A TWIST OF PROJECTIONS IN THE CALKIN ALGEBRA

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The reader is referred to [1] for terminology. Let $\pi\colon \mathcal{B}(H)\to\mathcal{C}(H)$ be the natural projection to the Calkin algebra. For $a\in\mathcal{B}(H)$ we sometimes write $\dot{a}=\pi(a)$. For a projection P we write $P^1=P$ and $P^\perp=I-P$.

Theorem 1. *There is an orthogonal family $\mathcal{F}$ of $\aleph_1$ projections in the Calkin algebra $\mathcal{C}(H)$ that is not contained in $\pi[\mathcal{A}]$ for any atomic masa $\mathcal{A}$ of $\mathcal{B}(H)$. Moreover, there is no projection Q in $\mathcal{B}(H)$ such that both families $\{P\in\mathcal{F}\mid P(I-Q)$ is compact$\}$ and $\{P\in\mathcal{F}\mid PQ$ is compact$\}$ are uncountable.*

The proof is given after two preliminary lemmas. Fix an increasing sequence R_n of finite-dimensional projections in $\mathcal{B}(H)$ such that $\bigvee_n R_n=I$ and write $\|a\|_n=\|(I-R_n)a\|$.

Lemma 2. *Assume $\mathcal{F}$ is an uncountable family of projections in $\mathcal{B}(H)$ such that $\dot{P}\dot{Q}=\dot{Q}\dot{P}$ for all P,Q in $\mathcal{F}$ but there is $\varepsilon>0$ such that for any two distinct P,Q in $\mathcal{F}$ we have $\|PQ-QP\|>\varepsilon$.*

1. *Then for any atomic masa $\mathcal{A}$ in $\mathcal{B}(H)$ the set $\{P\in\mathcal{F}\mid\dot{P}\in\pi[\mathcal{A}]\}$ is countable.*
2. *If moreover there is an enumeration $(P_\xi)_{\xi<\omega_1}$ of $\mathcal{F}$ such that for every ξ and n the set $\{\eta<\xi\|\|P_\eta P_\xi\|_n<\varepsilon\}$ is finite, then there is no projection Q in $\mathcal{B}(H)$ such that both sets $\{\xi\mid P_\xi Q$ is compact$\}$ and $\{\xi\mid P_\xi(I-Q)$ is compact$\}$ are uncountable.*

Proof. (1) Assume the contrary, and fix a basis (e_i) that diagonalizes operators in $\mathcal{A}$, and for $m\in\mathbb{N}$ let E_m denote the closed linear span of $\{e_i\mid i\leq m\}$. For each $\dot{P}\in\pi[\mathcal{A}]$ there is $X_P\subseteq\mathbb{N}$ such that with Q^{X_P} being the projection to the closed span of $\{e_i\mid i\in X_P\}$ we have $\dot{Q}^{X_P}=\dot{P}$. For such P we can find $m=m(P)$ such that for all unit vectors x in the orthogonal of E_m we have (writing $Q=Q^{X_P}$) $\|Px-Qx\|<\varepsilon/12$. By possibly increasing m, we can also assume that for each unit y in the span of E_m we have (writing R for the orthogonal projection onto E_m) $\|RPy-Py\|<\varepsilon/12$, and therefore $\|PR-RPR\|<\varepsilon/12$. Fix m such that $m=m(P)$ for uncountably many P. Since $\mathcal{B}(E_m)$ is separable in the norm topology, we can further refine the set of P to find a projection $S\leq R$ such that $\|RPR-S\|<\varepsilon/12$ for all P.

Now consider any of the remaining P. We have $\|P(I-R)-Q^{X_P}(I-R)\|<\varepsilon/12$, and consequently for any $Y\subseteq[m,\infty)$ we have $\|PQ^Y-Q^{X_P}Q^Y\|<\varepsilon/12$. We may assume $X_P\subseteq[m,\infty)$ and therefore $Q^{X_P}(I-R)=Q^{X_P}$ for all P. For two of the remaining P and P' we have $\|PP'(I-R)-P'P(I-R)\|<\|PQ^{X_{P'}}-P'Q^{X_P}\|+2\varepsilon/12<\|Q^{X_P}Q^{X_{P'}}-Q^{X'_P}Q^{X_P}\|+4\varepsilon/12=4\varepsilon/12$.

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Also $\|(PP' - P'P)R\| < \|PRP'R - P'RPR\| + 2\varepsilon/12 < \|(RPR)(RP'R) - (RP'R)(RPR)\| + 4\varepsilon/12 < \|SS - SS\| + 8\varepsilon/12$. Hence $\|PP' - P'P\| < \|PP'R - P'PR\| + \|PP'(I-R) - P'P(I-R)\| < 12\varepsilon/12 = \varepsilon$ contradicting the assumption. □

Lemma 3. *Given any sequence P_i of projections in $\mathcal{B}(H)$ such that $\dot{P}_i$ and $\dot{P}_j$ commute for all j, there is an atomic masa $\mathcal{A}$ in $\mathcal{B}(H)$ such that $\pi[\mathcal{A}]$ contains all $\dot{P}_i$.*

