni b \mapsto f_b \in C_0(\hat{B})$, defined by $f_b(\phi) = \phi(b)$.*

Proof. Note that $\hat{B}$ is a closed subset of $\mathbb{C}^B$ (in most reasonable topologies). If B is unital, then every nonzero homomorphism has norm 1. Thus by the Alaoglu–Birkhoff theorem $\hat{B}$ is compact in the weak*-topology.

If B is not unital consider $B \oplus \mathbb{C}$ as in Lemma 1.2, and note that $B \hat{\oplus} \mathbb{C}$ is a one-point compactification of the (Hausdorff) space $\hat{B}$.

For the second part, see e.g., [3, I.3.1] or [2, Theorem 1.1.1]. □

The following is an immediate consequence.

Theorem 2.2 (Gelfand–Naimark)**.** *Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. If it is moreover unital, then X can be chosen to be compact.* □

Consider a unital C^*-algebra A. For $a \in A$ let its *spectrum* be defined by

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible}\}.$$

It is not difficult to see that this is always a compact subset of $\mathbb{C}$. We say a is *positive* if it is *self-adjoint* (i.e., $a^* = a$) and $\mathrm{sp}(a) \subseteq [0, \infty)$. We define a partial ordering on the self-adjoint elements of A via $a \leq b$ iff $b - a$ is positive. An $a \in A$ is *normal* if $aa^* = a^*a$. For a normal A consider the subalgebra $C^*(a, 1)$ generated by a and the unit 1. It is the norm closure of $\{p(a, a^*) \mid p$ is a polynomial in two variables with complex coefficients$\}$.

If $\lambda \in \mathrm{sp}(a)$, then $\phi_\lambda(a) = \lambda$ uniquely defines a *-homomorphism of $C^*(a, 1)$ into $\mathbb{C}$.

Lemma 2.3. *The C^*-algebra $C^*(a, 1)$ is isomorphic to $C(\mathrm{sp}(a))$.*

Proof. Let $B = C^*(a, 1)$. Then every $\phi \in \hat{B}$ is uniquely determined by $\lambda_\phi = \phi(a)$. Moreover, $\phi(a) = \lambda$ implies $\phi(\lambda 1 - a) = 0$, and therefore $\lambda_\phi \in \mathrm{sp}(a)$. The isomorphism we seek is given by $C^*(a, 1) \ni b \mapsto f_b \in C(\mathrm{sp}(a))$, where $f_b(\lambda) = \phi_\lambda(b)$.

If $b = p(a, a^*)$ then $\lambda \mapsto f_b(\lambda) = p(\lambda, \bar{\lambda})$ is clearly continuous and it is not difficult to check that f_b is continuous for every b. This map $b \mapsto f_b$ is a *-homomorphism, and by Lemma 3.1 it is automatically continuous. Finally, it is onto since every $\lambda \in \mathrm{sp}(a)$ uniquely defines $\phi_\lambda \in \hat{B}$ as above. □

By Lemma 3.1, *-isomorphisms of C^*-algebras are norm-preserving, and therefore we have the following.

Corollary 2.4. *If a is normal, then $\|a\| = \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(a)\}$.* □

Lemma 2.3 enables us to define $f(a) \in A$ for every normal a and continuous $f\colon \mathrm{sp}(a) \to \mathbb{C}$. Note that $\lambda 1 - f(a) = (\lambda - f)(a)$, hence $\mathrm{sp}(f(a)) \subseteq \mathrm{range}(f)$. In particular, if the range of f is included in $[0, \infty)$ then $f(a)$ is positive. Since a^*a is always normal and positive, we can define

$$|a| = \sqrt{a^*a}$$

for every $a \in A$.

3. Irreducible representations

A map π between C^*-algebras is a **-homomorphism* if it is additive, multiplicative and preserves the *-operation.

