THE FOURTH HEAD OF $\beta\mathbb{N}$

ILIJAS FARAH

Abstract. We start with a concrete problem about continuous maps on the Čech–Stone remainder of the natural numbers, $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, and gradually zoom out to the larger picture.

Start from $\mathbb{N}$, the space of natural numbers with the discrete topology, and consider its Čech–Stone compactification, $\beta\mathbb{N}$. This is the compactification of $\mathbb{N}$ such that every $f\colon \mathbb{N} \to [0,1]$ has a unique continuous extension $\tilde{f}\colon \beta\mathbb{N} \to [0,1]$. For the rest of this note all maps are continuous. In his introduction to $\beta\mathbb{N}$ ([28]), Jan van Mill called it a three-headed monster. The first head shows under the Continuum Hypothesis, CH, and it is 'smiling and friendly' since CH easily resolves problems about $\beta\mathbb{N}$ (more precisely, as easily as solutions to problems about $\beta\mathbb{N}$ get). The second head is the 'ugly head of independence' as Paul Erdös used to call it (the head which, in van Mill's own words, 'constantly tries to confuse you'). The smallest, third, head is the ZFC-head of $\beta\mathbb{N}$. It provides those few facts about $\beta\mathbb{N}$ that can be resolved without applying additional set-theoretic axioms. Ever since Shelah's groundbreaking results discussed below we are witnessing the emergence of the fourth head of $\beta\mathbb{N}$: A coherent theory of $\beta\mathbb{N}$ deduced from forcing axioms (or Ramseyan axioms) with strong rigidity phenomena for $\beta\mathbb{N}$ and similar Čech–Stone compactifications.

The reader is assumed to have only basic familiarity with the topology of $\beta\mathbb{N}$ and axiomatic set theory (see e.g., [11]).

1. Trivial continuous maps

Let us start with a concrete problem, naturally stated as a problem about the Čech–Stone remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. A map $f\colon \mathbb{N}^* \to \mathbb{N}^*$ is *trivial* if there is a function $h\colon \mathbb{N} \to \beta\mathbb{N}$ such that $f = \tilde{h} \restriction \mathbb{N}^*$. Assuming CH, it is very easy to construct nontrivial maps and even nontrivial autohomeomorphisms of $\mathbb{N}^*$. In [29], Shelah constructed a model of ZFC in which all autohomeomorphisms of $\mathbb{N}^*$ are trivial (of course, assuming there is a model of ZFC). In other words, he showed that a nontrivial autohomeomorphism of $\mathbb{N}^*$ cannot be constructed without using some additional set-theoretic axioms.

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Question 1. *Is it possible to construct a nontrivial map $f\colon \mathbb{N}^* \to \mathbb{N}^*$ without using additional set-theoretic axioms?*

Another way of stating this question (and similarly all the questions stated below) is: Assuming there is a model of ZFC, is there a model of ZFC in which there are no nontrivial maps $f\colon \mathbb{N}^* \to \mathbb{N}^*$? To an untrained eye this question may appear to be ad hoc, but please read on.

The existence of a nontrivial $f\colon \mathbb{N}^* \to \mathbb{N}^*$ implies the existence of both a nontrivial surjection $f\colon \mathbb{N}^* \to \mathbb{N}^*$ and a nontrivial injection $f\colon \mathbb{N}^* \to \mathbb{N}^*$ (dualize examples of [13, §3.2]). However, a nontrivial $f\colon \mathbb{N}^* \to \mathbb{N}^*$ that is both a surjection and an injection cannot be constructed without using additional set-theoretic axioms, by Shelah's result.

Question 2 (Dow)**.** *Is it possible to construct a nonseparable extremally disconnected image of* $\mathbb{N}^*$ *without using additional set-theoretic axioms?*

If Question 1 has a negative answer, so does Question 2. This is because the assumption that all maps $f\colon \mathbb{N}^* \to \mathbb{N}^*$ are trivial implies every extremally disconnected image of $\mathbb{N}^*$ is separable (see [13, Proposition 4.11.7]). Under the Proper Forcing Axiom, PFA,[1] all extremally disconnected images of $\mathbb{N}^*$ have countable cellularity (see e.g., [34]). M. Bell has constructed a nonseparable zero-dimensional image of $\mathbb{N}^*$ with countable cellularity ([2]).

