

THE EXTENDER ALGEBRA AND Σ_1^2 -ABSOLUTENESS

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Abstract. We present a self-contained account of Woodin's extender algebra and its use in proving absoluteness results, including a proof of the Σ_1^2 -absoluteness theorem. We also include a proof that the existence of an inner model with Woodin limit of Woodin cardinals implies the existence of divergent models of AD^+ .

This note provides an introduction to Woodin's extender algebra and a proof (due to Steel and Woodin independently) of Woodin's Σ_1^2 -absoluteness theorem, using the extender algebra, from a large cardinal assumption. Unlike the published accounts of this proof, the present account should be accessible to a set theorist familiar with forcing and basic large cardinals (*e.g.*, [Kun80], [Kan95]). In particular, no familiarity with the inner model theory is required. A more comprehensive account of the extender algebra can also be found in [Ste10, §7.2] and [Nee04, §4] and the reader is invited to consult these excellent sources for more information and other applications. A strengthening of the Σ_1^2 absoluteness in terms of determinacy of games of length ω_1 was proved by Neeman ([Nee07]) also using the extender algebra. In [Lar09], the extender algebra was used to prove the consistency of Woodin's Ω -conjecture.

Organization of the paper. In §1 we introduce infinitary propositional logic $\mathcal{L}_{\delta,\gamma}$. In §2 we review basics of elementary embeddings, Woodin cardinals and extenders. The extender algebra is introduced in §3, where we also prove that it is powerfully δ -cc and introduce iteration trees and the iteration game. A variety of genericity iteration theorems is proved in §4 and their applications to absoluteness are proved in §5. Divergent models of AD^+ are constructed in §6, and §7 has no real content.

Acknowledgments. Over a period of six years this paper evolved from a note not intended for publication to its present form. This subsection is appropriately long for a paper that took six years to write. My interest in the extender algebra was initiated by a conversation with John Steel in

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The definition of the extender algebra and every uncredited result presented here is due to W.H. Woodin, but some of the proofs and formulations, as well as the exposition, are mine. Mistakes, obscurities, and dwelling on trivialities are also all mine.

Notation and terminology. Our notation is mostly standard but inner model theorist may want to take note of two exceptions. First, π usually denotes a transitive collapse instead of its inverse. Second, to an ultrafilter \mathcal{U} we associate quantifier $(\mathcal{U}x)$ so that $(\mathcal{U}x)\phi(x)$ stands for ‘the set of x satisfying $\phi(x)$ belongs to \mathcal{U} .’

All forcing considered here is set forcing.

§1. An infinitary propositional logic. The extender algebra is the Lindenbaum algebra of certain theory in an infinitary propositional logic. The logic is described here and the theory will be described in §3.

1.1. Logic $\mathcal{L}_{\delta,\gamma}$. For regular cardinals $\gamma \leq \delta$ we shall define the infinitary propositional logic $\mathcal{L}_{\delta,\gamma}$. The interesting cases are $\gamma = \omega$ and $\gamma = \delta$, but it will be easier to develop the basic theory in the two cases parallelly. Let $\mathcal{L}_{\delta,\gamma}$ be the propositional logic with γ variables a_ξ , for $\xi < \gamma$, which in addition to the standard propositional connectives $\vee, \wedge, \rightarrow, \leftrightarrow$, and \neg allows infinitary conjunctions of the form $\bigwedge_{\xi < \kappa} \phi_\xi$ and infinitary disjunctions of the form $\bigvee_{\xi < \kappa} \phi_\xi$ for all $\kappa < \delta$.

In addition to the standard axioms and rules of inference for the finitary propositional logic, for each $\kappa < \delta$ and formulas ϕ_ξ , for $\xi < \kappa$ the logic $\mathcal{L}_{\delta,\gamma}$ has axioms $\vdash \bigvee_{\xi < \kappa} \neg \phi_\xi \leftrightarrow \neg \bigwedge_{\xi < \kappa} \phi_\xi$ and $\vdash \bigwedge_{\xi < \kappa} \phi_\xi \rightarrow \phi_\eta$, for every

¹It wasn't.

$\eta < \kappa$, as well as the infinitary rule of inference: from $\vdash \phi_\xi$ for all $\xi < \kappa$ infer $\vdash \bigwedge_{\xi < \kappa} \phi_\xi$. The provability relation for $\mathcal{L}_{\delta,\gamma}$ will be denoted by $\vdash_{\delta,\gamma}$ or simply \vdash if δ and γ are clear from the context. Each proof in $\mathcal{L}_{\delta,\gamma}$ is a well-founded tree and the assertion that ϕ is provable in $\mathcal{L}_{\delta,\gamma}$ is upwards absolute between transitive models of ZFC. Since adding a new bounded subset of δ adds new formulas and new proofs to $\mathcal{L}_{\delta,\gamma}$, it is not obvious that the assertion that ϕ is not provable in $\mathcal{L}_{\delta,\gamma}$ is upwards absolute between transitive models of ZFC. This is, nevertheless, true: see Lemma 1.2.

Every $x \in \mathcal{P}(\gamma)$ naturally defines a model for $\mathcal{L}_{\delta,\gamma}$ via $v_x(a_\xi) = \mathbf{true}$ iff $x(\xi) = 1$ for $\xi < \gamma$. Define $A_\phi = A_{\phi,\delta,\gamma}$ via

$$A_\phi = \{x \in \mathcal{P}(\gamma) : x \models \phi\}.$$

When $\gamma = \omega$ then these are the so-called ∞ -**Borel** sets (see *e.g.*, [Woo10, §9.1]). Note that the sets of the form A_ϕ for $\phi \in \mathcal{L}_{\omega_1,\omega}$ are exactly the Borel sets.² Also note that $x \models \phi$ is absolute between transitive models of ZFC containing x and ϕ .

1.2. Completeness of $\mathcal{L}_{\delta,\gamma}$. The following two lemmas are standard.

- LEMMA 1.1. (1) $\vdash \phi$ implies $\models \phi$ for every formula ϕ in $\mathcal{L}_{\delta,\gamma}$.
(2) $\delta > 2^\gamma$ implies that $\models \phi$ but not $\vdash \phi$ for some formula ϕ in $\mathcal{L}_{\delta,\gamma}$.

PROOF. Clause (1) can be proved by recursion on the rank of the proof.
(2) For $x \subseteq \gamma$ let $\phi_x = \bigvee_{\xi < \gamma} a_\xi^x$, where $a_\xi^x = a_\xi$ if $\xi \in x$ and $a_\xi^x = \neg a_\xi$ if $\xi \notin x$. Then $y \models \phi_x$ if and only if $y = x$, and therefore the formula $\phi = \bigwedge_{x \subseteq \gamma} \neg \phi_x$ is not satisfiable. However, ϕ is satisfiable in every forcing extension in which there exists a new subset of γ . \dashv

LEMMA 1.2. For every ϕ in $\mathcal{L}_{\delta,\gamma}$ the following are equivalent.

- (1) $\vdash \phi$.
- (2) $A_\phi = \mathcal{P}(\gamma)$ in all generic extensions.
- (3) $A_\phi = \mathcal{P}(\gamma)$ in the extension by $\text{Col}(\omega, \kappa)$ for a large enough κ .

PROOF. Clause (1) is upwards absolute and by recursion on the rank of the proof it easily implies (2). Also, (2) trivially implies (3).

Assume (1) fails for ϕ and let κ be the cardinality of the set of all subformulas of ϕ . We may assume a_ξ is a subformula of ϕ if and only if $\xi < \kappa$.

We claim that for any two formulas ψ_1 and ψ_2 in $\mathcal{L}_{\delta,\gamma}$ such that $\psi_1 \not\vdash \phi$ we have either $\psi_1 \wedge \psi_2 \not\vdash \phi$ or $\psi_1 \wedge \neg \psi_2 \not\vdash \phi$. Otherwise we have $\psi_1 \vdash \psi_2 \rightarrow \phi$ and $\psi_1 \vdash \neg \psi_2 \rightarrow \phi$ and therefore $\psi_1 \vdash \phi$, a contradiction.

In the extension by $\text{Col}(\omega, \kappa)$ enumerate all subformulas of ϕ as ψ_n , for $n \in \omega$, and also enumerate $\{a_\xi : \xi < \kappa\}$ as b_n , for $n \in \omega$. Recursively

²Keep in mind that, in spite of the connection with Borel sets, this logic is **not** the ‘usual’ $L_{\omega_1,\omega}$ ([Kei71]). Our $\mathcal{L}_{\omega_1,\omega}$ happens to be the propositional fragment of the latter. This is really an accident since in each of the two notations ‘ ω ’ signifies a different constraint.

pick an increasing sequence of $\mathcal{L}_{\delta,\gamma}$ -theories \mathcal{T}_n , for $n \in \omega$, satisfying the following requirements: (i) $\mathcal{T}_0 = \{\neg\phi\}$ and each \mathcal{T}_n is a finite consistent set of ground-model formulas in $\mathcal{L}_{\delta,\gamma}$. (ii) either $b_n \in \mathcal{T}_n$ or $\neg b_n \in \mathcal{T}_n$. (iii) if ψ_n is of the form $\bigwedge_{\xi < \lambda} \sigma_\xi$ for some λ then either $\psi_n \in \mathcal{T}_n$ or $\neg\sigma_\xi \in \mathcal{T}_n$ for some $\xi < \lambda$. (iv) if ψ_n is of the form $\bigvee_{\xi < \lambda} \sigma_\xi$ for some λ then either $\neg\psi_n \in \mathcal{T}_n$ or $\sigma_\xi \in \mathcal{T}_n$ for some $\xi < \lambda$.

By the above, if \mathcal{T}_n satisfies the requirements then \mathcal{T}_{n+1} as required can be chosen. After all \mathcal{T}_n have been chosen define $x \subseteq \gamma$ by $x = \{\xi : a_\xi \in \bigcup_n \mathcal{T}_n\}$. Then $x \models \neg\phi$ can be proved by recursion on the rank of ϕ , using the fact that every infinitary conjunction and every infinitary disjunction appearing in ϕ is computed correctly. Therefore in this extension (3) fails. \dashv

Recall that the **Lindenbaum algebra** of a $\mathcal{L}_{\delta,\gamma}$ -theory \mathcal{T} is the Boolean algebra of all of equivalence classes of formulas in $\mathcal{L}_{\delta,\gamma}$ with respect to the equivalence relation $\sim_{\mathcal{T}}$ defined by $\phi \sim_{\mathcal{T}} \psi$ if and only if $\mathcal{T} \vdash \phi \leftrightarrow \psi$. We shall denote this algebra by $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ and consider its positive elements as a forcing notion (see Lemma 1.4 below).

LEMMA 1.3. For every $\mathcal{L}_{\delta,\gamma}$ -theory \mathcal{T} such that $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ has the δ -chain condition $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ is a complete Boolean algebra.

PROOF. Immediate, since both $\mathcal{B}_{\delta,\gamma}$ and \mathcal{T} are δ -complete. \dashv

For $x \subseteq \gamma$ such that $x \models \mathcal{T}$ define an ultrafilter of $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ by (here $[\phi]$ stands for the equivalence class of ϕ)

$$\Gamma_x = \{[\phi] \in \mathcal{B}_{\delta,\gamma}/\mathcal{T} : x \models \phi\}.$$

Note that for every generic $\Gamma \subseteq \mathcal{B}_{\delta,\gamma}/\mathcal{T}$ there is the unique $x \subseteq \gamma$ such that $\Gamma_x = \Gamma$, defined as $x = \{\xi : a_\xi \in \Gamma\}$.

LEMMA 1.4. Assume \mathcal{T} is a theory in $\mathcal{L}_{\delta,\gamma}$ and M is a transitive model of a large enough fragment of ZFC such that $\{\delta, \gamma, \mathcal{T}\} \subseteq M$, and $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ has the δ -chain condition in M . Then for every $x \in \mathcal{P}(\gamma)$ we have $x \models \mathcal{T}^M$ if and only if x is $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ -generic over M .

PROOF. If x is $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ -generic over M then a proof by induction on the complexity shows that $x \models \phi$ for all $[\phi] \in \Gamma$, hence $\{\phi \in \mathcal{L}_{\delta,\gamma} : x \models \phi\}$ includes \mathcal{T}^M .

Now assume $x \models \mathcal{T}^M$. Let ϕ_ξ ($\xi < \kappa$) be a maximal antichain of $(\mathcal{B}_{\delta,\gamma}/\mathcal{T})^M$ that belongs to M . By the δ -chain condition $\kappa < \delta$ and therefore $\bigvee_{\xi < \kappa} \phi_\xi$ is in $\mathcal{L}_{\delta,\gamma}^M$. By the maximality of the antichain we have $\mathcal{T}^M \models \bigvee_{\xi < \kappa} \phi_\xi$. Therefore $x \models \bigvee_{\xi < \kappa} \phi_\xi$. This implies $x \models \phi_\xi$ for some $\xi < \kappa$ and Γ_x intersects the given maximal antichain. Since the antichain was arbitrary, we conclude x is generic. \dashv

§2. Elementary embeddings. In the present section we recall some basic facts about elementary embeddings and extenders.

2.1. Extenders I. We only sketch the bare minimum of the theory of extenders. For more details see [MS94] or [Kan95, §26]. An **extender** E is a set that codes an elementary embedding $j_E: \mathbf{V} \rightarrow M$. For every elementary embedding $j: \mathbf{V} \rightarrow M$ such that with $\kappa = \text{crit}(j)$ we have $\kappa < \lambda < j(\kappa)$ there is an extender E in $\mathbf{V}_{\lambda+1}$ such that j_E and j coincide up to \mathbf{V}_λ in the sense that $j(A) \cap \mathbf{V}_\lambda = j_E(A) \cap \mathbf{V}_\lambda$ for all $A \subseteq \mathbf{V}_\kappa$. The model M is constructed as a direct limit of ultrapowers of V and it is denoted by $\text{Ult}(\mathbf{V}, E)$.

All extenders E used in this paper are such that $j_E(\kappa) > \lambda$, where $\kappa = \text{crit}(E)$ and λ is the strength of E . Such extenders are called **short** extenders. Long extenders are needed to describe stronger large cardinal embeddings.

A **generator** of an elementary embedding $j: \mathbf{V} \rightarrow M$ is an ordinal ξ such that there are an inner model N and elementary embeddings i_1 and i_2 such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{i_1} & N \\ & \searrow j & \downarrow i_2 \\ & & M \end{array}$$

commutes and $\text{crit}(i_2) = \xi$. For example, the critical point κ is the least generator and a counting argument shows that an elementary embedding such that $j(\kappa) \geq (2^\kappa)^+$ must have other generators. A **generator** of an extender E is a generator of j_E . The **strength** of E is the largest λ such that $\mathbf{V}_\lambda \subseteq M$, where $j_E: \mathbf{V} \rightarrow M$.

The only properties of extenders used in the present paper are (E1) and (E2).

- (E1) If E is an extender with $\kappa = \text{crit}(j_E)$ then $\text{Ult}(N, E)$ can be formed for every model N such that $(\mathbf{V}_{\kappa+1})^N = \mathbf{V}_{\kappa+1}$. Also, $\text{Ult}(N, E) \supseteq \mathbf{V}_\lambda$, where λ is the strength of E and whether or not ξ is a generator of E depends only on E and $\mathbf{V}_{\kappa+1}$.

For a proof of (E1) see [MS94, Lemma 1.5], or note that this is immediate from Definition 2.3 below since all ultrafilters E_s concentrate on $[\kappa]^{<\omega}$. While (E1) is a property of all extenders, (E2) below is not. However, we shall consider only the extenders satisfying (E2).