Lemma 3.1. *Every *-homomorphism of C^*-algebras is automatically continuous, and even a contraction. In particular, every *-isomorphism of C^*-algebras is norm-preserving.*

Proof. We may assume C*-algebras are unital and π sends the unity to the unity. If $a - \lambda I$ is invertible, then its inverse is mapped to the inverse of $\pi(a) - \lambda I$, so for a self-adjoint a we have $\mathrm{sp}(a) \supseteq \mathrm{sp}(\pi(a))$. By Corollary 2.4 we have $\|a\| \geq \|\pi(a)\|$ for every self-adjoint a.

To check π is continuous, it suffices to show that $\|\pi(a)\| \leq \|a\|$ for every a. But $\|a\|^2 = \|aa^*\| = \|\pi(aa^*)\| = \|\pi(a)\|^2$. □

A *representation* (π, H) of A is a *-homomorphism $\pi\colon A \to B(H)$. It is *irreducible* if the only closed subspaces of H invariant under all operators in $\pi(A)$ are $\{0\}$ and H. This is equivalent to asserting that the only projections of H that commute with $\pi(A)$ are the trivial ones. It is also equivalent to asserting that for every nonzero $\eta \in H$ the subspace generated by $\pi[A]\eta$ is dense in H.

A linear functional $\phi\colon A \to \mathbb{C}$ is a *state* if (i) it is *positive*: $\phi(x^*x) \geq 0$ for all $x \in A$ and (ii) $\|\phi\| = 1$. Note that it is not required that ϕ is multiplicative, but it agrees with the conjugation in as much that it is required to be positive.

Let $S(A)$ denote the collection of all states of A. This is clearly a convex set in the dual space of the Banach space A. On $S(A)$ we can define a partial ordering via $\phi \leq \psi$ if and only if $\phi(x^*x) \leq \psi(x^*x)$ for all x. A state ϕ is *pure* if it is an extreme point in $S(A)$.

Lemma 3.2. *A state ϕ is pure if and only if for every $\psi \in S(A)$ such that $\psi \leq \phi$ we have $\psi = t\phi$ for some $t \in [0, 1]$.* □

Lemma 3.3. *Pure states of $C_0(X)$ are exactly the evaluation functions $f \mapsto f(\lambda)$.*

Proof. Fix a functional ϕ. By Riesz representation theorem, ϕ is the Radon integral with respect to a Radon measure ν_ϕ on X. Since in $C_0(X)$ the elements of the form a^*a are exactly the nonnegative real-valued functions, ϕ is positive exactly when ν_ϕ is strictly positive, and it is a state if and only if $\nu_\phi(X) = 1$. Finally, ϕ is pure if and only if X cannot be partitioned in two measurable sets of positive measure. □

If Λ is a directed set then a net $\{e_\lambda\}_{\lambda\in\Lambda}$ in A is an *approximate identity* if (i) $1 \geq e_\lambda \geq 0$, (ii) $\lambda < \mu$ implies $e_\lambda \leq e_\mu$, and (iii) $\lim_\lambda \|ae_\lambda - a\| = 0$ for all $a \in A$. We typically use the directed ordering of A itself and let $\lambda = e_\lambda$. For example, all finite-dimensional projections form an approximate identity in $K(H)$.

Lemma 3.4. *Every C^*-algebra A has an approximate identity.*

Proof. See e.g., [2, Theorem 1.8.2]. □

Lemma 3.5. *Assume ϕ is a linear functional on a C^*-algebra A. If $\lim_\lambda \phi(e_\lambda) = 1$ for some approximate identity in A, then ϕ is a state. Conversely, if ϕ is a state then $\lim_\lambda \phi(e_\lambda) = 1$ for every approximate identity of A.*

In particular, if B is a C^*-subalgebra of A then every state on B can be extended to a state on A.

Proof. For the direct implication in the first part see e.g., [3, I.9.9]; the converse is immediate. The second part follows by the Hahn–Banach theorem. □

A representation (π, H) is *cyclic* if there is $\eta \in H$ such that $\pi[A]\eta$ is dense in H. Such an η is called a *cyclic vector.*

Theorem 3.6.

