NONHOMOGENEITY IN PRODUCTS WITH $\beta\mathbb{N}$-SPACES

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This note is a contribution to the study of large compact homogeneous spaces (see [5], [7]) and homogeneity properties of products of compact spaces in particular. It is an application of results from [4] and an attempt to draw more attention to these results.

Infinite products of zero-dimensional first-countable spaces are homogeneous ([2]), but the homogeneity of certain spaces cannot be improved by taking powers, or even products. Consider the following property of a space X.

(*) $X \times Y$ is not homogeneous for any compact space Y.

This property is shared by $(\omega_1 + 1)$ ([3]) and the 'topologist's sine curve' (Motorov proved it is not a retract of any compact homogeneous space). In [5] Kunen pointed out that it is not known whether (*) holds for all – or any – infinite F-space(s) and proved a weakening of (*) for infinite F-spaces. His theorem implies, for example, that a product of any family of (nontrivial) F-spaces is never homogeneous. Instead of F-spaces we work with a slightly larger class of compact $\beta\mathbb{N}$-*spaces* introduced in [1] (see below for definitions). Kunen's result and its proof remain correct if 'F-space' is replaced by '$\beta\mathbb{N}$-space' everywhere.

Theorem 1. *If X is an infinite connected $\beta\mathbb{N}$-space then the product $X \times Y$ is not homogeneous for any compact space Y. In particular, the Čech–Stone remainder of the half-line has the property (*).*

The main ingredients of the proof are the standard proof of the nonhomogeneity of $\mathbb{N}^*$ (Lemma 3 below) and the analysis of maps from products of compact spaces into $\beta\mathbb{N}$-spaces (Theorem 4 below). Some support for the conjecture that Theorem 1 continues to hold if connectedness is dropped from its assumptions is given by the following theorem, proved at the very end of this note.

Theorem 2. *Assume Y is a compact space such that $\mathbb{N}^* \times Y$ is homogeneous. If $\alpha \geq 0$ is such that Y is homeomorphic to $(\mathbb{N}^*)^\alpha \times Y'$ for some Y', then Y' is homeomorphic to $\mathbb{N}^* \times Y''$ for some Y''.*

Therefore, if (*) fails for an infinite compact $\beta\mathbb{N}$-space X then any compact space Y for which $X \times Y$ is homogeneous has to have rather unusual properties. I don't know whether there is a space Y that satisfies the conclusion of Theorem 2. Alan Dow pointed out that if X is any finite Hausdorff space then $X^\alpha \times \mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ if and only if α is finite.

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Definitions and background. All spaces are assumed to be Hausdorff. Following [1], we say that a space X is a $\beta\mathbb{N}$-*space* if for every countably infinite relatively discrete $D \subseteq X$ such that $\overline{D}$ is compact we have that $\overline{D}$ is homeomorphic to $\beta\mathbb{N}$. If X is compact this is equivalent to stating every countable relatively discrete $D \subseteq X$ is C^*-embedded, and in particular all F-spaces are $\beta\mathbb{N}$-spaces.

The projection of $X \times Y$ to X is denoted by p_X or by p_1, the projection to Y is denoted by p_Y or by p_2, etc. A function $f\colon X \times Y \to A$ *does not depend on the X-coordinate* if there is $g\colon Y \to A$ such that $f(x,y) = g(y)$ for all x, y. Similarly, $f\colon \prod_{i\in I} X_i \to A$ *depends on at most one coordinate* if there is $i \in I$ and $g\colon X_i \to A$ such that, with p_i being the projection to the i-th coordinate, we have $f(x) = g(p_i(x))$ for all x.

If p is an ultrafilter on $\mathbb{N}$ then a point a is a *nontrivial p-limit* in space X if there is a discrete sequence d_n $(n \in \mathbb{N})$ in X such that $\lim_{n\to p} d_n = x$. The well-known Lemma 3 below was used in Kunen's proof and it will be used in proofs of this note. Proofs of its parts (1) and (2) can be found in [6, Theorem 3.1.6] and [6, Theorem 3.4.1] or [5, Lemma 4], respectively.