- (E2) The strength of E is greater than the supremum of the generators of E .

It is important to note that the strength depends only on E , and not on the model to which E was applied. In particular, if E is an extender in M

with $\kappa = \text{crit}(j_E)$, λ is the strength of E , and $M \cap \mathbf{V}_{\kappa+1} = N \cap \mathbf{V}_{\kappa+1}$ then we have $\text{Ult}(M, E) \cap \mathbf{V}_\lambda = \text{Ult}(N, E) \cap \mathbf{V}_\lambda$.

2.2. Woodin cardinals. If A is a set such that $j: \mathbf{V} \rightarrow M$ satisfies $\mathbf{V}_\lambda = (\mathbf{V}_\lambda)^M$ and $j(A) \cap \mathbf{V}_\lambda = A \cap \mathbf{V}_\lambda$ then we say j is an A, λ -**strong** embedding. This may differ from the standard definition of a λ -strong embedding but I will consider only λ such that $\kappa + \lambda = \lambda$ for $\kappa = \text{crit}(j)$, in which case there is no difference. If j_E is an A, λ -strong then we say E is A, λ -strong. A cardinal δ is a **Woodin cardinal** if for every $A \subseteq \mathbf{V}_\delta$ there is $\kappa < \delta$ such that there are A, λ -strong elementary embeddings with critical point κ for an arbitrarily large $\lambda < \delta$. We say that A **reflects** to κ . Note that the Woodinness of δ is witnessed by the extenders in \mathbf{V}_δ , and therefore δ is Woodin in V if and only if it is Woodin in $\mathbf{L}(\mathbf{V}_\delta)$. Moreover, it suffices to consider only the extenders that satisfy property (E2).

A cardinal κ is λ -**strong** if it is a critical point of an \emptyset, λ -strong elementary embedding $j: \mathbf{V} \rightarrow M$. In particular every Woodin cardinal δ is a limit of cardinals each of which is λ -strong for all $\lambda < \delta$.

2.3. Extenders II. A reader not interested in extenders per se may want to skip the rest of this section on the first reading and take (E1) and (E2) for granted. The actual definition of an extender is, strictly speaking, not necessary for our present purpose. However, this notion is central³ in the theory and we include it for the reader's convenience. Every extender is of the form as described in Example 2.1.

EXAMPLE 2.1. Assume $j: \mathbf{V} \rightarrow M$ is an elementary embedding with $\text{crit}(j) = \kappa$. Fix λ such that $\kappa \leq \lambda < j(\kappa)$. Typically, we take λ such that $M \supseteq \mathbf{V}_\lambda$. For $s \in [\lambda]^{<\omega}$ define $E_s \subseteq [\kappa]^m$ (where $m = |s|$) by

$$X \in E_s \text{ if and only if } s \in j(X).$$

Then $E(j, \lambda) = \langle E_s : s \in [\lambda]^{<\omega} \rangle$ is a (κ, λ) -extender. Ordinal λ is called the **length** of the extender E and is denoted by $\text{lh}(E)$. In all of our applications $\text{lh}(E)$ will equal the strength of E .

The following facts can be found *e.g.*, in [Ste10]. For an extender E we write

$$\kappa_E = \text{crit}(E)$$

(the **critical point** of E , *i.e.*, the least ordinal moved by j_E) and

$$\lambda_E = \sup\{\eta : \mathbf{V}_\eta \subseteq (\mathbf{V}_\eta)^M\}$$

(the **strength** of E). Note that if $j: \mathbf{V} \rightarrow M$ is an elementary embedding such that $\text{crit}(j) = \kappa$, the strength of j is λ , and λ is a strong limit cardinal, then the (κ, λ) extender E defined in Example 2.1 satisfies $\text{Ult}(\mathbf{V}, E) \cap \mathbf{V}_\lambda = M \cap \mathbf{V}_\lambda$ and $j(A) \cap \mathbf{V}_\lambda = j_E(A) \cap \mathbf{V}_\lambda$ for all A and is therefore λ -strong.

³after all, it is the **extender algebra**

The assumption that λ is strong limit is needed to code subsets of $\mathcal{P}(\alpha)$ by sets of ordinals for every $\alpha < \lambda$.

Fix a cardinal κ . For $m \in \omega$ and $s \subseteq m$ with $|s| = n$ consider the projection map $\pi = \pi_{m,s}: [\kappa]^m \rightarrow [\kappa]^s$ defined by

$$\pi(\langle \xi_i : i < m \rangle) = \langle \xi_i : i \in s \rangle.$$

More generally, if $s \subseteq t$ are finite sets of ordinals (listed in the increasing order) then the projection $\pi = \pi_{t,s}: [\kappa]^t \rightarrow [\kappa]^s$ is defined by

$$\pi(\langle \xi_i : i \in t \rangle) = \langle \xi_i : i \in s \rangle.$$

Recall that if \mathcal{U} and \mathcal{V} are ultrafilters on sets I and J , respectively, then we write $\mathcal{U} \leq_{\text{RK}} \mathcal{V}$ if and only if there is $h: J \rightarrow I$ such that

$$X \in \mathcal{U} \text{ if and only if } h^{-1}(X) \in \mathcal{V}.$$

In this situation we say \mathcal{U} is **Rudin–Keisler reducible to** \mathcal{V} and that h is the **Rudin–Keisler reduction of \mathcal{U} to \mathcal{V}** .

Respecting the notation commonly accepted in the theory of large cardinals, we denote an ultrapower of a structure M associated to an ultrafilter \mathcal{U} by $\text{Ult}(M, \mathcal{U})$. Its elements are the equivalence classes of $f \in M^I \cap M$,

$$[f]_{\mathcal{U}} = \{g \in M^I \cap M : (\mathcal{U}i)f(i) = g(i)\}$$

and the membership relation is defined by $[f]_{\mathcal{U}} \in [g]_{\mathcal{U}}$ if and only if $(\mathcal{U}i)f(i) \in g(i)$.

If \mathcal{U} is \aleph_1 -complete then $\text{Ult}(M, \mathcal{U})$ is well-founded whenever M is well-founded, and we identify $\text{Ult}(M, \mathcal{U})$ with its transitive collapse.

Assume \mathcal{U} and \mathcal{V} are ultrafilters on index-sets I and J , respectively, and $\mathcal{U} \leq_{\text{RK}} \mathcal{V}$ is witnessed by a reduction $h: J \rightarrow I$. Then for any structure M we can define a map $j_h: M^I/\mathcal{U} \rightarrow M^J/\mathcal{V}$ by

$$j_h([f]_{\mathcal{U}}) = [f \circ h]_{\mathcal{V}}$$

In the following lemma M is any structure, \mathcal{U} and \mathcal{V} are arbitrary ultrafilters, and the proof is straightforward.

LEMMA 2.2. If $\mathcal{U} \leq_{\text{RK}} \mathcal{V}$, h is the Rudin–Keisler reduction, and j_h is defined as above, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{j_{\mathcal{U}}} & \text{Ult}(M, \mathcal{U}) \\ & \searrow j_{\mathcal{V}} & \downarrow j_h \\ & & \text{Ult}(M, \mathcal{V}) \end{array}$$

commutes and j_h is an elementary embedding. \dashv

For a finite set s and $i < |s|$ let s_i denote its i -th element. As common in set theory, we start counting at 0.

DEFINITION 2.3. Assume $\kappa < \lambda$ are uncountable cardinals. A (κ, λ) -**extender** is $E: [\lambda]^{<\omega} \rightarrow \mathbf{V}_{\kappa+2}$ such that for all s and t in $[\lambda]^{<\omega}$ we have

- (a) E_s is a nonprincipal κ -complete ultrafilter on $[\kappa]^{|s|}$.
- (b) If $s \subseteq t$ then $\pi_{t,s}$ is a Rudin–Keisler reduction of E_s to E_t .
- (c) **Normality:** If $s \in [\lambda]^{<\omega}$, $i < |s|$, and $f: [\kappa]^{|s|} \rightarrow \kappa$ is such that $f(u) < u_i$ for E_s many u , then there exist $\xi < s_i$ and j such that $f \circ \pi_{a \cup \{\xi\}, a}(u) = u_j$ for $E_{s \cup \{\xi\}}$ many u .
- (d) **Countable completeness:** if $s(n) \in [\lambda]^{<\omega}$ and $X(n) \in E_{s(n)}$ for all $n < \omega$ then there is increasing $h: \bigcup_n s(n) \rightarrow \kappa$ such that $h'' s(n) \in X(n)$ for all n .

Note that, with the above notation for a (κ, λ) -extender, we have $\kappa_E = \kappa$ but not necessarily $\lambda_E = \lambda$. However, the length and strength of each extender in all of our applications will coincide.

If E is an extender, then the models

$$M_s = \text{Ult}(\mathbf{V}, E_s)$$

are, by the κ -completeness of E_s , well-founded. By Lemma 2.2, they form a directed system under the embeddings (writing $j_{t,s}$ for $j_{\pi_{t,s}}$)

$$j_{t,s}: M_t \rightarrow M_s.$$

The direct limit of this system will be denoted by $\text{Ult}(\mathbf{V}, E)$ and identified with its transitive collapse if it is well-founded.

It is not difficult to show that if E satisfies (a), (b) and (c) of Definition 2.3 then the corresponding ultrapower $\text{Ult}(\mathbf{V}, E)$ is well-founded if and only if E satisfies (d) as well.

If E is a (κ, λ) -extender, let M be a model to which E can be applied and let $\kappa < \xi \leq \lambda$. One defines $E \upharpoonright \xi = \langle E_s : s \in [\xi]^{<\omega} \rangle$. Like in Lemma 2.2 one can define an elementary embedding i such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{j_{E \upharpoonright \xi}} & \text{Ult}(M, E \upharpoonright \xi) \\ & \searrow j_E & \downarrow i \\ & & \text{Ult}(M, E) \end{array}$$

commutes. If $\text{crit}(i) = \xi$ then we say ξ is a **generator** of E .

LEMMA 2.4. If $j: \mathbf{V} \rightarrow M$ is a λ -strong embedding with $\text{crit}(j) = \kappa$. Let $E = E(j, \lambda)$ be as in Example 2.1. Then

- (1) E satisfies (E1) and (E2).
- (2) For every $X \subseteq \kappa$ we have $j(X) \cap \lambda = j_E(X) \cap \lambda$.

In particular, if δ is a Woodin cardinal then there is a family $\vec{E} \subseteq \mathbf{V}_\delta$ of extenders satisfying (E1) and (E2) such that δ is Woodin in $\mathbf{L}[\vec{E}]$.

PROOF. Clause (E1) is a consequence of the fact that $\text{Ult}(N, E)$ depends only on $\mathbf{V}_{\kappa+1} \cap N$. Clearly every generator of a (κ, λ) -extender is $\leq \lambda$, and therefore (E2) follows. \dashv

Lemma 2.5 below will not be needed elsewhere in the present paper. It is included only as an illustration that the countable completeness of the extenders is a necessary requirement for wellfoundedness of the ultrapower.

LEMMA 2.5. Assume κ is a measurable cardinal. Then there is $\lambda > \kappa$ and $E: [\lambda]^{<\omega} \rightarrow \mathbf{V}_{\kappa+2}$ such that

- (a) E_s is a nonprincipal κ -complete ultrafilter on κ^s ,
- (b) If $s \subseteq t$ then $\pi_{t,s}$ is a Rudin–Keisler reduction of E_s to E_t

such that the ultrapower $\prod_E V$ is ill-founded.

PROOF. Let $j: \mathbf{V} \rightarrow M$ be an elementary embedding with $\text{crit}(j) = \kappa$. Let $\lambda = \sup\{n \in \omega : j^n(\kappa)\}$. For all X , n and k we have the following.

$$j^{n+k}(X) \cap j^k(\kappa) = j^k(j^n(X) \cap \kappa) = j^k(X \cap \kappa) = j^k(X) \cap j^k(\kappa).$$

Because of this for a finite $s \subseteq \lambda$ the following defines a subset E_s of $\kappa^{|s|}$:

$$X \in E_s \text{ if and only if } s \in j^n(X)$$

where n is such that $\max(s) < j^n(\kappa)$. Then E_s is clearly a κ -complete ultrafilter.

The map $E, [\lambda]^{<\omega} \ni s \mapsto E_s \in \mathbf{V}_{\kappa+1}$ satisfies (E1) and (E2). We can therefore form the direct limit of the ultrapowers $\prod_{E_s} V$, $s \in [\lambda]^{<\omega}$. However, this ultrapower is not well-founded. Let $s(n) = \{j^i(\kappa) : i < n\}$ and $M_n = \text{Ult}(\mathbf{V}, E_{s(n)})$. Then $j_{s(n), s(n+1)}(\kappa) > \kappa$, and therefore in the direct limit we have a decreasing ω -sequence of ordinals. \dashv

The problem with E defined in Lemma 2.5 is that the ultrapowers are iterated the ‘wrong way.’ Let us consider this example a little more closely. If $s(n) = \{j^m(\kappa) : m < n\}$ then $E_{s(n)}$ is the set of all $X \subseteq \kappa^n$ such that

$$(\mathcal{U}\xi_0)(\mathcal{U}\xi_1) \dots (\mathcal{U}\xi_{n-1})(\xi_{n-1}, \dots, \xi_1, \xi_0) \in X$$

Then the set of all decreasing n -tuples of ordinals $< \kappa$ belongs to $E_{s(n)}$ for each n .

§3. The extender algebra. Assume \vec{E} is a family of extenders in $\mathbf{V}_{\delta+2}$. Typically, \vec{E} will be a subset of \mathbf{V}_δ and it will witness that δ is a Woodin cardinal. Let $\mathcal{T}_{\delta, \gamma}(\vec{E})$ be the deductive closure in $\mathcal{L}_{\delta, \gamma}$ of all sentences of the form

$$\Psi(\vec{\phi}, \kappa, \lambda): \quad \bigvee_{\xi < \kappa} \phi_\xi \leftrightarrow \bigvee_{\xi < \lambda} \phi_\xi$$

for a sequence $\vec{\phi} = \langle \phi_\xi \mid \xi < \delta \rangle$ in $\mathcal{L}_{\delta,\gamma}$ such that $\phi_\xi \in \mathbf{V}_\kappa$ for all $\xi < \kappa$, ordinals $\kappa < \lambda$, and extender \vec{E} with $\text{crit}(E) = \kappa$ that is $\vec{\phi}, \lambda$ -strong.⁴

LEMMA 3.1. If $E \in \vec{E}$ and $\text{crit}(E) = \kappa$ then for every $f: \kappa \rightarrow \mathcal{L}_{\delta,\gamma} \cap \mathbf{V}_\kappa$ we have $\mathcal{L}_{\delta,\gamma} \vdash \bigvee_{\xi < \kappa} f(\xi) \rightarrow j_E(f)(\kappa)$.

PROOF. If λ is the strength of E then with $\vec{\phi} = \langle j_E(f)(\xi) : \xi < \lambda \rangle$ we have that E is $\vec{\phi}, \lambda$ -strong and $\Psi(\vec{\phi}, \kappa, \lambda)$ implies the above formula. \dashv

In the following it may be worth emphasizing that x is assumed to belong to the same inner model as \vec{E} (cf. Theorem 4.1).

LEMMA 3.2. For every real x in $\mathbf{L}[\vec{E}]$ we have $x \models \mathcal{T}_{\delta,\omega}(\vec{E})$. In particular, $\mathcal{T}_{\delta,\omega}(\vec{E})$ is a consistent theory.