1. *Suppose $\phi \in S(A)$. Then there is a representation (π_ϕ, H_ϕ) of A and a unit cyclic vector $v_\phi \in H$ such that $\langle \pi_\phi(a)v_\phi, v_\phi\rangle = \phi(a)$ for all $a \in A$.*
2. *If B is a subalgebra of A and the restriction of ψ to B is nonzero, then $v_\psi \in H_\psi$ and H_ψ is a subspace of H_ϕ.*

Proof. (1) On $A \times A$ let $\langle a, b\rangle = \phi(b^*a)$. Then $\langle \cdot, \cdot\rangle$ has all the properties of the scalar product except perhaps the positivity. Let $N_\phi = \{a \in A \mid \phi(a^*a) = 0\}$. This is a left ideal on A and $\langle \cdot, \cdot\rangle$ defines a scalar product on $A/N_\phi \times A/N_\phi$. Let H_ϕ be the completion of this space and let $\pi_\phi(a)$ in $B(H_\phi)$ be defined by $\pi_\phi(a)\bar{x} = \bar{a}\bar{x}$. (Here $\bar{a}$ denotes the N_ϕ-equivalence class of a.) [Note that in case when π_ϕ is irreducible, A/N_ϕ is already complete.]

Using Lemma 3.4, fix an approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A. Then $\{e_\lambda/N_\phi\}$ is a Cauchy net in A/N_ϕ. Let v_ϕ be its limit. Then $(\pi_\phi(a)v_\phi, v_\phi) = \lim_\lambda(\pi_\phi(a)e_\lambda, e_\lambda) = \pi_\phi(a)$.

(2) Straightforward. □

Corollary 3.7 (Gelfand–Naimark–Segal)**.** *Every C^*-algebra A is isomorphic to a subalgebra of $B(H)$ for some Hilbert space H.*

Proof. By Lemma 3.5 for every nonzero $a \in A$ there is a state $\phi_a \in S(A)$ such that $\phi_a(a)$ is nonzero. By Theorem 3.6 there is a representation (π_a, H_a) such that $\pi_a(a)$ is nonzero. Let $H = \bigoplus_{a\in A\setminus\{0\}} H_a$ and let $\pi = \bigoplus_{a\in A\setminus\{0\}} \pi_a$. □

Lemma 3.8. *If η is a cyclic unit vector for (π, H) then $\omega_\eta \colon A \to \mathbb{C}$ defined by*

$$\omega_\eta(a) = (\pi(a)\eta, \eta)$$

is a state.

Proof. It is clearly a homomorphism. $\omega_\eta(a^*a) = (\pi(a)\eta, \pi(a)\eta) \geq 0$. Also $\|\omega_\eta\| = \sup\{\|\omega_\eta(a)\| \mid \|a\| = 1\} \leq 1$. □

Theorem 3.9. *Suppose (H_1, π_1) and (H_2, π_2) are cyclic representations of A with cyclic vectors η_1 and η_2. Then the following are equivalent:*

1. *There is an unitary $u\colon H_1 \to H_2$ such that $u(\eta_1) = \eta_2$ and $u\pi_1(\cdot)u^* = \pi_2(\cdot)$.*
2. *$\langle \pi_1(a)\eta_1, \eta_1\rangle = \langle \pi_2(a)\eta_2, \eta_2\rangle$ for every $a \in A$.*

Proof. For the nontrivial direction, assume $\langle \pi_1(a)\eta_1, \eta_1\rangle = \langle \pi_2(a)\eta_2, \eta_2\rangle$ for every $a \in A$ and define u on $\pi_1[A]\eta_1$ via $u(\pi_1(a)\eta_1) = \pi_2(a)\eta_2$. We claim that u is well-defined. If $\pi_1(a)\eta_1 = \pi_1(b)\eta_1$, then $\pi_1(a - b)\eta_1 = 0$ hence $\pi_2(a - b)\eta_2 = 0$ and $\pi_2(a)\eta_2 = \pi_2(b)\eta_2$.