An $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$ is *trivial* if it is of the form $\tilde{h}$ for some $h\colon \mathbb{N}^\kappa \to \mathbb{N}^\lambda$. Note that $\tilde{h}$ need not exist for an arbitrary h; see e.g., [16]. The triviality of all $f\colon \mathbb{N}^* \to \mathbb{N}^*$ is equivalent to its self-strengthening asserting that every map $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$ is trivial, where κ and λ are any two cardinals, finite or infinite (see [16]). This is a consequence of a phenomenon conjectured by van Douwen ([4]) and proved in [16]: If $f\colon (\mathbb{N}^*)^\kappa \to K$ for a compact space K, then the domain can be decomposed into finitely many clopen sets so that f depends on at most one coordinate on each one of the pieces. This remains true if $\mathbb{N}^*$ is replaced with any $\beta\mathbb{N}$*-space*: a space with the property that the closure of any infinite discrete subspace is homeomorphic to $\beta\mathbb{N}$.

An appealing variation on Question 2 is the following (a copy of $\mathbb{N}^*$ in a compact space X is *nontrivial* if it is nowhere dense and not of the form $\overline{D}\setminus D$ for a countable discrete $D \subseteq X$).

Question 3 (van Douwen)**.** *Is it possible to construct a nontrivial copy of* $\mathbb{N}^*$ *inside* $\mathbb{N}^*$ *without using additional set-theoretic axioms?*

This is closely related to Question 1. If every $f\colon \mathbb{N}^* \to \mathbb{N}^*$ is trivial then every copy of $\mathbb{N}^*$ inside $\mathbb{N}^*$ is trivial. Conversely, if all copies of $\mathbb{N}^*$ inside $\mathbb{N}^*$ are trivial and all autohomeomorphisms of $\mathbb{N}^*$ are trivial, then all injections $f\colon \mathbb{N}^* \to \mathbb{N}^*$ are trivial, and therefore all maps $f\colon \mathbb{N}^* \to \mathbb{N}^*$ are trivial. Under CH nontrivial copies of $\mathbb{N}^*$ exist in abundance. A natural way of assuring that a copy X of $\mathbb{N}^*$

[1]The exact statements PFA, OCA and MA can be found in [34], [20], [13], or any up-to-date text on combinatorial Set Theory. Todorcevic's OCA is different from its namesake introduced in [1]. Note that PFA implies both OCA and MA.

is nontrivial is to make it into a P-set (i.e., a set such that every G_δ superset of X includes an open neighbourhood of X). Todorcevic's Open Coloring Axiom, OCA, implies that no copy of $\mathbb{N}^*$ is a P-set ([21] for consistency, [25] from OCA; see also [13, Corollary 3.5.5]). As pointed out by A. Dow, it is not easy to find a nontrivial copy of $\mathbb{N}^*$ anywhere.

K.P. Hart noted that every $f\colon \mathbb{N}^* \to 2^\kappa$ is trivial, because every continuous $f\colon \mathbb{N}^* \to \{0,1\}$ extends to $\beta\mathbb{N}$. Therefore every copy of $\mathbb{N}^*$ in any 2^κ is trivial.

2. A partial result

Assuming MA and OCA the following was proved in [13], building on the work of Shelah–Steprāns, Todorcevic, Just and Velickovic (a map $f\colon \alpha^* \to \gamma^*$ is *trivial* if $f = \tilde{h} \restriction \alpha^*$ for some $h\colon \alpha \to \beta\gamma$).