PROOF. Fix a sequence $\vec{\phi}$ that reflects to κ and this is witnessed by extenders in \vec{E} . We need to check $\Psi(\vec{\phi}, \kappa, \lambda)$ for all $\lambda > \kappa$. We may assume $x \models \phi_\xi$ for some ξ , since otherwise $\Psi(\vec{\phi}, \kappa, \lambda)$ vacuously holds for all λ . Pick an extender $E \in \vec{E}$ such that j_E is $(\vec{\phi}, \lambda)$ -strong for some $\lambda > \xi$. Since x is a real it is not moved by any j and therefore by elementarity we have $x \models \phi_\eta$ for some $\eta < \kappa$ and the conclusion follows. \dashv

Note that if a cardinal γ is δ -strong (or equivalently, in the terminology introduced above, if δ reflects to γ) and this is witnessed by extenders in \vec{E} then $\gamma \not\models \mathcal{T}_{\delta,\gamma}(\vec{E})$, as can be seen by taking $\phi_\xi(x)$ to be a_ξ if $\xi \in x$ and $\neg a_\xi$ if $\xi \notin x$. However, if $\text{crit}(E) = \gamma$ then by the minimality of γ and elementarity we have $\gamma \models j_E(\mathcal{T}_{\delta,\gamma}(\vec{E}))$.

The **extender algebra** with γ generators corresponding to \vec{E} is the algebra

$$\mathcal{W}_{\delta,\gamma}(\vec{E}) = \mathcal{B}_{\delta,\gamma} / \mathcal{T}_{\delta,\gamma}(\vec{E}).$$

Most important instances of $\mathcal{W}_{\delta,\gamma}(\vec{E})$ are given by $\gamma = \omega$ and $\gamma = \delta$ but for convenience we develop the theory of $\mathcal{W}_{\delta,\gamma}(\vec{E})$ for an arbitrary $\gamma \leq \delta$.

LEMMA 3.3. If δ is a Woodin cardinal and \vec{E} is a system of extenders witnessing its Woodinness, then $\mathcal{W}_{\delta,\gamma}$ has the δ -chain condition and is therefore complete.

PROOF. Pick a sequence $\vec{\phi} = \{\phi_\xi : \xi < \delta\}$ in $\mathcal{L}_{\delta,\gamma}$. We want to prove that $\{[\phi_\xi] : \xi < \delta\}$ is not an antichain in $\mathcal{W}_{\delta,\gamma}$. The set

$$\mathbf{C} = \{\kappa < \delta : (\forall \xi < \kappa) \phi_\xi \in \mathbf{V}_\kappa\}$$

is a club in δ . Using some reasonable coding of pairs in \mathbf{V}_δ by elements of \mathbf{V}_δ find $X \subseteq \mathbf{V}_\delta$ that codes the pair $\mathbf{C}, \vec{\phi}$. Since δ is Woodin there exists an

⁴Early versions of this paper contained a nonstandard definition of theory $\mathcal{T}_{\delta,\gamma}(\vec{E})$. I have decided to adopt the standard, more flexible, definition.

X , λ -strong extender $E \in \vec{E}$ such that with $\kappa = \text{crit}(E)$ the set $[\kappa, \lambda) \cap \mathbf{C}$ is nonempty. By the elementarity of j_E this implies $\kappa \in \mathbf{C}$. Therefore Lemma 3.1 applies, and $[\phi_\kappa]$ is compatible with $[\phi_\xi]$ for some $\xi < \kappa$.

The completeness of $\mathcal{W}_{\delta, \gamma}$ now follows by Lemma 1.3. \dashv

A partial converse of Lemma 3.3, that if $\mathcal{W}_{\delta, \gamma}(\vec{E})$ has δ -cc and $\mathbf{V}_\delta^\#$ exists then δ is Woodin, was proved in [KZ06] and independently by Woodin. It is not known whether the converse of Lemma 3.3 is true.

The following lemma, which will play an important role in §6, was proved in [Hjo97B, Lemma 3.6] in the case when $\lambda = 2$. The general case requires no new ideas.

LEMMA 3.4 (Hjorth). if $\lambda < \delta$ then product of λ copies of $\mathcal{W}_{\delta, \gamma}$ has the δ -chain condition.

PROOF. Assume the contrary and let $A = \{ \langle [\phi_\xi(i)] : i < \lambda \rangle \mid \xi < \delta \}$ be an antichain in $(\mathcal{W}_{\delta, \gamma})^\lambda$. By argument as in the proof of Lemma 3.3 we can find κ such that $\phi_\xi(i) \in \mathbf{V}_\kappa$ for all i and all $\xi < \kappa$ and there is an A , λ -strong extender E with critical point κ ($\lambda = \kappa + 1$ suffices).

Then $\mathcal{U} = \{ X \subseteq \kappa : j_E(X) \ni \kappa \}$ is a normal κ -complete ultrafilter on κ . If there were $\xi < \eta < \kappa$ such that $[\phi_\xi(i)]$ and $[\phi_\eta(i)]$ are compatible for all $i < \lambda$, then A would not be an antichain.

Consider the map $f: [\kappa]^2 \rightarrow \lambda$ defined by

$$f(\{\xi, \eta\}) = \min\{i < \lambda : [\phi_\xi(i)] \text{ and } [\phi_\eta(i)] \text{ are incompatible}\}.$$

Since \mathcal{U} is a normal ultrafilter we have $\mathcal{U} \rightarrow (\mathcal{U})_\lambda^2$ and there are $X \in \mathcal{U}$ and $i < \lambda$ such that f has constant value i on $[X]^2$. Let

$$\psi_\xi = \begin{cases} \phi_\xi(i), & \text{if } \xi \in X \text{ or } \xi \geq \kappa \\ \perp, & \text{if } \xi \notin X \text{ and } \xi < \kappa \end{cases}$$

Then $\psi_\kappa = j(\vec{\psi})_\kappa$. Los's theorem implies that $[\psi_\kappa]$ is incompatible with $[\psi_\xi]$ for all $\xi < \kappa$. However, $\psi_\kappa = j(\vec{\psi})_\kappa$ and this contradicts Lemma 3.1. \dashv

The following curious fact (not needed in proofs of the main results of this note) was pointed out to me by Paul Larson.

LEMMA 3.5. Assume γ is any cardinal less than the least critical point of each extender in \vec{E} and \mathbb{P} is a forcing notion of cardinality γ . Then in $\mathcal{W}_{\delta, \gamma}(\vec{E})$ there is a condition p that forces that \mathbb{P} is a regular subordering of $\mathcal{W}_{\delta, \gamma}(\vec{E})$.

THE FIRST PROOF OF LEMMA 3.5. Pick a bijection between \mathbb{P} and γ and hence identify \mathbb{P} with $\langle \gamma, \leq_{\mathbb{P}} \rangle$ for some partial ordering $\leq_{\mathbb{P}}$ on γ . Let the sentence ϕ be the conjunction of axioms expressing the following.

- (a) The order on \mathbb{P} : $a_\xi \rightarrow a_\eta$ whenever $\xi \leq_{\mathbb{P}} \eta$,

- (b) The incompatibility relation on \mathbb{P} : $\bigwedge_{\zeta < \gamma} (\neg(a_\zeta \rightarrow a_\eta) \vee \neg(a_\zeta \rightarrow a_\xi))$, if $\xi \perp_{\mathbb{P}} \zeta$.
- (c) For every maximal antichain \mathcal{A} of \mathbb{P} , $\bigvee_{\xi \in \mathcal{A}} a_\xi$.

Then ϕ is in $\mathcal{L}_{\delta, \gamma}$ and since its size is below the least critical point of an extender in \vec{E} , it is consistent with $\mathcal{T}_{\delta, \gamma}(\vec{E})$. Therefore we may take p to be (the equivalence class of) ϕ . \dashv

We shall give another proof of Lemma 3.5 after Theorem 4.2.

3.1. Iteration trees. We say that $T = (\zeta, \leq_T)$ is a **tree order** on an ordinal ζ if

- (1) $T = (\zeta, \leq_T)$ is a tree with root 0,
- (2) \leq_T is coarser than \leq ,
- (3) every successor ordinal is a successor in \leq_T ,
- (4) if $\xi \leq \zeta$ is a limit ordinal then the set of \leq_T -predecessors of ξ is cofinal in ξ .

Consider a transitive model M of a large enough fragment of ZFC and a system \vec{E} of extenders in M . We allow M to be a proper class. As a matter of fact, it is typically going to be a proper class. Nevertheless, we omit the (straightforward) nuisances involved in formalization of the notion of an iteration tree in ZFC.

An \vec{E} -**iteration tree** is a structure consisting of $\langle T, M_\eta, E_\xi \mid \eta \leq \zeta, \xi < \zeta \rangle$, together with a commuting system of elementary embeddings $j_{\xi\eta}: M_\xi \rightarrow M_\eta$ for $\xi \leq_T \eta$ such that

- (5) \leq_T is a tree order on ζ ,
- (6) E_ξ is an extender in \vec{E}^{M_ξ} ,
- (7) If $\kappa = \text{crit}(E_\xi)$ then the immediate \leq_T -predecessor of $\xi + 1$ is the least ordinal η such that $M_\eta \cap \mathbf{V}_{\kappa+1} = M_\xi \cap \mathbf{V}_{\kappa+1}$,
- (8) If $\xi + 1$ is the immediate \leq_T successor of η then $M_{\xi+1} = \text{Ult}(M_\eta, E_\xi)$, hence $j_{\eta, \xi+1} = j_{E_\xi}$ as computed with respect to M_η ,
- (9) If ξ is a limit ordinal then M_ξ is the direct limit of M_η , for $\eta <_T \xi$, and
- (10) each M_ξ is well-founded.
- (11) If $\xi < \eta$ and ζ is the length of E_ξ then $M_\xi \cap \mathbf{V}_\zeta = M_\eta \cap \mathbf{V}_\zeta$.

It is usually not required that an iteration tree satisfies condition (7), and the iteration trees satisfying this condition are called **normal** iteration trees. A straightforward induction shows that condition (11) is a consequence of the previous conditions.

If E_ξ was always applied to M_ξ , then we would have a **linear** iteration that is moreover **internal**—*i.e.*, each extender used in the construction belongs to the model to which it is applied. The wellfoundedness of such an iteration follows from a rather mild additional condition about the extenders. On the other hand, the choice of condition (7) is behind the

power of the iteration trees (see the proof of Theorem 4.1). This condition also prevents the obstacle to wellfoundedness of the iteration exposed in Lemma 2.5 (see Lemma 3.6). It will be important that the extenders in \vec{E} have the property (E2) (see §2.1), that for every generator ξ of E we have $\text{Ult}(\mathbf{V}, E) \cap \mathbf{V}_\xi = \mathbf{V}_\xi$, or in other words, that every extender is ξ -strong for each of its generators ξ .

LEMMA 3.6. Assume $\langle T, M_\eta, E_\xi \mid \eta \leq \zeta, \xi < \zeta \rangle$, is an iteration tree such that every extender used in its construction satisfies (E2). Assume E_0 and E_1 are extenders used along the same branch of T and E_1 was used after E_0 . Then $\kappa = \text{crit}(E_1)$ is greater than the supremum of all generators of E_0 .

PROOF. Assume the contrary. Therefore κ is less or equal than some generator of E_0 , and (E2) implies that κ is smaller than the strength of E_0 . Let us first consider the case when E_1 was applied to $M_\beta = \text{Ult}(M_\alpha, E_0)$ for some α along the branch. If ξ is the strength of E_0 then we have $M_\beta \cap \mathbf{V}_\xi = M_\alpha \cap \mathbf{V}_\xi$. Therefore, since $\kappa < \xi$ and E_1 could be applied to M_β , it could be applied to M_α as well. This contradicts our assumption that E_1 was applied to M_β .

We may therefore assume E_1 was applied to a model M_γ that is a direct limit of other models on the branch, including $\text{Ult}(M_\alpha, E_0)$ for some α . The above and an induction argument show that $\text{crit}(j_{E_1})$ is greater than any generator of any extender used in the construction of this branch, including the generators of E_0 . \dashv

Note that an iteration tree $\langle T, M_\eta, E_\xi \mid \eta \leq \omega_1, \xi < \omega_1 \rangle$ has a branch of length ω_1 by (4). In Lemma 3.7 below, and elsewhere, we assume that all critical points of elementary embeddings $j_{\xi\eta}$ used in building an iteration tree are countable ordinals. Lemma 3.7 is an attempt to extract one of the key ideas from the proof of Theorem 4.1. It is part of the proof of the comparison lemma for mice, which at the level of generality we are dealing in now, is due to Martin and Steel [MS94]. Preprint [Schundated] was very helpful during the extraction of this lemma.

LEMMA 3.7. Assume $\langle T, M_\eta, E_\xi \mid \eta \leq \omega_1, \xi < \omega_1 \rangle$ is an iteration tree and $b \subseteq \omega_1$ is its cofinal branch. Assume $H \prec H_{(2^{\aleph_1})^+}$ is countable and it contains the iteration tree. Let \bar{H} be its transitive collapse and let $\pi: H \rightarrow \bar{H}$ be the collapsing map.

With $\alpha = H \cap \omega_1$ we have the following.

- (12) $\alpha \in b$ and M_α is the direct limit of M_ξ , $\xi \in b \cap \alpha$.
- (13) $M_{\omega_1} \cap H$ is the direct limit of $\langle M_\xi \cap H, \xi \in b \cap \alpha \rangle$,
- (14) π^{-1} and $j_{\alpha\omega_1}$ agree on $M_\alpha \cap \bar{H}$ and in particular
 - (a) $\pi^{-1}[M_\alpha \cap \bar{H}]$ is included in $M_{\omega_1} \cap H$, and
 - (b) $\text{crit}(j_{\alpha\omega_1}) = \alpha$,

- (15) If γ is the strength of the extender $E_{\alpha'}$ such that $\text{Ult}(M_\alpha, E_{\alpha'})$ is the successor of M_α along b then $j_{\alpha\omega_1} = j_{E_{\alpha'}} \circ i$ for an elementary embedding i such that $i \upharpoonright \mathbf{V}_\gamma$ is the identity,
- (16) $M_{\omega_1} \cap \mathcal{P}(\alpha + 1) = M_\alpha \cap \mathcal{P}(\alpha + 1)$.

PROOF. Clause (12) follows by (4) and (9). By elementarity in H it holds that M_{ω_1} is the direct limit of M_ξ , for $\xi \in b$, and therefore (13) follows. By applying this and (12), (14) follows as well. Let γ be as in (15). We have $j_{\alpha\omega_1} = j_{E_{\alpha'}} \circ i$ for some i . By Lemma 3.6 we have $\text{crit}(i) \geq \gamma$ and (15) follows.

Clause (16) is a consequence of (14). \dashv

3.2. The iteration game. In what follows elements of an iteration tree will be proper classes instead of sets. The arguments can be formalized within ZFC by using the standard reflection and compactness devices. We leave out the well-known details. We define a two-player game of transfinite length ζ in which the players build an iteration tree, starting from a model M and a system of extenders \vec{E} in M . Let $M_0 = M$. In his α -th move player I picks an extender E_α in \vec{E}^{M_α} such that the strength of E_α is greater than the strength of E_β for all $\beta < \alpha$. Then the referee finds the minimal $\beta \leq \alpha$ such that (writing $\text{crit}(E)$ for the critical point of the elementary embedding j_E) $\mathbf{V}_{\text{crit}(E_\alpha)+1} \cap M_\beta = \mathbf{V}_{\text{crit}(E_\alpha)+1} \cap M_\alpha$. Hence M_β is the earliest model in the iteration to which E_α can be applied. Referee then defines

$$M_{\alpha+1} = \text{Ult}(M_\beta, E_\alpha),$$

with $j_{\beta\alpha+1}$ being the corresponding embedding. The referee also extends the tree order T by adding $\alpha + 1$ as an immediate successor to β . At a limit stage α player II picks a maximal branch $\langle M_\xi \mid \xi \in b \rangle$ of T such that b is cofinal in α and lets M_α be the direct limit of the system $\langle M_\xi, j_{\xi,\eta} \mid \xi < \eta \in b \rangle$. If M_α is well-founded then we identify it with its transitive collapse.