To see u is unitary, note that $\|\pi_2(a)\eta_2\|^2 = \langle \pi_2(a)\eta_2, \pi_2(a)\eta_2\rangle = \langle \pi_2(a^*a)\eta_2, \eta_2\rangle$ and similarly $\|\pi_1(a)\eta_1\|^2 = \langle \pi_1(a^*a)\eta_1, \eta_1\rangle$. Therefore $\|\pi_1(a)\eta_1\| = \|\pi_2(a)\eta_2\|$ for all a, implying $\|u(\xi)\| = \|\xi\|$ for all ξ in the dense subspace $\pi_1[A]\eta_1$. □

Corollary 3.10. *If (π, H) is a cyclic representation of A then (π, H) is unitarily equivalent to (π_ρ, H_ρ) for some state $\rho\colon A \to \mathbb{C}$.*

Proof. Let $\rho(a) = \langle \pi(a)\eta, \eta\rangle$, where η is a unitary cyclic vector of (π, H). □

It is worth mentioning (yet not difficult to check; e.g., [3, I.9.8]) that a state ϕ is pure if and only if the associated representation π_ϕ is irreducible.

Corollary 3.11. *Every irreducible representation of a commutative C^*-algebra is one-dimensional.* □

Proof. By Corollary 3.10 and remark above we only need to check the pure states. By Theorem 2.2 we may assume B is $C_0(X)$ for some locally compact X. By Lemma 3.3, its only pure states are the evaluation functions $f \mapsto f(\lambda)$. □

4. Naimark's theorem

Theorem 4.1 (Naimark). *Let $K(H)$ be the C^*-algebra of compact operators on a Hilbert space H. Then (up to the unitary equivalence) $K(H)$ has only one irreducible representation.*

Proposition 4.2. *If ρ is a pure state of $K(H)$ then there is $\eta \in H$ such that $\rho(a) = \omega_\eta(a) = \langle a(\eta), \eta\rangle$.*

Proof of Theorem 4.1. Let (π', H') be an irreducible representation of $K(H)$. Let $\rho(a) = \langle \pi'(a)\xi, \xi\rangle$, where $\xi \in H'$ has $\|\xi\| = 1$. By Proposition 4.2 find $\eta \in H$ such that $\rho = \omega_\eta$. Now $(\pi', H') \sim_u (\pi_\rho, H_\rho) \sim_u (\pi_{\omega_\eta}, H_{\omega_\eta})$. The latter is unitarily equivalent to (id, H). □

Proof of Proposition 4.2. An operator $a \in B(H)$ is a *Trace Class Operator* if for some orthogonal basis E of H we have

$$\sum_{e \in E} \langle |a|e, e\rangle < \infty.$$

Recall that $|a| = \sqrt{a^*a}$ was defined after Lemma 2.3. For a trace class operator a define its *trace* as

$$\mathrm{tr}(a) = \sum_{e \in E} \langle ae, e\rangle.$$

Just like in the finite-dimensional case we have $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for any trace class operator a and any operator b. In particular, this sum does not depend on the choice of the orthonormal basis Every trace class operator is compact since it can be approximated by finite rank operators.

For norm one vectors η_1 and η_2 in H define a rank one operator $b_{\eta_1,\eta_2}\colon H \to H$ by

$$b_{\eta_1,\eta_2}(\xi) = \langle \xi, \eta_2 \rangle \eta_1.$$

Claim 1. *Given a functional $\rho \in K(H)^*$ there is a trace class operator u such that $\rho(a) = \mathrm{tr}(ua)$ for all $a \in K(H)$. If $\rho \geq 0$ then $u \geq 0$.*

Proof. For the existence, see e.g., [4, Theorem 3.4.13]. To see u is positive, pick $\eta \in H$. Then $ub_{\eta,\eta}(\xi) = u(\langle \xi, \eta \rangle \eta) = \langle \xi, \eta \rangle u(\eta) = b_{u(\eta),\eta}(\xi)$. Therefore

$$\begin{aligned} 0 \leq \rho(b_{\eta,\eta}) = \mathrm{tr}(ub_{\eta,\eta}) &= \mathrm{tr}(b_{u(\eta),\eta}) \\ &= \sum_{e \in E} \langle b_{u(\eta),\eta}(e), e \rangle = \sum_{e \in E} \langle ub_{\eta,\eta} e, e \rangle = \langle u(\eta), \eta \rangle. \end{aligned}$$