Lemma 3. (1) *There are Rudin–Keisler incomparable ultrafilters on* $\mathbb{N}$.
(2) *If p and q are Rudin-Keisler incomparable ultrafilters on $\mathbb{N}$, then a homeomorphism between compact $\beta\mathbb{N}$-spaces cannot send a nontrivial p-limit to a nontrivial q-limit.* □

The case of connected $\beta\mathbb{N}$-spaces

The following is an instance of a phenomenon first isolated by van Douwen ([1]).

Theorem 4. *If $\prod_{i\in I} X_i$ are compact spaces and Z is a $\beta\mathbb{N}$-space then for every continuous map $f\colon \prod_{i\in I} X_i \to Z$ there is a cover of $\prod_{i\in I} X_i$ by clopen sets such that the restriction of f to each of these sets depends on at most one coordinate.*

Proof. The case when $X_i = X_j$ for all i, j was proved in [4, Theorem 3], and the proof of the more general statement is identical. □

Lemma 5. *Assume $f\colon X \times Y \to A \times B$ is a homeomorphism such that $p_1 \circ f$ does not depend on the X-coordinate. Then there is a subspace B' of Y such that the restriction of $p_B \circ f$ to $X \times B'$ is a homeomorphism with range B.*

Proof. Fix $f_1\colon Y \to A$ such that $f(x,y) = f_1(y)$ for all x, y and fix $a \in A$. Let $B' = \{y \in Y \mid f_1(y) = a\}$. Then $f^{-1}(\{a\} \times B) = X \times B'$: $f(x,y) \in \{a\} \times B$ iff $y \in B'$. Therefore $b \mapsto f^{-1}(a,b)$ is a homeomorphism between B and $X \times B'$. □

Lemma 6. *Assume $f\colon X \times Y \to A \times B$ is a homeomorphism such that $p_1 \circ f$ does not depend on the X-coordinate and $p_1 \circ f^{-1}$ does not depend on the B-coordinate. Then $x \mapsto p_1(f(x,b))$ is a homeomorphism between X and A for any fixed $b \in Y$.*

Proof. We have maps $f_1\colon X \times Y \to A$ such that $f_1(x) = p_1 \circ f(x,y)$ for all y and $g_1\colon A \times B \to X$ such that $g_1(x) = p_1 \circ f^{-1}(x,y)$ for all y. Thus $g_1 \circ f_1$ is the identity on X and $f_1 \circ g_1$ is the identity on A. Both maps are continous, hence f_1 is a homeomorphism between X and A. □

Proof of Theorem 1. Assume $X \times Y$ is compact and homogeneous. Then each one of its connected components is homogeneous; by going to a connected component of Y we can assume Y is connected. Let $\mathcal{F}$ be a family of surjections $f\colon Y \to X$

such that for every finite $s \subseteq \mathcal{F}$ the map $y \mapsto \langle f(y) \mid f \in s\rangle$ is onto X^s; let $\mathcal{F}$ be a maximal family with this property (possibly $\mathcal{F} = \emptyset$). Then the range of the map $y \mapsto g(y) = \langle f(y) \mid f \in \mathcal{F}\rangle$ is dense in $X^{\mathcal{F}}$, and hence $g\colon Y \to X^{\mathcal{F}}$ is onto. By Lemma 3 fix Rudin–Keisler incomparable points p and q in $\mathbb{N}^*$ and a sequence $\{y^n\}$ in Y such that the sequence $\{f(y^n)\}$ is discrete in X for every $f \in \mathcal{F}$. Let $a = \lim_{n\to p} y^n$ and $b = \lim_{n\to q} y^n$, and let a_1, b_1 in X be a nontrivial p-limit and a nontrivial q-limit, respectively. By homogeneity, there is an autohomeomorphism h of $X \times Y$ such that $h(a_1, a) = (b_1, b)$. Consider $p_1 \circ h\colon X \times Y \to X$. By Theorem 4, we can cover $X \times Y$ by finitely many clopen sets such that on each one $p_1 \circ h$ depends on at most one coordinate. Since $X \times Y$ is connected, $p_1 \circ h$ depends on at most one coordinate.