Theorem 4 (OCA+MA)**.** *For any two locally compact countable spaces α and γ and $f\colon \alpha^* \to \gamma^*$ there is a clopen partition $\alpha^* = U \, dot\cup \, V$ such that $f \restriction U$ is trivial and $f[V]$ is nowhere dense.*

This easily implies that under OCA+MA $(\alpha^*)^\kappa$ maps onto $(\gamma^*)^\lambda$ if and only if this is witnessed by a trivial map. Also, $(\alpha^*)^\kappa$ and $(\gamma^*)^\lambda$ are homeomorphic if and only if this is witnessed by a trivial map. It is not difficult to characterize when such a trivial map exists; see [13, Theorem 4.5.1] for one-dimensional versions and [16, Theorem 4.6] for (a bit more difficult) higher-dimensional versions. Therefore Theorem 4 and analogous results reduce the highly complex problem of the existence of continuous maps between large topological spaces to a simple problem of countable combinatorics. The interest in our questions largely derives from such reductions of complexity. Just ([22]) first proved the consistency of the statement '$(\mathbb{N}^*)^d$ does not map onto $(\mathbb{N}^*)^{d+1}$ for any $d \in \mathbb{N}$.' He used a rather weak consequence of Theorem 4 proved in [24]. As an additional motivation for the program discussed here, the reader is invited to compare the complex proof of [22] with the straightforward calculation of the same result from a consequence of Theorem 4 given in [13, Theorem 4.6.1]; see also [16] and [17].

Parovičenko's theorem implies that under CH any two remainders α^* and γ^* of locally compact countable spaces are homeomorphic (as long as they are both nonempty. However, powers $(\alpha^*)^\kappa$ and $(\alpha^*)^\gamma$ are homeomorphic if and only if $\kappa = \gamma$ ([4]).

Question 5. *Is it possible to construct a nontrivial map between Čech–Stone remainders of locally compact countable spaces without using additional axioms of set theory?*

Again, the conclusion is equivalent to its self-strengthening asserting all maps between powers of such spaces are trivial. Admittedly, the restriction to the class of countable locally compact spaces (also known as 'countable ordinals') is ad hoc. An analogue of Theorem 4 holds for a slightly wider class of spaces; see [13, §4.10]. Pending an answer to Question 5, I will refrain from fantasizing about the widest class of spaces for which analogous rigidity results can be proved (but see [13, §§4.10–4.11]). It is nevertheless worth mentioning that PFA implies all autohomeomorphisms of D^* are trivial for every discrete space D ([36]).

3. Rigidity phenomena for quotients $\mathcal{P}(\mathbb{N})/I$

Via the Stone duality, all of the above discussion could be recast in terms of Boolean algebras. The space $\mathbb{N}^*$ is the Stone space of the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, where Fin is the ideal of finite subsets of $\mathbb{N}$. Hence Question 1 asks whether for every homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ there exists a sequence $\{\mathcal{U}_n\}$ of ultrafilters on $\mathbb{N}$ such that $\Phi([A]_{\mathrm{Fin}}) = [\{n \mid A \in \mathcal{U}_n\}]_{\mathrm{Fin}}$. Such a homomorphism is said to have an *additive lifting*. It is not difficult to see that each homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ for a countably generated ideal $\mathcal{I}$ has an additive lifting if and only if each homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ for a countably generated ideal $\mathcal{I}$ has an additive lifting (see [8, Theorem 3.3]). The algebraic reformulation suggests asking for which ideals $\mathcal{I}$ on $\mathbb{N}$ the following assertion is true:

($\mathrm{C}_{\mathcal{I}}$) Every homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has an additive lifting.

The topological dual, assertion that every continuous map g from a closed subset F of $\mathbb{N}^*$ into $\beta\mathbb{N}$ is trivial, is still meaningful for very simple subspaces F. For example, if F is an intersection (or a closure of the union) of countably many clopen sets then the analogue of Theorem 4 holds: g is a direct sum of a trivial map and a map with a nowhere dense range ([13, Theorem 3.9.2]). Again CH trivializes the question. I have conjectured that PFA implies ($\mathrm{C}_{\mathcal{I}}$) for every analytic ideal $\mathcal{I}$. (Consider $\mathcal{P}(\mathbb{N})$ with the Cantor-set topology; a set is *analytic* if it is a continuous image of the irrationals.) I reluctantly refrain from discoursing on this subject any further. The survey [19] of this conjecture is still up to date.