(In the last equality we change the basis to E' so that $\eta \in E'$.) □

Since u is a positive compact operator, it is diagonalizable (e.g., [4, Theorem 3.3.8]), so we can write $u = \sum_{e \in E} \lambda_e e^*$ with the appropriate choice of the basis E. Thus $\rho(a) = \mathrm{tr}(ua) = \mathrm{tr}(au) = \sum_{e \in E} \langle aue, e \rangle = \sum_{e \in E} \lambda_e \langle ae, e \rangle \geq \lambda_{e_0} \langle ae_0, e_0 \rangle$, for any $e_0 \in E$. Since ρ is a pure state, for each $e \in E$ there is $t_e \in [0,1]$ such that $t_e \rho(a) = \lambda_{e_0} \langle ae_0, e_0 \rangle$. Thus exactly one $t_e = t_{e_0}$ is nonzero, and $a \mapsto \lambda_{e_0} \langle ae_0, e_0 \rangle$. □

5. Rosenberg's theorem

Problem 5.1 (Naimark's problem)**.** If A is a C*-algebra all of whose irreducible representations are unitarily equivalent, is A isomorphic to some $K(H)$?

The affirmative answer for 'small' A is given by Theorem 5.3 and Corollary 5.7. Assuming $\diamond$, there is a unital C^*-algebra of density $\aleph_1$ all of whose irreducible representations are equivalent. Since for an infinite-dimensional H the algebra $K(H)$ is not unital, this gives a (consistent) negative solution to Naimark's problem ([1]).

Lemma 5.2. *Assume (π, H) is a faithful irreducible representation of A such that every irreducible representation of A is unitarily equivalent to it. If B is a maximal commutative subalgebra of A then for every $\phi \in \hat{B}$ there is $\eta_\phi \in H$ such that*

$$\phi(b)\eta_\phi = \pi(b)\eta_\phi.$$

Also, if $\phi \neq \psi$ then η_ϕ and η_ψ are orthogonal.

Proof. Fix ϕ. By Lemma 3.5 we can extend ϕ to a state $\tilde{\phi} \in S(A)$. Let v_ϕ, $(\pi_{\tilde{\phi}}, H_{\tilde{\phi}})$ be as in Theorem 3.6. By our assumption there is a unitary operator $u = u_\phi\colon H_{\tilde{\phi}} \to H$. Let $\xi_\phi = u(v_\phi)$. Since in an irreducible representation every nonzero vector is cyclic, by Theorem 3.9 we have $(\pi(a)\xi_\phi, \xi_\phi) = (\pi_{\tilde{\phi}}(a)v_\phi, v_\phi)$ for all $a \in A$. By Theorem 3.6 we have $\phi(b) = (\pi_{\tilde{\phi}}(b)v_\phi, v_\phi)$ for all $b \in B$, thus $\phi(b) = (\pi(b)\xi_\phi, \xi_\phi)$ for all $b \in B$. Since ξ_ϕ is a unit vector we have $\phi(b) = (\phi(b)\xi_\phi, \xi_\phi)$.

It remains to check that the space $\pi[B]\xi_\phi$ is one-dimensional. By (2) of Theorem 3.6 this space is isomorphic to H_ϕ from the irreducible representation (π_ϕ, H_ϕ).

Finally, since B is commutative, all of its irreducible representations are one-dimensional (Corollary 3.11).

For $b \in B$ we have $\phi(b)\langle \eta_\phi, \eta_\psi \rangle = \langle \phi(b)\eta_\phi, \eta_\psi \rangle = \langle \eta_\phi, \bar{\phi}(b)\eta_\psi \rangle = \psi(b)\langle \eta_\phi, \eta_\psi \rangle$. Pick $b \in B$ such that $\phi(b) \neq \psi(b)$; then the above implies $\langle \eta_\phi, \eta_\psi \rangle = 0$. □

Theorem 5.3 (Rosenberg). *If A is a C^* algebra with an irreducible representation (π, H) such that H is separable and every irreducible representation of A is unitarily equivalent to π, then A is isomorphic to the algebra of compact operators on a Hilbert space.*