The first player who disobeys the rules loses. Assume both players obeyed the rules of the iteration game. If M_α is ill-founded then the game is over and I wins. If all M_α are well-founded, then player II wins, and otherwise player I wins.

For definiteness, we call the above game the (\vec{E}, ζ) -iteration game in M . We shall suppress \vec{E} and M whenever they are clear from the context. An (\vec{E}, ζ) -iteration strategy is a winning strategy for player II in the iteration game of length ζ . A pair (M, \vec{E}) is (\vec{E}, ζ) -iterable if player II has a \vec{E}, ζ -winning strategy. An (\vec{E}, ζ) -iteration of M is an elementary embedding $j_{0\zeta} : M \rightarrow M_\zeta$ extracted from an \vec{E} -iteration tree on ζ . A model is **fully iterable** if it is (\vec{E}, ζ) -iterable for every ordinal ζ .

Universally Baire sets of reals were introduced in [FMW92] and are defined in §6.1.

THEOREM 3.8 (Martin–Steel, [MS94]). Assume there exist n Woodin cardinals and a measurable above them all. For every $a \in \mathbb{R}$ and every $m \leq n$ there exists an inner model containing a and m Woodin cardinals, denoted by $M_m(a)$. It is $(\omega_1 + 1)$ -iterable and its Woodin cardinals are countable ordinals in V .

If there are class many Woodin cardinals then every $M_n(a)$ is fully iterable in every forcing extension and the iteration strategy is coded by a universally Baire set of reals. \dashv

The assumption that there are class many Woodin cardinals is not optimal. For the case when $n = 1$ see the proof of Theorem 5.6 below.

The following lemma will be needed in §6.2.

LEMMA 3.9. Assume Σ is an ω_1 -iteration strategy that is universally Baire and moreover that all sets projective in Σ are universally Baire. Then Σ can always be extended to a full iteration strategy.

PROOF. Since the statement ‘ Σ is a winning iteration strategy’ is projective in Σ it is forcing-absolute. One constructs an iteration strategy $\bar{\Sigma}$ for player II by induction on limit ordinals $\alpha \geq \omega_1$. The recursive hypothesis is that every play of the iteration game in which player II obeyed $\bar{\Sigma}$ has the following property. If κ is the cardinality of the resulting iteration tree, then in the extension by Levy collapse $\text{Col}(\omega, \kappa)$ of κ to ω tree T is the result of a play of an iteration game in which player II has obeyed Σ .

Let T be an iteration tree of height α resulting from an iteration game in which player II has obeyed the extension of Σ constructed so far. Go to a forcing extension by Levy collapse $\text{Col}(\omega, |\alpha|)$, and let \mathbf{b} be the α -branch chosen by Σ .

We claim that for every $\beta < \alpha$ the condition whether $\beta \in \mathbf{b}$ is decided by the maximal condition. Otherwise, fix β and choose p_1 and p_2 such that $p_1 \Vdash \check{\beta} \in \check{\mathbf{b}}$ and $p_2 \Vdash \check{\beta} \notin \check{\mathbf{b}}$. Let $G_1 \subseteq \text{Col}(\omega, \alpha)$ be a generic filter such that $p_1 \in G_1$. By using the homogeneity of $\text{Col}(\omega, \alpha)$ in $\mathbf{V}[G_1]$ we can choose a generic filter G_2 such that $p_2 \in G_2$ and $\mathbf{V}[G_1] = \mathbf{V}[G_2]$. Therefore in $\mathbf{V}[G_1] = \mathbf{V}[G_2]$ we have that Σ does not choose a unique α -branch of T , a contradiction.

Therefore branch \mathbf{b} is decided in the ground model. We can therefore extend strategy $\bar{\Sigma}$ by having player II choose \mathbf{b} at the α -th stage. Since well-foundedness is absolute for forcing extensions, this defines a winning $(\alpha + 1)$ -iteration strategy. \dashv

§4. Genericity iterations. In the present section we formulate and prove results that make the extender algebra unique.

Assuming M is sufficiently iterable, we may talk about iteration strategies for player I. These are the strategies that, when played against player

II's winning strategy, produce models (necessarily well-founded) with desirable properties.

THEOREM 4.1. Assume (M, \vec{E}) is $(\omega_1 + 1)$ -iterable and \vec{E} witnesses a countable ordinal δ is a Woodin cardinal in M . Then for every $x \subseteq \omega$ there is a (well-founded) countable iteration $j: M \rightarrow M^*$ such that x is $j(\mathcal{W}_{\delta, \omega}(\vec{E}))$ -generic over M^* .

PROOF. By Lemma 3.3 and Lemma 1.4 we only need to assure $x \models j(\mathcal{T}_{\delta, \omega}(\vec{E}))$. Define a strategy for player I for building an \vec{E} -iteration tree with $M_0 = M$ as follows. Assume $\langle T, M_\xi, E_\xi \mid \xi \leq \alpha \rangle$ has been constructed. If $x \models j_{0\alpha}(\mathcal{T}_{\delta, \omega}(\vec{E}))$ then $j_{0\alpha}: M \rightarrow M_\alpha$ is the required iteration and we stop. Otherwise, let λ be the minimal cardinal such that there are $\vec{\phi}, \kappa$, and a $\vec{\phi}, \lambda$ -strong extender $E \in j_{0\alpha}(\vec{E})$ with $\text{crit}(E) = \kappa$ such that $x \not\models \Psi(\vec{\phi}, \kappa, \lambda)$. Fix such $\vec{\phi}, \lambda$ and E . We have $x \not\models \bigvee_{\xi < \kappa} \phi_\xi$ and $x \models \bigvee_{\xi < \lambda} \phi_\lambda$. and note that $\lambda < j_{0\alpha}(\delta)$. Then let player I play $E_\alpha = E$. Note that $j_{E_\alpha}(\kappa) \geq \lambda$.

This describes the iteration strategy for player I. We claim that if player II responds with his winning strategy then the process of building the iteration tree terminates at some countable stage. Assume otherwise. Let $\langle T, M_\xi, E_\xi \mid \xi < \omega_1 \rangle$ be the resulting iteration tree and let $b \subseteq \omega_1$ be its cofinal branch such that M_{ω_1} , the direct limit of $\langle M_\xi \mid \xi \in b \rangle$, is well-founded.

Fix a countable $H \prec H_{(2^{\aleph_1})^+}$ containing everything relevant. Let $\alpha = H \cap \omega_1$ and let \bar{H} be the transitive collapse of H , with $\pi^{-1}: \bar{H} \rightarrow H(\vartheta)$ the inverse of the collapsing map. Since the iteration did not stop at stage α , we can consider the extender E applied to M_α along the well-founded ω_1 -branch of T . Note that E may be different from E_α , the extender chosen in M_α during the run of the iteration game. Instead, E is equal to E_β for some $\beta \geq \alpha$.

By (15) of Lemma 3.7 we have that π^{-1} and $j_{\alpha\omega_1}$ agree on $M_\alpha \cap \bar{H}$ because i is the identity on \mathbf{V}_γ . In particular, $\text{crit}(E) = \alpha$.

By the choice of the iteration, E is $\vec{\phi}, \lambda$ -strong for some $\vec{\phi}$ and λ such that $\vec{\phi}$ reflects to α but $x \not\models \bigvee_{\xi < \alpha} \phi_\xi$ and $x \models \bigvee_{\xi < \lambda} \phi_\xi$. By definition we have $\phi_\eta \in M_\alpha \cap \mathbf{V}_\alpha$ for all $\eta < \alpha$, hence the formula $\bigvee_{\eta < \alpha} \phi_\eta$ belongs to $\mathbf{V}_{\alpha+1}$. By (12) of Lemma 3.7 there are $\xi <_T \alpha$ and $\vec{\psi} \in M_\xi$ such that $\vec{\phi} = j_{\xi\alpha}(\vec{\psi})$.

Since $j_{\alpha\omega_1}$ and j_E agree on $M_\alpha \cap \mathbf{V}_\lambda$, we have that $j_{\alpha\omega_1}(\vec{\phi})$ implies $\bigvee_{\xi < \lambda} \phi_\xi$, hence $x \models j_{\alpha\omega_1}(\vec{\psi})$ holds in H . Thus for some $\xi \in H \cap \omega_1$ we have $x \models \phi_\xi$ and therefore $x \models \bigvee_{\xi < \alpha} \phi_\xi$, a contradiction. \dashv

A number of extensions of Theorem 4.1 in different directions are in order (some of their obvious common generalization are being omitted).

Assume an extender E and a forcing \mathbb{P} are such that for some κ we have $\text{crit}(j_E) > \kappa$ and \mathbb{P} is in \mathbf{V}_κ . Then E still defines an extender in the extension by \mathbb{P} and this extender is λ -strong for every λ for which E is λ -strong. This is essentially a consequence of the Levy–Solovay result that a measurable cardinal cannot be destroyed by a small forcing ([Kan95]). The converse is also true ([HW00]): in a forcing extension $\mathbf{V}[G]$ by forcing $\mathbb{P} \in \mathbf{V}_\kappa$ every λ -strong elementary embedding of $\mathbf{V}[G]$ into an inner model with critical point κ is a lift of a ground model λ -strong elementary embedding with critical point κ . This is a bit deeper than the corresponding result for elementary embeddings that are not necessarily λ -strong.

In the following lemma and elsewhere we shall slightly abuse the notation and denote the extender obtained from E in a forcing extension by the same letter.

THEOREM 4.2. Assume (M, \vec{E}) is $(\omega_1 + 1)$ -iterable and \vec{E} witnesses a countable ordinal δ is a Woodin cardinal in M . Assume moreover $\kappa < \min\{\text{crit}(j_E) : E \in \vec{E}\}$ and $\mathbb{P} \in \mathbf{V}_\kappa \cap M$ is a forcing notion. If $G \subseteq \mathbb{P}$ in V is M -generic then for every $x \subseteq \omega$ there is a (well-founded) countable iteration $j: M[G] \rightarrow M^*[G]$ such that x is $j(\mathcal{W}_{\delta, \omega}(\vec{E}))$ -generic over $M^*[G]$.

PROOF. By the above discussion δ is still a Woodin cardinal in $M[G]$ as witnessed by \vec{E} . The proof is similar to the proof of Theorem 4.1. In his strategy, player I computes $\Psi(\vec{\phi}, \kappa, \lambda)$ in $M[G]$ instead of M . Since $G \in V$, this describes an iteration strategy of player I in V . Thus player II can respond to it by using his winning iteration strategy and the other details of the proof are identical.

As a matter of fact, a simple proof shows that the full iterability of M implies the full iterability of $M[G]$ (essentially using the same strategy). The point is that, since \mathbb{P} is small, the ultrapowers of M lift to ultrapowers of $M[G]$. Therefore this is a consequence of Theorem 4.1. \dashv

THE SECOND PROOF OF LEMMA 3.5. We now provide a different proof that if γ is any cardinal less than the least critical point of each extender in \vec{E} and \mathbb{P} is a forcing notion of cardinality γ then in $\mathcal{W}_{\delta, \gamma}(\vec{E})$ there is a condition p that forces that \mathbb{P} is a regular subordering of $\mathcal{W}_{\delta, \gamma}(\vec{E})$. The present proof will also show that there exist condition $q \in \mathcal{W}_{\delta, \gamma}(\vec{E})$ and condition r in \mathbb{P} that force that \mathbb{P} is forcing-equivalent to $\mathcal{W}_{\delta, \gamma}(\vec{E})$. Assume $G \subseteq \mathbb{P}$ is generic over M . We may assume $\mathbb{P} = (\gamma, <_{\mathbb{P}})$ for some ordering $<_{\mathbb{P}}$ on γ . Since δ is a countable ordinal and $\gamma < \delta$, by using Theorem 4.1 we can find an iteration $j: M \rightarrow M^*$ such that G is $j(\mathcal{W}_{\delta, \gamma}(\vec{E}))$ generic over M^* . By our assumption γ is smaller than the critical point of j and therefore $j(\mathbb{P}) = \mathbb{P}$. Hence in M^* there is q_0 in $j(\mathcal{W}_{\delta, \gamma}(\vec{E}))$ and $r \in \mathbb{P}$ that forces \mathbb{P} and $j(\mathcal{W}_{\delta, \gamma}(\vec{E}))$ are forcing equivalent. By the elementarity, this is true in M for \mathbb{P} and $\mathcal{W}_{\delta, \gamma}(\vec{E})$. \dashv

Here is yet another variation of genericity iterations, also due to Woodin. Its proof was sketched in the appendix to [NZ00] (not included in the published version [NZ01]).

THEOREM 4.3. Assume \mathbb{P} is a forcing notion of cardinality κ and \dot{x} is a \mathbb{P} -name for a real. Assume (M, \vec{E}) is $(\kappa^+ + 1)$ -iterable and \vec{E} witnesses a countable ordinal δ is a Woodin cardinal in M . Then there is a (well-founded) iteration $j: M \rightarrow M^*$ of length $< \kappa^+$ such that \mathbb{P} forces \dot{x} is $j(\mathcal{W}_{\delta, \omega}(\vec{E}))$ -generic over M^* .

PROOF OF THEOREM 4.3. has similar structure as the proof of Theorem 4.1. More precisely, player I and player II play the iteration game in which player II follows his winning strategy while player I at each move chooses a bad extender with the least possible strength. In the present situation an extender is bad if some condition in \mathbb{P} forces that \dot{x} violates an axiom associated with this extender. The game stops when \mathbb{P} forces that \dot{x} satisfies the theory corresponding to \vec{E} and therefore that it is $j(\mathcal{W}_{\delta, \omega}(\vec{E}))$ -generic over the iterate M^* .

Showing that the construction terminates before the κ^+ -th stage requires taking the elementary submodel H of cardinality κ and an appropriate analogue of Lemma 3.7. \dashv

In [NZ01] it was proved that if the forcing \mathbb{P} is proper then one can find a countable genericity iteration. The authors of [NZ01] also pointed out that this is not necessarily true when \mathbb{P} is only assumed to be semiproper.

4.1. Genericity iterations for subsets of ω_1 . We finally turn to applications of the algebra $\mathcal{W}_{\delta, \delta}$ with δ generators.

THEOREM 4.4. Assume (M, \vec{E}) is $(\omega_1 + 1)$ -iterable and \vec{E} witnesses a countable ordinal δ is a Woodin cardinal in M . Then for every $x \subseteq \omega_1$ there is a (well-founded) countable iteration $j: M \rightarrow M^*$ such that $x \cap j(\delta)$ is $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ -generic over M^* .

Furthermore, we can assure $j(\delta) \in \mathbf{C}$ for any club $\mathbf{C} \subseteq \omega_1$ given in advance.

PROOF. The strategy for player I is identical to the strategy used in the proof of Theorem 4.1, and the proof of the latter shows that the iteration terminates at some countable stage at which $x \cap j(\delta) \models j(\mathcal{T}_{\delta, \delta}(\vec{E}))$.