It seems plausible that all questions stated here have a positive answer, and more precisely that it follows from PFA.

4. Further results

At present it is unclear what the limitations of the rigidity phenomena described above are. It is, however, clear that they go a bit beyond the outlined framework. A small number of rigidity results for $\mathbb{N}^*$ were recently proved by combining lifting results with other techniques. For example, assuming OCA and MA, Dow and Hart have proved that a Čech–Stone remainder of a locally compact, σ-compact space X is a continuous image of $\mathbb{N}^*$ if and only if X is homeomorphic to a sum of $\mathbb{N}$ with a compact space ([9]). From the same assumptions they also deduced that the Stone space of the Lebesgue measure algebra is not a continuous image of $\mathbb{N}^*$ ([10]). Both statements contradict the conclusion of Parovičenko's theorem. I proved ([14, §8]) that OCA implies $\operatorname{Exp}(\mathbb{N}^*)$ is not a continuous image of $\mathbb{N}^*$, thus confirming a conjecture of M. Bell. Dow ([7]) proved that PFA implies every two-to-one image of $\mathbb{N}^*$ is *trivial*: whenever $f\colon \mathbb{N}^* \to X$ is such that each fibre has exactly two points then X is homeomorphic to $\mathbb{N}^*$ and moreover f has to be a trivial map in the sense of Question 1. Needles to say, CH implies the existence of nontrivial two-to-one maps. In Dow's result it is important that each fibre has exactly two points; van Douwen ([5]) has constructed a nontrivial $f\colon \mathbb{N}^* \to X$ such that each fibre has at most two points. It is not known whether it is possible that a two-to-one image of $\mathbb{N}^*$ is not homeomorphic to $\mathbb{N}^*$; curiously enough, a negative answer follows from CH ([12]).

Question 6. *Is it possible to construct a nontrivial n-to-one map on $\mathbb{N}^*$ for some $n \in \mathbb{N}$ without using additional axioms of set theory?*

This subject would not be complete without the insight provided by Murray Bell.

Question 7 (M. Bell)**.** *Is it possible to construct an extremally disconnected image of a zero set in $\mathbb{N}^*$ that is not an image of $\mathbb{N}^*$ without using additional set-theoretic axioms?*

Some partial answers to this question were obtained in [8]. The *rectangle algebra* is the algebra of subsets of $\mathbb{N}^{\mathbb{N}}$ generated by rectangles, $\prod_{i=1}^{\infty} A_i$.

Question 8 (M. Bell)**.** *Is it possible to show that the Stone space of the rectangle algebra is a continuous image of $\mathbb{N}^*$ without using additional set-theoretic axioms?*

I would conjecture that in both cases there is either a (properly defined) 'trivial' map witnessing the connection or appropriate axioms imply there is no surjection.

5. Conclusion

The main purpose of this note was to draw the attention of topologists and set theorists to an emerging canonical theory of spaces and Boolean algebras closely related to $\mathbb{N}^*$ and $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ respectively. Under CH, two such spaces that could possibly be related (via a homeomorphism or a surjection) are indeed related. A variant of this sweeping claim is a consequence of Woodin's Σ^2_1-absoluteness theorem ([37], [15]); see [13, §2.1] and also §5.1 below.

This note is about the other extreme situation. For spaces X and Y define the notion of 'trivial' map $f\colon X \to Y$. This notion should be simple so that deciding the existence of a trivial homeomorphism/surjection between X and Y is reasonably easy, and that the statement 'there is a trivial isomorphism (or surjection) between X and Y' is absolute between sufficiently closed models of ZFC. It should also be well-chosen so that forcing (or Ramseyan) axioms imply every isomorphism (or surjection) between X and Y has to be trivial. This 'ideal' scenario is rather flexible. For example, in the case when $X = Y$ it serves to completely describe the group of all autohomeomorphisms of X: Take the case when $X = \mathbb{N}^*$ ([29]), from where all this has started.