Proof. Let B be a maximal commutative subalgebra of A. By Lemma 5.2, for each $\phi \in \hat{B}$ there is an $\eta_\phi \in H$ and $X = \{\eta_\phi \mid \phi \in \hat{B}\}$ are pairwise orthogonal. Since $|X| < 2^{\aleph_0}$, $\hat{B}$ is a locally compact Hausdorff space of cardinality less than $2^{\aleph_0}$. It therefore has an isolated point, and the conclusion follows by Lemma 5.6 (3) proved below. □

Lemma 5.4. *Assume $p \in A$ is a projection and (π, H) is an irreducible representation of A. Then $(\pi \restriction pAp, (\pi(p))[H])$ is an irreducible representation of pAp.*

Proof. Pick a nonzero vector $\eta \in H' = (\pi(p))[H]$. Since π is irreducible, $\{(\pi(a))\eta \mid a \in A\}$ generates a dense subspace of H. Therefore $\{(\pi(p)\pi(a))\eta \mid a \in A\}$ generates a dense subspace of H'. □

Lemma 5.5. *Assume A is a C^* algebra that has a unique irreducible representation up to the unitary equivalence. Then A has the following properties.*

(1) *A is* simple*: its only closed *-ideals are $\{0\}$ and A.*

Proof. (1) For every $a \in A$ there is an irreducible representation ϕ such that $\|\phi(a)\| = \|a\|$, A has to be simple. Therefore A has an irreducible representation that anihilates A and one that does not. □

Lemma 5.6. *Assume A is a C^*-algebra that has only one irreducible representation π up to the unitary equivalence and satisfies at least one of the following conditions. Then A is isomorphic to $K(H)$ for some Hilbert space H.*

(1) *$\pi[A] \cap K(H)$ contains a nonzero vector.*
(2) *There is a projection p in A such that pAp is commutative.*
(3) *A has a maximal commutative subalgebra B such that $\hat{B}$ has an isolated point.*

Proof. (1) Since $\pi[A] \cap K(H)$ is nonempty, $\pi^{-1}[K(H)]$ is a nonzero ideal of A. By Lemma 5.5 (1), A is simple, hence $\pi[A] \subseteq K(H)$ and therefore $\pi[A] = K(H)$.

(2) Note that $\pi(p)$ is a projection of H. By Lemma 5.4, $(\pi \restriction pAp, (\pi(p))[H])$ is an irreducible representation of pAp. By Every irreducible representation of a commutative C^*-algebra is one-dimensional, and therefore $\pi(p)$ is a projection of rank 1. The conclusion follows by (1).

(3) By Lemma 2.1, B is isomorphic with $C_0(\hat{B})$. If $\phi_0 \in \hat{B}$ is an isolated point, then define $p_0 \in C_0(\hat{B})$ by $p_0(\phi_0) = 1$ and $p_0(\psi) = 0$ for $\psi \neq \phi_0$. We claim that p_0 commutes with every $b \in C_0(\hat{B})$: write $b = \lambda p_0 + c$, where $c(\phi_0) = 0$. Then $p_0 b = p_0(\lambda p_0 + c) = \lambda p_0 = (\lambda p_0 + c)p_0$. We can identify p_0 with an element of B, denoted by p. Since p commutes with every $b \in B$, by the maximality of B we have $pAp \subseteq B$. Therefore the conclusion follows by (2). □

Corollary 5.7. *Assume A is a C^*-algebra isomorphic to a closed subspace of $B(H)$ for a Hilbert space H that has an orthonormal basis of cardinality strictly less than the cardinality of the continuum. Then A is isomorphic to $K(H')$ for some H' if and only if all of its irreducible representations are unitarily equivalent.*

Proof. One direction is Naimark's theorem, and the other follows from the proof of Rosenberg's theorem: Let $\kappa < 2^{\aleph_0}$ be the density of H. If B is a maximal commutative subalgebra of A, then by Lemma 5.2 the cardinality of the locally compact space $\hat{B}$ is at most κ, and it therefore has an isolated point. □

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