Now we show how to assure $j(\delta) \in \mathbf{C}$. Let $c_\alpha \subseteq \omega$ be a real coding the α -th element of \mathbf{C} and let $\overline{\mathbf{C}}$ be a subset of ω_1 such that its intersection with the interval $[\omega\alpha, \omega(\alpha + 1))$ codes c_α . Identify x with a set of even ordinals and $\overline{\mathbf{C}}$ with a set of odd ordinals and let y be the result. By applying the first part of this theorem to y find an iteration $j: M \rightarrow M^*$ such that $x \cap j(\delta)$ and $\overline{\mathbf{C}} \cap j(\delta)$ are both $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ -generic over M^* . Let $G \subseteq [j(\mathcal{W}_{\delta, \delta}(\vec{E}))]$ be the generic filter defined by $x \cap j(\delta)$ and $\overline{\mathbf{C}} \cap j(\delta)$. Since $c_\xi \in M^*[G]$ for

all $\xi < j(\delta)$, we have that in $M^*[G]$ all ordinals below $j(\delta)$ -th element of \mathbf{C} are collapsed to \aleph_0 . By the $j(\delta)$ -cc of $j(\mathcal{W}_{\delta,\delta}(\vec{E}))$, $j(\delta) = \aleph_1^{M^*[G]}$. Assume $\mathbf{C} \cap j(\delta)$ was bounded and pick ξ such that $\sup \mathbf{C} \cap j(\delta) < \xi < j(\delta)$. Then c_ξ codes an element of \mathbf{C} greater than $j(\delta)$. But this means that $j(\delta)$ is collapsed to \aleph_0 in $M^*[G]$, contradicting the above. \dashv

The following theorem is an analogue of Theorem 4.1 for an arbitrary set of ordinals and it will be used in the proof of Theorem 5.6. More useful versions (Theorem 4.6 and Theorem 4.7) require stronger large cardinal assumptions.

THEOREM 4.5. Assume (M, \vec{E}) is fully iterable and \vec{E} witnesses a countable ordinal δ is a Woodin cardinal in M . Then for every set of ordinals x there is a (well-founded) iteration $j: M \rightarrow M^*$ of length $< |x|^+$ such that x is $j(\mathcal{W}_{\delta,\delta}(\vec{E}))$ -generic over M^* .

PROOF. Let $E_0 \in \vec{E}$ be an extender with minimal strength. Iterate E_0 to obtain an iteration $j_0: M \rightarrow M_0$ such that $x \subseteq j_0(\delta)$. Since this is a linear iteration, it is well-founded. By the minimality, the strength of every extender in $j_0(\vec{E})$ is greater than the strength of all iterates of E_0 used in j_0 . Therefore for every iteration $i: M_0 \rightarrow M^*$ of $(M_0, j_0(\vec{E}))$ we have that $i \circ j_0: M \rightarrow M^*$ is an iteration of (M, \vec{E}) , and therefore $(M_0, j_0(\vec{E}))$ is fully iterable.

From this point on the proof is analogous to the proof of Theorem 4.1. Define a strategy for player I for building an iteration tree starting with M_0 as follows. Assume $\langle T, M_\xi, E_\xi \mid \xi \leq \alpha \rangle$ has been constructed. If $x \models (j_{0\alpha} \circ j_0)(\mathcal{T}_{\delta,\delta}(\vec{E}))$ then $j_{0\alpha} \circ j_0: M \rightarrow M_\alpha$ is the required iteration and we stop. Otherwise, let λ be the minimal cardinal such that there are $\vec{\phi}, \kappa$, and a $\vec{\phi}, \lambda$ -strong extender $E \in (j_{0\alpha} \circ j_0)(\vec{E})$ with $\text{crit}(E) = \kappa$ such that $x \not\models \Psi(\vec{\phi}, \kappa, \lambda)$. Fix such $\vec{\phi}, \lambda$ and E . We have $x \not\models \bigvee_{\xi < \kappa} \phi_\xi$ and $x \models \bigvee_{\xi < \lambda} \phi_\xi$. and note that $\lambda < (j_{0\alpha} \circ j_0)(\delta)$. Then let player I play $E_\alpha = E$. Note that $(j_{E_\alpha} \circ j_0)(\kappa) \geq \lambda$.

This describes the iteration strategy for player I. A proof that if player II responds with his winning strategy then the process of building the iteration tree terminates at some stage before $|x|^+$ is identical to the proof of Theorem 4.1, using the variant of Lemma 3.7 for an iteration tree of height $|x|^+$. \dashv

The iterability assumption of Theorem 4.6 below is almost the strongest known result of its kind. Its consistency modulo large cardinals was proved by Neeman [Nee02A]. If CH holds and $\{x_\xi : \xi < \omega_1\}$ is an enumeration of all reals we write

$$\mathbb{R} \upharpoonright \gamma = \{x_\xi : \xi < \gamma\}.$$

The extender algebra with δ generators need not satisfy (3) of the following theorem, even when it collapses δ to \aleph_1 .

THEOREM 4.6. Assume (M, \vec{E}) is $(\omega_1 + 1)$ -iterable and \vec{E} witnesses δ is a Woodin limit of Woodin cardinals in M . Furthermore assume CH holds and A is a set of reals. Then there are a (well-founded) iteration $j: M \rightarrow M^*$ of countable length and an M^* -generic filter $G \subseteq j(\mathcal{W}_{\delta, \delta}(\vec{E}_0))$ such that the following hold:

- (1) $\mathbb{R}^{M^*[G]} = \mathbb{R} \upharpoonright j(\delta)$,
- (2) $A \cap (\mathbb{R} \upharpoonright j(\delta)) \in M^*[G]$, and
- (3) Every real in $M^*[G]$ is M^* -generic for a forcing of cardinality $< j(\delta)$.
- (4) There is $B \subseteq \omega_1$ such that $N = \mathbf{L}[M, B \upharpoonright \alpha]$ satisfies $\mathbb{R}^{M^*[G]} = \mathbb{R}^N$

PROOF. Let Σ be the winning strategy for player II in the iterarion game for M, \vec{E} . It is naturally identified with a set of reals and therefore, using CH, with a subset of ω_1 . Fix a subset B of ω_1 so that its intersection with ω codes M and B codes \mathbb{R} , A and Σ . Let \mathbf{C} be the club of all $\alpha < \omega_1$ such $B \upharpoonright \alpha$ is an elementary submodel of B .

Applying Theorem 4.4 find an iteration $j: M \rightarrow M^*$ such that (with $\alpha = j(\delta)$) we have that $B \cap \alpha$ is $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ -generic over M^* and α is a limit point of club \mathbf{C} . Therefore for cofinally many $\xi < \alpha$ we have $B \upharpoonright \xi \prec B$. Let G be the generic filter defined by $B \upharpoonright \alpha$. Note that (2) is immediate. We shall now prove (4). Note that the converse inclusion is immediate.

Only the direct inclusion requires a proof. Let $\langle T, M_\eta, E_\xi \mid \eta \leq \alpha, \xi < \alpha \rangle$ be the iteration tree constructed during the course of finding M^* . Since the iteration strategy of player I is definable and $\Sigma \upharpoonright \alpha$ is an elementary submodel of Σ , this tree (except possibly its final α -branch) can be constructed in N .

Now assume \dot{y} is a nice $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ -name for a real. By α -cc there is a Woodin cardinal $\gamma < \alpha$ in M^* such that all conditions in \dot{y} use only generators less than γ . Let $\vec{E}(\gamma)$ be the extenders in $j(\vec{E})$ whose strength is below γ . Then $\vec{E}(\gamma)$ witnesses Woodinness of γ in M^* . Also, $G \upharpoonright \gamma$ satisfies all axioms of theory $\mathcal{T}(\vec{E}(\gamma))$ in M^* . Let $\beta < \alpha$ be large enough so that $M_\beta \cap \mathbf{V}_{\gamma+1} = M^* \cap \mathbf{V}_{\gamma+1}$ and therefore all axioms of $\mathcal{T}(\vec{E}(\gamma))$ are satisfied in M_β .

Therefore we have proved

- (5) $G \upharpoonright \gamma$ is generic for $\mathbb{P} = \mathcal{W}_{\gamma, \gamma}(\vec{E}_\gamma)$ over both M_β and M^* .

Since all conditions occurring in \dot{y} belong to \mathbb{P} , the interpretation of \dot{y} with respect to the filter in \mathbb{P} defined by $G \upharpoonright \gamma$ over M_β is definable from M and $G \upharpoonright \alpha$ and therefore belongs to N . Since this interpretation coincides with the interpretation of \dot{y} with respect to the filter in $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ defined

by G , and since \dot{y} was an arbitrary name for a real, the direct inclusion of (4) follows.

Since N is a definable inner model in $M^*[G]$, (1) follows.

It remains to prove (3). The elementarity of $\mathbb{R} \upharpoonright \alpha$ in \mathbb{R} implies that $j(\mathcal{W}_{\delta,\delta}(\vec{E}))$ collapses all cardinals below α to \aleph_0 . By (5) each x_ξ , for $\xi < \gamma$, is generic over M^* for a small forcing. Since δ is a limit of Woodins all reals in $\mathbb{R} \upharpoonright \alpha$ are generic over M^* for a small forcing notion. \dashv

The assumption of the existence of an $(\omega_1 + 1)$ -iterable model with a measurable Woodin cardinal used in the following theorem is presently beyond reach of the inner model theory.

THEOREM 4.7. Assume (M, \vec{E}) is $(\omega_1 + 1)$ -iterable and \vec{E} witnesses a countable ordinal δ is a measurable Woodin cardinal in M . Then for every $x \subseteq \omega_1$ there is a (well-founded) ω_1 -iteration $j: M \rightarrow M^*$ such that x is $j(\mathcal{W}_{\delta,\delta}(\vec{E}))$ -generic over M^* .

PROOF. This proof is an extension of the proof of Theorem 4.4. We need to assure $x \models j(\mathcal{T}_{\delta,\delta}(\vec{E}))$ for an ω_1 -iteration j . Define a strategy of player I for building an iteration tree starting with $M_0 = M$ as follows. Assume $\langle T, M_\eta, E_\xi \mid \eta \leq \alpha, \xi < \alpha \rangle$ has been constructed.

(a) Assume there is a sequence $\vec{\phi}$ in M_α that reflects to some κ but $x \not\models \Psi(\vec{\phi}, \kappa, \lambda)$ for some λ satisfying $\lambda < j_{0\alpha}(\delta)$ and $\lambda > \sup_{\xi < \alpha} (\lambda_{E_\xi})$. Choose the minimal λ with this property, fix the appropriate $\vec{\phi}$ and κ so that $x \not\models \Psi(\vec{\phi}, \kappa, \lambda)$. Then let E_α be a $(\vec{\phi}, \lambda)$ -strong extender in M_α such that $\text{crit}(E_\alpha) = \kappa$. Note that $j_{E_\alpha}(\kappa) \geq \lambda$.

(b) Now assume α is countable and $x \models j_{0\alpha}(\mathcal{T}_{\delta,\delta}(\vec{E}))$. Since δ is a measurable Woodin cardinal, use normal measure U on $j_{0\alpha}(\delta)$ to define $M_{\alpha+1} = \text{Ult}(M_\alpha, U)$ and let $j_{\alpha,\alpha+1}: M_\alpha \rightarrow M_{\alpha+1}$.

This describes the iteration strategy for player I. Now consider the iteration tree formed when player II plays his winning strategy against the iteration strategy just defined for player I. By Theorem 4.4, the set C of all stages α in the iteration such that $x \models j_{0\alpha}(\mathcal{T}_{\delta,\delta}(\vec{E}))$ and $\alpha = j_{0\alpha}(\delta)$ is unbounded, and it is therefore a club. Note that $x \cap \alpha$ is $j_{0\alpha}(\mathcal{W}_{\delta,\delta}(\vec{E}))$ -generic for all $\alpha \in C$.

By the choice of extenders and Lemma 3.6, for $\alpha \in C$ the critical points of embeddings constructed after α th stage will never drop below $j_{0\alpha}(\delta)$.

The critical sequence defines a club in ω_1 and an ω_1 -iteration $\langle N_\xi, j_{\xi\eta} \mid \xi \leq \eta \leq \omega_1 \rangle$ such that $N_0 = M$, for $\xi < \eta$ the embedding $j_{\xi\eta}: N_\xi \rightarrow N_\eta$ has α_ξ as its critical point, and $x \cap \alpha_\xi \models j_{0\xi}(\mathcal{T}_{\delta,\delta}(\vec{E}))$ for all ξ . Since the critical points are increasing, this implies that $x \models j_{0\omega_1}(\mathcal{T}_{\delta,\delta}(\vec{E}))$, as required. \dashv

§5. Absoluteness. By ‘ M_1 exists’ we denote the statement ‘There exists an inner model of the form $\mathbf{L}[\vec{E}]$ with a Woodin cardinal whose Woodinness is witnessed by extenders in \vec{E} .’ Similarly, if a is a real then ‘ $M_1(a)$ exists’ denotes the statement ‘There exists an inner model of the form $\mathbf{L}[\vec{E}]$ with a Woodin cardinal whose Woodinness is witnessed by extenders in \vec{E} and which contains a .’ Variants such as ‘ M_n exists’ or ‘ M_1 exists and it is fully iterable’ are interpreted similarly. The following lemma and its variations will be used tacitly in order to furnish the assumptions of genericity iteration theorems.

LEMMA 5.1. Assume M_1 exists and it is fully iterable. Then there exists a fully iterable inner model with a Woodin cardinal δ such that δ is a countable ordinal in V .

PROOF. Take a countable elementary submodel N of M_1 . Theorem 5.6 implies that N has a sharp. We can therefore find an iteration of N of length Ord whose resulting model is an inner model with a Woodin cardinal which is a countable ordinal in V . The iteration strategy for N is obtained by copying the iteration strategy of M_1 (see [Sarto appearA]). \dashv

5.1. Absoluteness in $\mathbf{L}(\mathbb{R})$. The following result and its proof are a prototype for the main result of this note, Theorem 5.9.

THEOREM 5.2. Assume $M_1(a)$ is fully iterable in all forcing extensions for all $a \in \mathbb{R}$. Then all Σ_3^1 statements are forcing absolute.

PROOF. A Σ_3^1 statement $\phi(a)$ with parameter $a \in \mathbb{R}$ is of the form $(\exists x)\psi(x, a)$ where ψ is Π_2^1 . To ϕ we associate a sentence ϕ^* of $M_1(a)$ (see Theorem 3.8) stating that there exists a forcing notion \mathbb{P} forcing ϕ . We claim that ϕ holds (in V) if and only if ϕ^* holds in $M_1(a)$. This will suffice since the iterability of $M_1(a)$ is not changed by forcing. If ϕ^* holds in $M_1(a)$ then since $M_1(a) \cap \mathbf{V}_{\delta+1}$ is countable we can find an $M_1(a)$ -generic filter $G \subseteq \mathbb{P}$ for \mathbb{P} referred to in ϕ^* . If $M_1(a)[G] \models \psi(x, a)$ then by Shoenfield’s absoluteness theorem $\psi(x, a)$ holds in V . For the converse implication, assume $\psi(x, a)$ holds in some forcing extension of V for some $x \in \mathbb{R}$. Since $M_1(a)$ is fully iterable in all forcing extensions, apply Theorem 4.1 to find a countable iteration $j: M_1(a) \rightarrow M^*$ such that x is generic over M^* . Then (again using Shoenfield) $M^* \models \phi^*$, and by elementarity $M_1(a) \models \phi^*$. \dashv

The assumptions of Theorem 5.2 are far from optimal. By a result of Martin and Solovay ([MS69]), if κ is a measurable cardinal then Σ_3^1 sentences are forcing absolute for forcing notions in \mathbf{V}_κ . As a matter of fact, all Σ_3^1 sentences are absolute between all forcing extensions of V if and only if all sets have sharps (Martin–Steel, Woodin; see *e.g.*, [Ste10] for terminology). One important fact about Theorem 5.2 and its extension,

Theorem 5.4 below, is that its proof is susceptible to far-reaching generalizations. Also, Corollary 5.3 below is worth mentioning. In the language of Woodin's Ω -logic (see [Woo10, §10], [Woo01], [BCL06], or [Lar09]), it states that if the function $x \mapsto M_1(x)^\#$ is universally Baire then it is an Ω -proof for all true Σ_3^1 statements (see §5.3). As common in inner model theory, by $M_1(x)^\#$ we denote an iterable transitive model of ZFC that contains x and has a top measure (for more on mice see [Sch01]).