Assume first $p_1 \circ h$ depends only on the X-coordinate; thus there is a continuous map $h_1\colon X \to X$ such that $h_1(x) = p_1 \circ h(x, y)$ for all x and y. Also, $p_1 \circ h^{-1}$ depends only on one coordinate. This easily has to be the X-coordinate. But then Lemma 6 implies h_1 is an autohomeomorphism of X mapping a nontrivial p-limit to a nontrivial q-limit, which contradicts Lemma 3. Therefore $p_1 \circ h$ depends only on the Y-coordinate, and we have a continuous $h_2\colon Y \to X$ such that $h(x, y) = h_2(y)$ for all x, y.

By Lemma 5 there is a subspace Y' of Y such that the restriction of $p_Y \circ h$ to $X \times Y'$ is a homeomorphism with Y. Let us identify $X \times Y'$ with Y. By Theorem 4 and connectedness, each $f \in \mathcal{F}$ depends on at most one coordinate on $X \times Y'$. Since $f(a_1, a)$ is a nontrivial q-limit, Lemma 3 implies that f depends only on Y'-coordinate. Hence the projection p_1 of $X \times Y'$ to X is not in $\mathcal{F}$. We claim that $\mathcal{F} \cup \{p\}$ contradicts the maximality of $\mathcal{F}$. If $c \in X$ and $d \in X^{\mathcal{F}}$ we need to find a point mapped to (c, d) by $p \times g$. By the property of $\mathcal{F}$, there is $(x, y) \in X \times Y'$ such that $g(x, y) = d$ and therefore $(c, y) \mapsto (c, d)$. □

Dropping connectedness

A space Y has X as a *factor* if Y is homeomorphic to $X \times Z$ for some Z. Consider a homeomorphism $f\colon \prod_{\xi<\kappa} X_\xi \to \prod_{\xi<\lambda} Y_\xi$ and x in its domain. Let $I = \{\xi < \kappa \mid x(\xi) \text{ is not isolated}\}$ and $J = \{\xi < \lambda \mid f(x)(\xi) \text{ is not isolated}\}$. We say f is *trivial at the point* $x \in \prod_{\xi<\kappa} X_\xi$ if there is a bijection $\alpha\colon J \to I$, open sets $U_{\alpha(\xi)} \ni x(\alpha(\xi))$ ($\xi \in I$), $V_\xi \ni f(x)(\xi)$ ($\xi \in J$) and homeomorphisms $g_\xi\colon U_{\alpha(\xi)} \to V_\xi$ such that (letting $U_\xi = \{x(\xi)\}$ for $\xi \in \kappa \setminus I$) for all $z \in \prod_{\eta<\kappa} U_\eta$ we have $f(z)(\xi) = g_\xi(z(\alpha(\xi))$ for all $\xi < \kappa$. If for some $I \subseteq \lambda$ we relax the conditions by allowing $U_{\alpha(\xi)}$ and V_ξ to be singletons (not necessarily open) for $\xi \notin I$ and keep the other conditions unchanged then we say f is *trivial at x and coordinates in I and $\alpha[I]$*.

The following is a straightforward consequence of Theorem 4 and I was quite surprised to find out that it was not stated in [4].