In most situations it is sufficient to prove that sufficiently strong forcing (or Ramseyan) axioms imply that if there is a homeomorphism (or surjection) $f\colon X \to Y$ then a trivial homeomorphism (or surjection) exists. In many concrete cases this is a theorem; see [13, §2.1]. The existence of a trivial connecting map is typically a $\mathbf{\Sigma}^1_2$ statement, and therefore absolute by Shoenfield's Absoluteness Theorem.

Having a general lifting theorem greatly simplifies the question whether two spaces are homeomorphic or otherwise related. Compare e.g., [23], where a weak lifting theorem was supplemented by a technical tour de force argument and the proof of the same result in [13, Corollary 3.4.4, Proposition 1.13.13] where a strong lifting theorem was supplemented by straightforward computations. Isolating the notion of 'trivial' connecting map is not necessary to prove rigidity results. Take, for example, the Dow–Hart result on the Stone space of the Lebesgue measure algebra ([10]).

The problems of determining the relation between lifting statements such as those considered above and other set-theoretic statements are difficult and well-studied, but this is an another story (see e.g., [33], [32], [31]).

5.1. **Metamathematics.** The phenomenon that CH resolves so many questions about $\mathbb{N}^*$ has a metamathematical explanation or two. Model-theoretically (see [3] for model-theoretic background), $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a countably saturated Boolean algebra, hence CH implies it is a saturated model of the (complete) theory of atomless Boolean algebras. Clopen algebras of other Parovičenko spaces are also countably saturated, and this allows one to apply back-and-forth methods to relate these and similar algebras.

Another explanation is of a different nature. Instead of giving a technical device for constructing maps, it implies that maps that can be constructed in some models of set theory can also be constructed using CH. Hence CH is an optimal assumption for finding such maps. Let X and Y be spaces whose basic open sets can be coded by real numbers; for example, Čech–Stone compactifications of countable, locally compact spaces, as well as their finite powers, are of this form. A continuous map $f\colon X \to Y$ can be coded by a set of pairs of basic open subsets of X and Y, and therefore by a set of real numbers, C_f. Statements like 'f is onto' or 'f is a homeomorphism' are *projective in* C_f: they can be expressed using quantification over the real numbers only. Thus 'α^* and γ^* are homeomorphic' (for countable ordinals α and γ) is equivalent to a statement of the form $(\exists C \subseteq \mathbb{R})\phi(C)$ for a statement ϕ projective in C. A statement of this syntactical form is called a Σ^2_1-*statement*. Using a large cardinal assumption, Woodin proved ([37], see [15] or [27]) that if a Σ^2_1 statement can be forced then it holds in every forcing extension that satisfies CH.

Consequences of OCA+MA fit together forming a coherent picture of $\mathbb{N}^*$ and related spaces, with their rigidity properties maximized (see [13, §2.1] for an overview). A satisfactory metamathematical explanation of this phenomenon is yet to be found, but the current state of our understanding suggests that the ability to make gaps in quotient algebras indestructible is of central importance. The gap-freezing technique was developed in [34, §8] (cf. [1]) and first employed in this context in [30]. See [6] for analysis of gaps in $[\kappa]^\omega/\operatorname{Fin}$ or [18] for gaps in quotients of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$; for an approach compatible with CH see [35].

Notably, OCA and MA both hold in Woodin's canonical model for the negation of CH ([38]; see [26]). A discussion of the wider metamathematical context is beyond the scope of this article and it appears elsewhere ([19], [13, §2.1]).

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 35, Belgrade

E-mail address: ifarah@mathstat.yorku.ca

URL: http://www.math.yorku.ca/∼ifarah