COROLLARY 5.3. Assume N is a transitive model of a large enough fragment of ZFC that is closed under the map $x \mapsto M_1(x)^\#$. Then N is Σ_3^1 -correct.

PROOF. For every real x in N , N can reconstruct $M_1(x)$ and its iteration strategy. Therefore the proof of Theorem 5.2 applies inside N . \dashv

It should be noted that M_1 is not necessarily Σ_3^1 correct. For example, the fine-structural version of M_1 has a Δ_3^1 well-ordering of the reals. Similarly, being closed under the sharps (*i.e.*, under the map $x \mapsto x^\#$) does not guarantee Σ_3^1 correctness since $L[\mu]$ has a Δ_3^1 well-ordering of the reals. Nevertheless, M_2 is Σ_4^1 correct (see [Ste95B]). Here we shall only show that a proof analogous to the above gives a weaker result.

THEOREM 5.4. Assume $M_n(a)$ is fully iterable in all forcing extensions for every $a \in \mathbb{R}$. Then all Σ_{n+2}^1 statements are forcing absolute.

PROOF. We first prove the case $n = 2$. A Σ_4^1 sentence ϕ has the form $(\exists x)(\forall y)\psi(x, y, a)$ for some real parameter a and a Π_2^1 formula ψ . Let $M_2(a)$ be the minimal model for two Woodin cardinals containing a fully iterable in all forcing extensions (Theorem 3.8), and let $\delta_0 < \delta_1$ be its Woodin cardinals.

To ϕ associate a sentence ϕ^* stating that there is a forcing \mathbb{P} in \mathbf{V}_{δ_0+1} and a \mathbb{P} -name \dot{x} for a real such that for every forcing $\dot{\mathbb{Q}} \in \mathbf{V}_{\delta_1+1}$ and a $\dot{\mathbb{Q}}$ -name \dot{y} for a real we have

$$\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}} \psi(\dot{x}, \dot{y}, a).$$

We claim that ϕ holds in V if and only if ϕ^* holds in $M_2(a)$.

Assume $M_2(a) \models \phi^*$. Find an $M_2(a)$ -generic $G \subseteq \mathbb{P}$ and let $x = \text{int}_G(\dot{x})$. Let y be any real. Let \vec{E}_1 be the system of extenders witnessing δ_1 is Woodin in $M_2(a)$. We may assume $\min\{\text{crit}(j_E) : E \in \vec{E}_1\} > \delta_0$. Using the full iterability of $M_2(a)$ and Theorem 4.2 find a well-founded iteration $j : M_2(a)[G] \rightarrow M^*[G]$ such that y is $j(\mathcal{W}_{\delta, \omega}(\vec{E}_1))$ -generic over $M^*[G]$. Then by elementarity we have $M^*[G][y] \models \psi(x, y, a)$ and by Shoenfield's absoluteness theorem $\psi(x, y, a)$ holds. Since y was arbitrary, we have proved ϕ holds in V .

Now assume ϕ holds in V and let $x \in \mathcal{P}(\omega)$ be such that $(\forall y)\psi(x, y, a)$. By Theorem 4.1 we can find an iteration $j_0: M_2(a) \rightarrow M^*$ such that x is $j_0(\mathcal{W}_{\delta, \omega}(\vec{E}_0))$ -generic over M^* . Fix $y \in \mathcal{P}(\omega)$. By Theorem 4.2 there is an iteration $j_1: M^*[x] \rightarrow M^{**}[x]$ such that y is $(j_1 \circ j_0)(\mathcal{W}_{\delta, \omega}(\vec{E}_1))$ -generic over $M^{**}[x]$. Then $\mathbf{V} \models \phi(x, y, a)$ and by the Shoenfield's absoluteness theorem $M^{**}[x] \models \phi(x, y, a)$. Since y was arbitrary, this shows ϕ^* holds in $M_2(a)$.

This concludes proof of the theorem in the case when we have two Woodin cardinals. In general case, to a Σ_{n+2}^1 formula $(\exists x_1)(\forall x_2) \dots \psi(x_1, \dots, x_n, a)$ with ψ being Σ_2^1 one associates a formula ϕ^* of $M_n(a)$ stating

$$(\exists \mathbb{P}_1 \in \mathbf{V}_{\delta_0+1})(\exists \dot{x}_1)(\forall \mathbb{P}_2 \in \mathbf{V}_{\delta_1+1})(\forall \dot{x}_2) \dots \Vdash_{\mathbb{P}_1 * \dot{\mathbb{P}}_2 * \dots * \dot{\mathbb{P}}_n} \psi(\dot{x}_1, \dots, x_n, a)$$

and proves that $\mathbf{V} \models \phi$ is equivalent to $M_n(a) \models \phi^*$ as above. \dashv

REMARK 5.5. The following was pointed out by Menachem Magidor. In Theorem 5.4 it is not sufficient to assume that there are n Woodin cardinals and a measurable above. Such an assumption can hold if $V = \mathbf{L}[\vec{E}]$ for a system of extenders \vec{E} . However a forcing extension of $\mathbf{L}[\vec{E}]$ in which a large enough cardinal is collapsed to \aleph_0 is of the form $\mathbf{L}[x]$, for a real x . Such an extension satisfies the projective statement ‘there exist a real x and a $\Delta_2^1(x)$ well-ordering of \mathbb{R} .’

However, the existence of a proper class of Woodin cardinals gives such an absoluteness, and more (see [Lar04]).

By Remark 5.5 the existence of $M_n(a)$ is a strictly weaker assumption than its iterability in all forcing extensions, needed in Theorem 5.2. We write M_n for $M_n(0)$. Recall that a set of ordinals X **has a sharp** if there is a nontrivial elementary embedding of $\mathbf{L}[X]$ into itself.

The following result is also due to Woodin.

THEOREM 5.6. The following are equivalent.

- (1) M_1 exists and it is fully iterable in all forcing extensions.
- (2) Every set has a sharp and there is a proper class model with a Woodin cardinal.

PROOF. (2) implies (1) is a difficult result using Mitchell–Steel constructions and we only prove the implication from (1) to (2). We only need to show that the full iterability of M_1 implies that every set has a sharp. Assume the contrary, and let X be a set without a sharp. Then the Covering Lemma holds in $\mathbf{L}[X]$, hence the successor λ^+ of some singular cardinal λ such that $X \in \mathbf{V}_\lambda$ is correctly computed in $\mathbf{L}[X]$ ([Kan95]). By Theorem 4.5 there is an iteration $j: M_1 \rightarrow M^*$ such that X is generic over M^* and $j(\delta) < \lambda^+$. Then $M^*[X]$ correctly computes λ^+ , since it includes $\mathbf{L}[X]$. On the other hand, by the chain condition of the extender algebra $j(\delta)$ remains a cardinal in $M^*[X]$. A contradiction. \dashv

I learned the above proof that (1) implies (2) from Ralf Schindler. This proof also shows that if M_{n+1} is fully iterable then for every set X there is an inner model including $\mathbf{L}[X]$ with n Woodin cardinals that has a sharp.

5.2. Absoluteness for $H(\aleph_2)$. Theorem 5.9 below was proved independently by Steel and Woodin and a proof of its strengthening due to Neeman can be found in [Nee04]. Theorem 5.9 implies that the existence of a model with a measurable Woodin cardinal that is fully iterable in all forcing extensions would provide another proof of Woodin's Σ_1^2 -absoluteness theorem ([Woo85]; see also [Doe10], [Lar04], or [Far07C]). The proof of this theorem will use the following standard forcing fact. The assumption that $\mathbf{V}_\delta \cap M$ is countable is used only to assure the existence of generic objects.

LEMMA 5.7. Assume M is a transitive model of a large enough fragment of ZFC such that $M \cap \mathbf{V}_\delta$ is countable. Assume \mathbb{P} and \mathbb{Q} belong to $M \cap \mathbf{V}_\delta$ and \mathbb{P} is a regular subalgebra of \mathbb{Q} . If $G \subseteq \mathbb{P}$ is M -generic, then there is an M -generic $H \subseteq \mathbb{Q}$ such that $G \in M[H]$. \dashv

LEMMA 5.8. Assume \mathbb{P} is a forcing notion in M with δ -cc of cardinality δ . If $j: M \rightarrow M^*$ is an elementary embedding with M^* a definable class in M and $\text{crit}(j) = \delta$, then \mathbb{P} is a regular subordering of $j(\mathbb{P})$ in M .

PROOF. We may assume $\mathbb{P} \subseteq \mathbf{V}_\delta$. Let \mathcal{A} be a maximal antichain in \mathbb{P} . By the δ -cc we have $\mathcal{A} \in \mathbf{V}_\delta$, and therefore $j(\mathcal{A}) = \mathcal{A}$. By the elementarity, \mathcal{A} is a maximal antichain in $j(\mathbb{P})$ in M^* . Since being a maximal antichain is absolute, \mathcal{A} is a maximal antichain of $j(\mathbb{P})$ in M . \dashv

THEOREM 5.9. Assume there exists a model \mathcal{M}_{mw} with a countable ordinal δ that is a measurable Woodin cardinal in \mathcal{M}_{mw} which is fully iterable in all forcing extensions. Then to every Σ_1^2 statement ϕ we can associate a statement ϕ^* such that if $\mathbf{V} \models \phi$ then $\mathcal{M}_{\text{mw}} \models \phi^*$ and if $\mathcal{M}_{\text{mw}} \models \phi^*$ and CH holds then $\mathbf{V} \models \phi$.

PROOF. The sentence ϕ is of the form $(\exists X \subseteq \mathbb{R})\psi(X)$ where ψ is a statement of $(H(\aleph_1), X, \in)$. To it we associate ϕ^* stating that some condition in $\mathcal{W}_{\delta, \delta}$ forces ϕ and $|\check{\delta}| = \aleph_1$.

In order to prove that ϕ implies $\mathcal{M}_{\text{mw}} \models \phi^*$, assume $X \subseteq \mathbb{R}$ is such that $\psi(X)$ holds. Go to a forcing extension of V with the same reals that satisfies CH. Fix $Y \subseteq \omega_1$ that codes X and all reals. Using Theorem 4.7 find an iteration $j: \mathcal{M}_{\text{mw}} \rightarrow \mathcal{M}$ of length ω_1 such that Y is $j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ -generic over \mathcal{M} . This forcing has $j(\delta)$ -chain condition and it collapses all cardinals below $j(\delta)$ to ω . Therefore $\mathbb{R}^{\mathcal{M}_{\text{mw}}[Y]} = \mathbb{R}$, hence $\mathcal{M}_{\text{mw}}[Y] \models \psi(X)$.

We now assume CH and $\mathcal{M}_{\text{mw}} \models \phi^*$ and prove ϕ . In \mathcal{M}_{mw} fix a condition p in $\mathcal{W}_{\delta, \delta}$ and a name \dot{X} such that p forces $\psi(\dot{X})$. By using CH we can enumerate \mathbb{R} as r_ξ , for $\xi < \omega_1$. In this proof we shall write M_0 instead of \mathcal{M}_{mw} . We shall now describe an iteration strategy for player I. Along

with the iteration, player I constructs generic filters. After player I's strategy is described, we shall run it against player II's winning strategy for the iteration game and argue that the run of the game produces a well-founded iteration $j: M_0 \rightarrow M^*$ and $G \subseteq j(\mathcal{W}_{\delta, \delta}(\vec{E}))$ generic over M^* such that $p = j(p)$ is in G and that $M^*[G]$ contains all reals. All extenders E used in player I's strategy defined below will satisfy $\text{crit}(E) \geq \delta$, therefore assuring $j(p) = p$.

Since $M_0 \cap \mathbf{V}_{\delta+1}$ is countable, in V we can find a $G_0 \subseteq \mathcal{W}_{\delta, \delta}(\vec{E})$ generic over M_0 and containing p . Now use the extender in M_0 with critical point δ to find $j_0: M_0 \rightarrow M_1$. By Lemma 5.8, G_0 can be extended to a generic filter for $j_0(\mathcal{W}_{\delta, \delta}(\vec{E}))$.

We want to find a generic $G_1 \subseteq j_0(\mathcal{W}_{\delta, \delta}(\vec{E}))$ extending G_0 and such that $r_0 \in M_1[G_1]$. Let δ_0 be the least Woodin cardinal in M_1 greater than δ whose Woodinness is witnessed by (an initial segment of) $j_0(\vec{E})$. Let \vec{E}_1 consist of generators in $j_0(\vec{E})$ witnessing Woodinness of δ_0 whose critical points exceed δ . Now player I attempts to find an iteration $i_0: M_1 \rightarrow M_1^*$ using the extenders in \vec{E}_1 such that r_0 is generic over $i_0(\mathcal{W}_{\delta_0, \omega}(\vec{E}_1))$. (Recall that player II continues playing his winning strategy for the iteration game corresponding to \vec{E} from \mathcal{M}_{mw} , and therefore by Theorem 4.1 after countably many stages we will assure that r_0 is generic.) Assume player I has succeeded in finding i_0 . Since the critical point never drops below δ , this is an iteration of M_1 resulting in some M_1^* . Since p forces that $(i_0 \circ j_0)(\mathcal{W}_{\delta, \delta}(\vec{E}))$ collapses 2^{δ_0} to \aleph_0 , $i_0(\mathcal{W}_{\delta, \delta, \omega}(\vec{E}_1))$ can be embedded as a regular subalgebra of the former below p . We can therefore use Lemma 5.7 to find a generic $G_1 \subseteq (i_0 \circ j_0)(\mathcal{W}_{\delta, \delta}(\vec{E}))$ including G_0 such that $r_0 \in (i_0 \circ j_0)(M_1)[G_1]$.

We proceed in this manner. At the α th stage we have an iteration $j_\alpha^0: M_0 \rightarrow M_\alpha$ and G_α is a generic filter for $j_\alpha^0(\mathcal{W}_{\delta, \delta}(\vec{E}))$. Player I uses the extender in M_α with critical point $j_\alpha^0(\delta)$ to find $j_\alpha^1: M_\alpha \rightarrow M_\alpha^1$. Let δ_α be the least Woodin cardinal in M_α^1 above $j_\alpha^0(\delta)$. Choose a system of extenders $\vec{E}_{\alpha+1} \subseteq (j_\alpha^1 \circ j_\alpha^0)(\vec{E})$ which witnesses δ_α is Woodin and such that critical points of all extenders in $\vec{E}_{\alpha+1}$ exceed $j_\alpha^0(\delta)$. Using the genericity iteration theorem, player I finds an iteration j_α^2 of M_α^1 such that r_α is generic over $j_\alpha^2(\vec{E}_{\alpha+1})$. Using Lemma 5.8 to absorb this forcing in quotient, Find an iteration $j_\alpha^3: M_\alpha^1 \rightarrow M_\alpha^*$ with critical point $j_\alpha^0(\delta)$ such that the filter G_α can be extended to a generic ultrafilter $G_{\alpha+1}$ included in $j_\alpha(\mathcal{W}_{\delta, \delta}(\vec{E}))$, with $j_\alpha = j_\alpha^3 \circ j_\alpha^1 \circ j_\alpha^0$.