Lemma 7. *Assume X_ξ ($\xi < \kappa$) and Y_ξ ($\xi < \lambda$) are compact spaces such that all X_ξ for $\xi \neq 0$ and all Y_ξ for $\xi \neq 0$ are $\beta\mathbb{N}$-spaces. Assume $f\colon \prod_{\xi<\kappa} X_\xi \to \prod_{\xi<\lambda} Y_\xi$ is a homeomorphism and $x \in \prod_{\xi<\kappa} X_\xi$ is such that none of the points $x(\xi)$, $f(x)(\xi)$ for $\xi \neq 0$ is isolated, and neither $x(0)$ nor $f(x)(0)$ has a clopen neighborhood that has an infinite $\beta\mathbb{N}$-space as a factor. Then f is trivial at x and coordinates in $\lambda \setminus \{0\}$ and $\kappa \setminus \{0\}$.*

Proof. By Theorem 4, for each $\xi \in \lambda \setminus \{0\}$ there is a clopen $W_\xi \ni x$ such that the restriction of $p_\xi \circ f$ to W_ξ depends on at most one coordinate; call it $\alpha(\xi)$. Let $I = \{\xi \in \lambda \setminus \{0\} \mid \alpha(\xi) \neq 0\}$. Again by Theorem 4, for $\xi \in I$ find clopen $W'_\xi \ni f(x)$ such that the restriction of $p_{\alpha(\xi)} \circ f^{-1}$ to W'_ξ depends on at most one coordinate, $\beta(\xi)$. Since neither one of $f(x)(\xi)$ or $x(\xi)$ is isolated, we must have $\beta(\xi) = \xi$. Shrink $W_\xi \cap f^{-1}[W'_\xi]$ to a clopen $U_{\alpha(\xi)} \ni x(\xi)$ and $W'_\xi \cap f[W_\xi]$ to a clopen $V_\xi \ni f(x)(\xi)$ so that there is a homeomorphism $g_\xi \colon U_{\alpha(\xi)} \to V_\xi$ satisfying $f(y)(\xi) = g_\xi(y(\alpha(\xi))$ for all $y \in p^{-1}_{\alpha(\xi)} U_{\alpha(\xi)}$.

For $\xi \in \kappa \setminus \{0\}$ there are a clopen $W \ni x$ and $\alpha'(\xi) < \lambda$ such that $\pi_\xi \circ f^{-1}$ depends at most on $\alpha'(\xi)$-th coordinate. Assume for a moment $\alpha'(\xi) = 0$. Then Lemma 5 implies that $p_0[W]$ has $(p_\xi \circ f^{-1})(W)$ as a factor. Since $x(\xi)$ is not an isolated point, this is a nontrivial $\beta\mathbb{N}$-space, contradicting our assumption on Y_0. Therefore $\alpha'(\xi) \neq 0$ and we have $\xi = \alpha(\alpha'(\xi))$, hence the range of α includes $\kappa \setminus \{0\}$. A symmetric argument shows that $\alpha(\xi) \neq 0$ for all ξ, and therefore α is a bijection between $\lambda \setminus \{0\}$ and $\kappa \setminus \{0\}$ and the proof is complete. □

Lemma 8. *Assume X is an infinite compact $\beta\mathbb{N}$-space and Y is a compact space that has a clopen subset U homeomorphic to a product of a (possibly empty) family of infinite $\beta\mathbb{N}$-spaces and a space that is not homeomorphic to a direct sum of spaces of the form $V \times Y'$ for V a clopen subset of X. Then $X \times Y$ is not homogeneous.*

Proof. Let U be homeomorphic to $\prod_{\xi<\kappa} X_\xi$ such that each X_ξ $(\xi > 0)$ is an infinite $\beta\mathbb{N}$-space and some $x(0) \in X_0$ does not have a clopen neighborhood that has an infinite $\beta\mathbb{N}$-space as a factor.

Pick $x \in X \times Y$ so that each of its coordinates (except the one at X_0) is a nontrivial p-limit and $y \in X \times Y$ so that each of its coordinates (except the one at X_0) is a nontrivial q-limit. Assume f is an autohomeomorphism of $X \times Y$ such that $f(x) = y$. By Lemma 7 we have a homeomorphism between compact $\beta\mathbb{N}$-spaces that sends a p-limit to a q-limit, contradicting Lemma 3. □

Any two nonempty clopen subsets of $\mathbb{N}^*$ are homeomorphic, and Theorem 2 is a consequence of Lemma 8.

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