Fix the least Woodin cardinal in M_α^* above $j_\alpha^0(\delta)$. Using the algebra with ω generators on this cardinal and Theorem 4.1 find an iteration of $i_\alpha: M_\alpha^* \rightarrow M_\alpha^{**}$ that makes r_α generic for an algebra that is a regular

subalgebra of $(i_\alpha \circ j_\alpha)(\mathcal{W}_{\delta,\delta}(\vec{E}))$. Again player I plays only the extenders E with $\text{crit}(E) \geq j_\alpha^0(\delta)$ so the responses of player II result in an iteration $i_\alpha: M_\alpha^* \rightarrow M_\alpha^{**}$, as required. By Lemma 5.8, we can find a generic filter $G_{\alpha+1} \subseteq (i_\alpha \circ j_\alpha)(\mathcal{W}_{\delta,\delta}(\vec{E}))$ extending G_α such that r_α belongs to $M_\alpha^{**}[G_{\alpha+1}]$.

At a limit stage of the construction player II chooses a maximal branch of the iteration tree constructed so far. By the δ -cc, every maximal antichain of the image of $\mathcal{W}_{\delta,\delta}(\vec{E})$ in this model belongs to some earlier model. Therefore the direct limit of the G_ξ corresponding to the models on the branch is generic.

This describes a game which produces an iteration $j: M_0 \rightarrow M^*$ such that $j(\delta) = \aleph_1$ and a $G \subseteq j(\mathcal{W}_{\delta,\delta}(\vec{E}))$ generic over M^* and containing $j(p) = p$. By the choice of p and elementarity $\psi(X)$ holds in $M^*[G]$. Since the model $M^*[G]$ also contains all reals, $\psi(X)$ holds in V . \dashv

COROLLARY 5.10 (Steel, Woodin). Assume there exists a fully iterable model \mathcal{M}_{mw} with a countable ordinal δ that is a measurable Woodin cardinal in \mathcal{M}_{mw} . Then every Σ_1^2 statement true in some forcing extension of V is true in every forcing extension of V that satisfies CH. \dashv

5.3. A fairly complicated set of natural numbers. The following was pointed out by John Steel. Recall that if κ is a limit of Woodin cardinals then a set is κ -universally Baire if and only if it is $< \kappa$ -homogeneously Suslin (*e.g.*, [Lar04, Theorem 3.3.13]). By Hom_∞ we denote the pointclass of all homogeneously Suslin sets. A sentence ψ is $\Sigma_1^2(\text{Hom}_\infty)$ if there is $A \in \text{Hom}_\infty$ and formula $\phi(X, Y)$, where X and Y are second-order variables, such that ψ is of the following form:

$$(\exists X \in \text{Hom}_\infty)(H(\aleph_1), \in, A, X) \models \phi.$$

By a result of Woodin ([Ste09, Theorem 5.1]), if there are class many Woodin cardinals then every $\Sigma_1^2(\text{Hom}_\infty)$ sentence is absolute between forcing extensions. We shall need only the lightface version of this result.

Let \mathcal{O}_∞ denote the set of all $\Sigma_1^2(\text{Hom}_\infty)$ truths, with no real or Hom_∞ parameters. By identifying ϕ with its Gödel number $\ulcorner \phi \urcorner$ we consider \mathcal{O}_∞ as a set of natural numbers. By [Ste09, Theorem 5.1], if there exist class many Woodin cardinals \mathcal{O}_∞ is invariant under forcing.

COROLLARY 5.11. Assume there exist class many Woodin cardinals. Let Γ be the set of all Σ_1^2 sentences ϕ that hold in forcing extensions that satisfy CH. If there exists a mouse with a measurable Woodin cardinal and Hom_∞ iteration strategy, then Γ is many-one reducible to \mathcal{O}_∞ .

PROOF. By Theorem 5.9, truth of a sentence ϕ in Γ is equivalent to the existence of an iterable mouse with a Hom_∞ iteration strategy such that $(\phi$ holds in a forcing extension that satisfies CH) M . \dashv

In the presence of class many Woodin cardinals, every generic absoluteness result proved using genericity iterations along the lines of Theorem 5.9 or Theorem 5.2 implies many-one reducibility of the relevant set Γ to \mathcal{O}_∞ , provided there are class many Woodin cardinals and the relevant mouse has a Hom_∞ iteration strategy. Proofs of Σ_1^2 absoluteness using stationary tower ([Lar04]) or Levy collapse followed by forcing with a saturated ideal ([Far07C]) do not seem to produce many-one reduction of the relevant set of sentences to \mathcal{O}_∞ .

Similarly, determinacy proofs produce many-one reductions of game-quantifier truths to \mathcal{O}_∞ . For example, Neeman's [Nee04] produces Hom_∞ strategies for open ω_1 -games whose payoff set is \mathbb{R}_1^1 in the codes by transforming a Hom_∞ iteration strategy for the mouse. It therefore shows that $\mathcal{D}_{\text{open-}\omega_1}\mathbb{R}_1^1$ truth is many-one reducible to \mathcal{O}_∞ .

Recall the definition of a proof in Woodin's Ω -logic (see [Woo10, §10], [Woo01], [BCL06], or [Lar09]). One first assumes there are class many Woodin cardinals, and therefore a set of reals is universally Baire if and only if it is homogeneously Suslin. If a set of reals A is universally Baire and M is a transitive model of a large enough fragment of ZFC then we say M is **A -closed** if for every $\mathbb{P} \in M$ and every \mathbb{P} -name τ for a real such that $\tau \in M$ the set $\{p \in \mathbb{P} : p \Vdash \tau \in A\}$ belongs to M . See [Woo10, Definition 10.141] or [BCL06, §2.2] for more details. For production of A -closed models see Lemma 6.2 and Lemma 7.2. An **Ω -proof** of ϕ is a universally Baire set A such that for every A -closed model M of a large enough fragment of ZFC we have $M \models \phi$. If ϕ has an Ω -proof then we write $\vdash_\Omega \phi$. Consider the set of all theorems of Ω -logic,

$$\mathcal{O}^\Omega = \{\phi : \vdash_\Omega \phi\}.$$

This $\Sigma_1^2(\text{Hom}_\infty)$ set is easily seen to be many-one equivalent to \mathcal{O}_∞ .

Woodin's Ω -conjecture (see [Woo01], [Lar09]) asserts that \mathcal{O}^Ω is the set of Gödel numbers of statements true in all forcing extensions. Therefore Ω -conjecture implies, and is essentially equivalent to, the assertion that all generic absoluteness comes via many-one reductions to \mathcal{O}_∞ .

§6. Divergent models of AD^+ . In the present section we prove another previously unpublished theorem due to (surprise, surprise) Woodin. Recent sweeping results of Grigor Sargsyan added to the importance of this theorem by putting it in a chain of implications leading to dramatic lowering of the consistency strength of the axiom ' $\text{AD}_\mathbb{R} + \Theta$ is regular' (see [Sarto appearA] or [Woo10, §1.6] for more details). I would like to thank Grigor for suggesting that I include this proof and for clarifying a number of details, in particular the ones sketched in §6.2. I am indebted to

Hugh Woodin for sketching the proof during a train ride from Oberwolfach in January 2011. Koellner's note [Koeundated] was quite helpful in reconstructing this proof.

Two inner models N_1 and N_2 of AD^+ are said to be **incompatible** if they have the same reals but $\mathcal{P}(\mathbb{R})^{N_1} \not\subseteq \mathcal{P}(\mathbb{R})^{N_2}$ and $\mathcal{P}(\mathbb{R})^{N_2} \not\subseteq \mathcal{P}(\mathbb{R})^{N_1}$. The terminology is justified by the fact that no inner model that includes both N_1 and N_2 can satisfy Wadge determinacy.

THEOREM 6.1. Assume there exists Woodin limit of Woodin cardinals. Then in some forcing extension of V there are incompatible AD^+ models.

The remainder of this section contains the proof of Theorem 6.1.

6.1. Universally Baire sets and absoluteness. A set of reals A is κ -**universally Baire** if there are trees S and T on $\omega \times \lambda$ for some cardinal λ such that S projects to A , T projects to $\mathbb{R} \setminus A$, and in every forcing extension by $\mathbb{P} \in \mathbf{V}_\kappa$ trees S and T project to complementary sets of reals. Set A is **universally Baire** if it is κ -universally Baire for all κ . In forcing extension we use these trees as a code for A and identify set A with the projection of T . We therefore define $A^{V[G]}$ to be $p[T]^{V[G]}$ for a generic filter $G \subseteq \mathbb{P}$. In [Woo10] set $A^{V[G]}$ is denoted A_G . Good sources for universally Baire sets is [Lar04] and [Woo10, §10].

Given a κ -universally Baire set A we shall need a sufficiently closed with respect to A countable transitive model of a large enough fragment of ZFC.

LEMMA 6.2. Assume A is κ -universally Baire and $\vartheta > |\mathbf{V}_\kappa|$. Expand the language of ZFC by adding a predicate for A . Then for club many $M \prec H(\vartheta)$ the transitive collapse \bar{M} of M satisfies the following.

- (1) \bar{M} is **A -correct**: $A^{\bar{M}} = \mathbb{R}^{\bar{M}} \cap A$, and
- (2) \bar{M} is **A -absolute**: If \mathbb{P} is a forcing notion in $(\mathbf{V}_{\bar{\kappa}})^{\bar{M}}$ and $G \subseteq \mathbb{P}$ is \bar{M} -generic, then $A^{\bar{M}[G]} = \mathbb{R}^{\bar{M}[G]} \cap A$.

PROOF. Let M be an elementary submodel of a large enough $H(\vartheta)$ containing trees S and T that witness κ -universal Baireness of A and let \bar{M} be its transitive closure. We also denote images of S, T , and κ under the transitive collapse by \bar{S}, \bar{T} , and $\bar{\kappa}$. Let $A^{\bar{M}}$ denote the projection of \bar{S} in \bar{M} . In \bar{M} interpret A to be the projection of \bar{T} . Since the reals are not moved by the collapsing map, the elementarity of M implies (1) and (2). \dashv

Model \bar{M} in the conclusion of Lemma 6.2 is **A -closed** (see §5.3).

If A is a set of reals and there is a nontrivial elementary embedding of $\mathbf{L}(A, \mathbb{R})$ into itself then $(A, \mathbb{R})^\#$ is the theory of the first ω many indiscernibles for $\mathbf{L}(A, \mathbb{R})$ with parameters from \mathbb{R} and predicate for A . It is naturally identified with a set of reals.

Theorem 6.3 is due to Woodin. Its variant for projective sets was proved in [Woo82]. A very similar result also appears in [Woo83, Theorem 1.13]

(also compare [Woo83, Theorem 1.14] with Corollary 6.4 below) and a variant of part (2) for $\mathbf{L}(\mathbb{R})$ and proper forcing was proved in [NZ01].

Let us remind the reader that, while large cardinals imply that for a pointclass Γ all sets of reals in $\mathbf{L}(\Gamma, \mathbb{R})$ are universally Baire, these sets are not universally Baire in $\mathbf{L}(\Gamma, \mathbb{R})$. A forcing notion \mathbb{P} is **weakly proper** if every countable set of ordinals in the extension is included in a ground-model countable set. In particular every proper forcing is weakly proper.

THEOREM 6.3. Suppose A is a set of reals such that all sets in $\mathbf{L}(A, \mathbb{R})$ are κ -universally Baire and $(A, \mathbb{R})^\#$ exists.

- (1) Then for every forcing notion $\mathbb{P} \in \mathbf{V}_\kappa$ and V -generic filter $G \subseteq \mathbb{P}$ there is a generic elementary embedding $j_G: \mathbf{L}(A, \mathbb{R})^\mathbf{V} \rightarrow \mathbf{L}(A, \mathbb{R})^{\mathbf{V}[G]}$.
- (2) If in addition AD and DC hold in $\mathbf{L}(A, \mathbb{R})$ and \mathbb{P} is weakly proper then the canonical j_G fixes all ordinals.

PROOF. (1) This is essentially [Woo10, Theorem 2.30], where analogous statement was proved under the assumption that all sets in $\mathbf{L}(A, \mathbb{R})$ were weakly homogeneously Suslin. In this case the assumption that $(A, \mathbb{R})^\#$ exists is automatic. The present theorem is proved using appropriately modified versions of [Woo10, Lemma 2.27–2.29], but we shall sketch a proof for the convenience of a reader. If $B \subseteq \mathbb{R}$ is in $\mathbf{L}(A, \mathbb{R})$ then we have trees T and S such that $p[T] = B$ and $p[S] = \mathbb{R} \setminus B$ and $p[T]$ and $p[S]$ cover \mathbb{R} in forcing extension by \mathbb{P} . We therefore let $j_G(B) = p[T]^{\mathbf{V}[G]}$. Note that $(A, \mathbb{R})^\#$ is a countable union of sets in $\mathbf{L}(A, \mathbb{R})$ (the n th set codes the theory of the first n indiscernibles), and is therefore universally Baire. If U and V are trees witnessing universal Baireness of $(A, \mathbb{R})^\#$ then an application of Lemma 6.2 easily shows that $p[U]^{\mathbf{V}[G]}$ still codes the first order diagram of

$$(H(\aleph_1)^{\mathbf{V}[G]}, \in, j_G(C))$$

where C ranges over universally Baire sets in $\mathbf{L}(A, \mathbb{R})$. Therefore map $j_G: (H(\aleph_1), \in, A)^\mathbf{V} \rightarrow (H(\aleph_1), \in, A)^{\mathbf{V}[G]}$ is elementary and so is its canonical extension $\bar{j}_G: \mathbf{L}(A, \mathbb{R})^\mathbf{V} \rightarrow \mathbf{L}(A, \mathbb{R})^{\mathbf{V}[G]}$.

(2) Let j_G be as defined in the proof of (1). The fact that j_G does not move ordinals is essentially [Woo10, Theorem 10.63]. Proof that j_G does not move ordinals relies on a result of Steel ([Woo10, Theorem 3.40]). \dashv

Another way to prove variants of Theorem 6.3 (1) in case when the generic filter of \mathbb{P} is determined by a single real x_G is to construe $\mathbf{L}(A, \mathbb{R})^{\mathbf{V}[G]}$ as a generic ultrapower of $\mathbf{L}(A, \mathbb{R})^\mathbf{V}$ associated with

$$\mathcal{U}(x_G) = \{B \in (\mathcal{P}(\mathbb{R}) \cap \mathbf{L}(A, \mathbb{R}))^\mathbf{V} : x_G \in B\}.$$

By using universal Baireness of sets in $\mathbf{L}(A, \mathbb{R})$, $\mathcal{U}(x_G)$ is a generic ultrafilter in $(\mathcal{P}(\mathbb{R}) \cap \mathbf{L}(A, \mathbb{R}))^\mathbf{V}$. In general case one can construe $\mathbf{L}(A, \mathbb{R})^{\mathbf{V}[G]}$ as a direct limit of ultrapowers of this sort (see [Woo10, p. 752]).

By a result of Woodin ([Woo10]) there exists a semiproper forcing \mathbb{P} such that the generic embedding as in Theorem 6.3 necessarily moves ordinals.

COROLLARY 6.4. Assume A satisfies the assumption of Theorem 6.3 (2) for an uncountable cardinal κ . In addition assume \mathbb{P} is a forcing notion of cardinality \aleph_1 such that every real added by \mathbb{P} is added by a countable regular subordering. Then the identity map is an elementary embedding of $\mathbf{L}(\Gamma, \mathbb{R})^{\mathbf{V}}$ into an inner model N that contains all reals.

PROOF. If $\kappa > \aleph_1$ and \mathbb{P} is weakly proper then this is an immediate consequence of Theorem 6.3, but it is exactly the case when $\kappa = \aleph_1$ that we shall need. By recursion we can extract a regular subordering \mathbb{Q} of \mathbb{P} that is an increasing ω_1 -chain of countable regular suborderings and such that the quotient \mathbb{P}/\mathbb{Q} does not add any new reals. We can write \mathbb{Q} as a direct limit of regular suborderings \mathbb{Q}_ξ , for $\xi < \omega_1$, each of which is forcing-equivalent to \mathbb{C} .

By iterating Theorem 6.3 ω_1 times we can conclude that $\mathbf{L}(A, \mathbb{R})^{\mathbf{V}}$ is an elementary submodel of $N = \mathbf{L}(A, \mathbb{R})^{V^{\mathbb{Q}}}$. After forcing by quotient \mathbb{P}/\mathbb{Q} no reals are added and therefore $\mathbf{L}(A, \mathbb{R})^{V^{\mathbb{P}}} = \mathbf{L}(A, \mathbb{R})^{V^{\mathbb{Q}}}$. This model contains all reals of the extension and therefore satisfies the conclusion of the corollary. \dashv

In the following lemma we assume \mathbb{P} is a forcing notion with the following properties:

- (1) $|\mathbb{P}| = \delta$,
- (2) \mathbb{P} collapses δ to \aleph_1 ,
- (3) for every \mathbb{P} -name \dot{x} for a real there exists a regular subordering \mathbb{Q} of \mathbb{P} such that \dot{x} is a \mathbb{Q} -name and $|\mathbb{Q}| < \delta$,
- (4) $\mathbb{P} \times \mathbb{P}$ has δ -chain condition.

LEMMA 6.5. Assume \mathbb{P} is as above and \dot{B} is a \mathbb{P} -name for a set of reals such that \mathbb{P} forces AD+DC hold in $\mathbf{L}(\dot{B}, \mathbb{R})^{V^{\mathbb{P}}}$, all sets in $\mathbf{L}(\dot{B}, \mathbb{R})$ are δ -universally Baire, and $(\dot{B}, \mathbb{R})^\#$ exists. Then $\mathbb{P} \times \mathbb{P}$ forces that there is an elementary embedding which fixes all reals and all ordinals from $\mathbf{L}(\dot{B}, \mathbb{R})^{V^{\mathbb{P}}}$ into an inner model of $V^{\mathbb{P} \times \mathbb{P}}$ that contains all reals.

PROOF. Since every real added by \mathbb{P} is added by a small forcing and since \mathbb{P} collapses its own cardinality to \aleph_1 , this is an immediate consequence of Corollary 6.4. \dashv

6.2. An iterable model with Woodin limit of Woodin cardinals.

As the final preparation for the proof of Theorem 6.1 we outline some properties of Neeman's mouse. Assume $W = \mathbf{L}[\vec{E}]$ is an inner model with cardinal δ which is a Woodin limit of Woodin cardinals as constructed in [Nee02A]. Then W has the following properties.

- (5) W is an extender model, $W = \mathbf{L}[\vec{E}]$, where \vec{E} witnesses Woodinness of both δ and cofinally many cardinals below δ .
- (6) For every δ -universally Baire set of reals A in W the model $\mathbf{L}(A, \mathbb{R})$ satisfies \mathbf{AD}^+ .
- (7) The transitive collapse of every countable $X \prec (\mathbf{V}_{\delta+1})^W$ is fully iterable with a δ -universally Baire iteration strategy.

Clause (5) is a result of [MS94]. Clause (6) is a result of Neeman [Nee02]. Clause (7) can be proved by using methods from [Ste] where similar statements were proved.

We shall need conditions (10) and (11) given below, while (8) and (9) will not be used outside of the present subsection. Let us call a countable model of the form $\mathbf{L}[\vec{E}]$ that contains a Woodin limit of Woodins **premouse**. Every premouse M has the canonical relative constructibility well-ordering $<_M$ of the reals. We shall refer to an iterable premouse as a **mouse**.

Recall that Γ_{uB} denotes the class of all δ -universally Baire sets of reals. A sentence ψ is $\Sigma_1^2(\Gamma_{\text{uB}})$ if there is $A \in \Gamma_{\text{uB}}$ and formula $\phi(X, Y)$, where X and Y are second-order variables, such that ψ is of the following form:

$$(\exists X \in \Gamma_{\text{uB}})(H(\aleph_1), \in, A, X) \models \phi.$$

Under our assumption that δ is a limit of Woodin cardinals $< \delta$ -homogeneously Suslin is equivalent to δ -universally Baire and this pointclass coincides with $\Sigma_1^2(\text{Hom}_{<\delta})$ considered in §5.3.

- (8) If M and N are $(\omega_1 + 1)$ -iterable mice then either $\mathbb{R}^M \subseteq \mathbb{R}^N$ and $<_M$ is an initial segment of $<_N$ or $\mathbb{R}^N \subseteq \mathbb{R}^M$ and $<_N$ is an initial segment of $<_M$.
- (9) The set of $(\omega_1 + 1)$ -iterable mice is $\Sigma_1^2(\Gamma_{\text{uB}})$.

Given M and N as in (8), a standard comparison argument (that can be reconstructed from the proof of Theorem 4.1⁵) shows that M and N have countable iterations $j: M \rightarrow M^*$ and $k: N \rightarrow N^*$ such that M^* is an initial segment of N^* or N^* is an initial segment of M^* . In either case, the well-orderings $<_{M^*}$ and $<_{N^*}$ coincide. Since the reals of a mouse and their well-ordering are unaffected by iteration, (8) follows.

Now we prove (9). A set Σ is an ω_1 -iteration strategy if and only if for all countable iteration trees T obtained when player II obeys Σ all branches chosen by Σ result in well-founded models. Therefore the assertion that Σ is an ω_1 -iteration strategy is projective in Σ . Therefore stating that a countable transitive model M is of the form $\mathbf{L}[\vec{E}]$ and is ω_1 -iterable with a δ -universally Baire iteration strategy is $\Sigma_1^2(\Gamma_{\text{uB}})$.

By Lemma 3.9, every such model is $\omega_1 + 1$ -iterable, and this completes the proof of (9).

⁵Actually the proof of Theorem 4.1 was based on the proof of Comparison Lemma

- (10) There is a $\Sigma_1^2(\Gamma_{\text{uB}})$ good well-ordering of the reals in W .
- (11) For every real r there is a δ -universally Baire set A such that r is ordinal definable in $\mathbf{L}(A, \mathbb{R})$.

By (7) every real belongs to an $\omega_1 + 1$ -iterable mouse with a δ -universally Baire strategy. Since canonical well-orderings of the reals in such mice cohere by (8), we can define well-ordering by letting $x < y$ if there exists an $\omega_1 + 1$ -iterable mouse M with a δ -universally Baire iteration strategy such that $x <_M y$. By (9) this is a $\Sigma_1^2(\Gamma_{\text{uB}})$ statement.

We finally prove (11). Let M be an $\omega_1 + 1$ -iterable mouse containing real x and let Σ be its δ -universally Baire strategy. Let α be such that x is the α -th real in $<_M$. By (8), x is the α -th real in every mouse N such that $x \in N$ and Σ is an $\omega_1 + 1$ -iteration strategy for N . We want to show that x is ordinal-definable from α in $\mathbf{L}(\Sigma \upharpoonright \omega_1, \mathbb{R})$. Since $\Sigma \upharpoonright \omega_1$ need not be δ -universally Baire in $\mathbf{L}(\Sigma \upharpoonright \omega_1, \mathbb{R})$, Lemma 3.9 does not apply. Nevertheless, $\Sigma \upharpoonright \omega_1$ can be extended to an $\omega_1 + 1$ -iteration strategy as follows. Since $\mathbf{L}(\Sigma \upharpoonright \omega_1, \mathbb{R})$ is a model of AD, \aleph_1 is a measurable cardinal in it. In particular it has the tree property and after an iteration game of length ω_1 player II can choose an ω_1 branch of the iteration tree. As a direct limit of well-founded models of length ω_1 , the model corresponding to this branch is well-founded.

6.3. Proof of Theorem 6.1. Assume that V is Neeman's model described in §6.2 in which δ is a Woodin limit of Woodin cardinals and continue numbering of formulas started in §6.2. Let M be the transitive collapse of an elementary submodel X of $H(\lambda)$ for a large enough $\lambda > \kappa$. An initial segment of M is an initial segment of $X \cap \mathbf{V}_{\kappa+1}$ and it therefore satisfies (7). This iteration strategy clearly gives an iteration strategy for M . Pick a Cohen real c over M . Now fix a δ -universally Baire set A as in (11) so that c is ordinal definable in $\mathbf{L}(A, \mathbb{R})$. Since W satisfies CH, both A and $A^\#$ can be identified with subsets of ω_1 .

Let $\mathbb{R} \upharpoonright \alpha$ be the first α reals in the well-ordering of the reals as in (10) and write $B \upharpoonright \alpha = B \cap (\mathbb{R} \upharpoonright \alpha)$ for $B \subseteq \mathbb{R}$. By applying Theorem 4.6 to a $C \subseteq \omega_1$ that codes $A, A^\#$ and trees witnessing \aleph_1 -universal Baireness of all sets in $\mathbf{L}(A, \mathbb{R})$, we can find an iteration $j: M \rightarrow M^*$ such that with $\alpha = j(\delta)$ the following hold:

- (12) $\mathbf{L}(C \upharpoonright \alpha, \mathbb{R} \upharpoonright \alpha) \prec \mathbf{L}(C, \mathbb{R})$,
- (13) the pair $(C \upharpoonright \alpha, \mathbb{R} \upharpoonright \alpha)$ is $j(\mathcal{W}_{\delta, \delta})$ -generic over M^* ,
- (14) every real in the extension is added by a small forcing,
- (15) the reals of the forcing extension from (13) are equal to $\mathbb{R} \upharpoonright \alpha$.

We also assure that α is large enough to have $c \in \mathbb{R} \upharpoonright \alpha$. Then for generic $G \subseteq j(\mathcal{W}_{\delta, \delta})$ as in (13) we have that in $M^*[G]$ real c is ordinal definable over a model of the form $\mathbf{L}(B, \mathbb{R})$ for some \aleph_1 -universally Baire set B , and this model contains all reals of the extension by (15). This model is a model

of AD^+ as witnessed by $A^\#$. Note that, since M and M^* have the same reals, c is Cohen over M^* .

By elementarity between M and M^* we can fix a condition p in $(\mathcal{W}_{\delta,\delta})^M$ which forces that there exist a real c , an \aleph_1 -universally Baire set B , and a countable ordinal α such that

- (16) c is the α -th real in the well-ordering of ordinal definable reals in $\mathbf{L}(B, \mathbb{R})$,
- (17) $\mathbf{L}(B, \mathbb{R}) \models \text{AD}^+$,
- (18) every new real belongs to a forcing extension by a forcing of cardinality $< \delta$,
- (19) c does not belong to any dense G_δ set coded in M ,

Let $G_1 \times G_2$ be a $\mathcal{W}_{\delta,\delta} \times \mathcal{W}_{\delta,\delta}$ -generic filter over M below (p, p) . We therefore have an \aleph_1 -universally Baire set B_1 and real c_1 in $M[G_1]$ such that in $\mathbf{L}(\mathbb{R}, B_1)^{M[G_1]}$ conditions (16)–(19) hold. We also have a universally Baire set B_2 and real c_2 in $M[G_2]$ such that (16)–(19) hold with the same α .

By Hjorth's Lemma 3.4 product $\mathcal{W}_{\delta,\delta} \times \mathcal{W}_{\delta,\delta}$ has δ -cc and $\mathcal{W}_{\delta,\delta}$ satisfies the assumptions of Lemma 6.5. By applying Lemma 6.5 twice, we find elementary embeddings

$$j_1: \mathbf{L}(\mathbb{R}, B_1)^{M[G_1]} \rightarrow \mathbf{L}(\mathbb{R}, B'_1)^{M[G_1][G_2]}$$

and

$$j_2: \mathbf{L}(\mathbb{R}, B_2)^{M[G_2]} \rightarrow \mathbf{L}(\mathbb{R}, B'_2)^{M[G_2][G_1]}.$$

Both j_1 and j_2 fix all reals and all ordinals.

Now assume that in $M[G_1][G_2]$ there are no divergent models of AD^+ . This applies to images of j_1 and j_2 . Therefore one of these models includes the other, and in this model both c_1 and c_2 are the α -th ordinal definable real in the well-ordering of ordinal-definable reals. We therefore must have that $c_1 = c_2$. This implies $c_1 \in M$, contradicting the fact that c_1 is a Cohen real over M .

§7. Other applications. A positive answer to the following (modulo sufficient large cardinals) was conjectured by John Steel (see also [Woo02]).

QUESTION 7.1. Assume ϕ is a Σ_2^2 sentence such that $\text{CH} + \phi$ holds in some forcing extension. Is it true that ϕ holds in every forcing extension that satisfies \diamond ?

Several partial positive results were proved in [DS10] and in [FKLM08]. By a result of Woodin (see [KLZ10]), if there is a measurable Woodin cardinal then there is a forcing \mathbb{P} that simultaneously forces every Σ_2^2 sentence ϕ that holds in some forcing extension satisfying CH . The forcing \mathbb{P} is the iteration of the collapse of 2^{\aleph_0} to \aleph_1 and another forcing notion. It is not known whether the collapse of 2^{\aleph_0} to \aleph_1 alone suffices for this conclusion.

The extender algebra has many other applications but it is time to finish (*cf.* [Mil95, p. 155]). For example, in [Lar09] a proof that the Ω -conjecture is true in many inner models (for example, Neeman's model briefly described in §6.2) was presented. One of the key points in the proof is Lemma 7.2 below. If A is a universally Baire set of reals and M is a transitive model of a large enough fragment of ZFC, we say that M is **A -closed** if for every generic extension $M[G]$ of M we have that $A \cap M[G] \in M[G]$. This condition is closely related to the conclusion of Lemma 6.2, the only difference being that in the present situation the language of ZFC is not expanded by adding a predicate for A . Note that the assumption of the following lemma is a consequence of clause (7) in §6.2.

LEMMA 7.2. Assume that δ is a Woodin cardinal and $\kappa > \delta$ is an inaccessible cardinal such that the transitive collapse of every elementary submodel $H \prec \mathbf{V}_\kappa$ is fully iterable with a universally Baire iteration strategy. Then for every universally Baire set of reals A there is a fully iterable A -closed transitive model. \dashv

The proof of Lemma 7.2 is identical to the proof of Lemma 6.2. Its conclusion, together with an application of Theorem 4.1, implies Ω -conjecture (see [Lar09, Theorem 9.2]).

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