Proof. This is a consequence of the fact that every projection in $\mathcal{C}(H)$ is equal to $\dot{P}$ for some projection P in $\mathcal{B}(H)$ ([1, Lemma 3.1]). We will recursively choose projections Q_i in $\mathcal{B}(H)$ and orthonormal basis e_i so that for all $i \leq j$ we have $\dot{Q}_i = \dot{P}_i$ and $Q_i(e_j) \in \{e_j, 0\}$. Assume Q_i and e_i ($i < n$) have been chosen to satisfy these requirements. Let R be the projection to the orthogonal of $\{e_i \mid i < n\}$ and for each $\alpha \in \{1, \perp\}^n$ let $R_\alpha = R_\alpha(n) = R\prod_{i<n} Q_i^{\alpha(i)}$. For each $\alpha \in \{1, \perp\}^n$ we have that $\dot{P}_n$ and $\dot{R}_\alpha$ commute, hence by [1, Lemma 3.1] there is a projection P_α in $\mathcal{B}(R_\alpha[H])$ such that $\dot{P}_\alpha = \dot{P}_n\dot{R}_\alpha$, and $\dot{P}_n = \sum_\alpha \dot{P}_n\dot{R}_\alpha$. Now let e_n be any unit vector in some $R_\alpha Q_n H$; then it is orthogonal to all e_i ($i < n$) and it satisfies $Q_i e_n \in \{e_n, 0\}$ for all $i \leq n$. We can also assure (e_n) span a dense subspace of H by assuring each unit vector in a prescribed countable norm-dense subset of the unit ball is approximated by linear combinations of (e_n) up to an arbitrarily small error. Let $X_i = \{n \mid n \geq i$ and the unique $\alpha \in \{1, \perp\}^n$ such that $e_n \in R_\alpha(n)$ satisfies $\alpha(i) = 1\}$. Fix $i \in \mathbb{N}$. Clearly Q^{X_i} (as defined in the proof of Lemma 2) satisfies $\dot{Q}^{X_i} = \dot{P}_i$ and it is diagonalized by (e_n).

(2) Assume the contrary and fix Q. Since PQ is compact if and only if QP is compact if and only if $\|PQ\|_n \to 0$ as $n \to \infty$, by a counting argument we may find n such that the sets $\{\xi \mid \|(I-Q)P_\xi\|_n < \varepsilon/2\}$ and $\{\xi \mid \|P_\xi Q\|_n < \varepsilon/2\}$ are both uncountable. Fix ξ in the latter such that there are infinitely many $\eta < \xi$ in the former set. For such ξ and η we have $\|P_\xi P_\eta\|_n = \|P_\xi Q\|_n + \|(I-Q)P_\eta\|_n < \varepsilon$, a contradiction. □

Proof of Theorem 1. We shall recursively construct projections P_ξ ($\xi < \omega_1$) in $\mathcal{B}(H)$ that has the following properties for all $\xi \neq \eta$.

(1) $P_\xi P_\eta$ is compact,
(2) $\|P_\xi P_\eta - P_\eta P_\xi\| > 1/4$,
(3) For every $\xi < \omega_1$ and $n \in \mathbb{N}$ the set $\{\eta < \xi \mid \|P_\eta P_\xi\|_n < 1/2\}$ is finite.

At finite stages of the recursion we need to take extra care to assure that $I - \bigvee_{i=0}^{n-1} P_i$ is not compact; straightforward details are suppressed.

Assume P_ξ ($\xi < \zeta$) have been chosen to satisfy all relevant instances of (1)–(2). By Lemma 3 find an orthonormal basis (e_n) that diagonalizes all P_ξ ($\xi < \zeta$). Re-enumerate these projections as Q_i ($i \in \mathbb{N}$) and fix $X_i \subseteq \mathbb{N}$ such that $\dot{Q}_i = \dot{Q}^{X_i}$ (with Q^{X_i} as defined in the proof of Lemma 2) for all i. Let S_i denote the projection to the span of $(e_j \mid j \leq i)$. Now recursively define the following:

(4) a sequence (l_i) in $\mathbb{N}$ such that $\|(Q_j - Q^{X_j})(I - S_{l_i})\| < 1/8$ for all $j \leq i$ and such that for every unit vector v in the span of $\{e_n \mid n \geq i\}$ we have $\|v\|_i > 1/\sqrt{2}$.
(5) Distinct $n_i(0)$ and $n_i(1)$ in $[l_i, l_{i+1})$ satisfying
    (a) For each $j < i$ we have $\{n_i(0), n_i(1)\} \cap X_j = \emptyset$, and
    (b) $\{n_i(0), n_i(1)\} \cap X_i = \{n_i(0)\}$.

The construction is straightforward. When these objects are constructed we let $f_i = \frac{1}{\sqrt{2}}(e_{n_i(0)} + e_{n_i(1)})$; these vectors form an orthonormal sequence and we let P_ζ be the projection to the closed subspace spanned by this sequence. Note that $P_\zeta Q_i = P_\zeta Q_i S_{l_i} + P_\zeta Q_i(I - S_{l_i})$ for each i, and each of these two operators is compact. Similarly, $Q_i P_\zeta$ is compact, hence (1) holds.

Now we check (2) for P_ζ and Q_i. Note that $Q^{X_i} f_i = e_{n_i(0)}/\sqrt{2}$, hence $\|Q_i(f_i) - e_{n_i(0)}/\sqrt{2}\| < 1/8$. Also $P_\zeta e_{n_i(0)} = f_i/\sqrt{2}$, hence $\|P_\zeta Q_i f_i - f_i/2\| < 1/8$. On the other hand $Q_i P_\zeta f_i = Q_i f_i = e_{n_i(0)}/\sqrt{2}$, hence $\|P_\zeta Q_i - Q_i P_\zeta\| > \|e_{n_i(0)}/\sqrt{2} - f_i/2\| - 2/8 = 1/4$.

For (3), since $P_\zeta Q_i e_{n_i(0)} = f_i/\sqrt{2}$, by the last part of (4) for $n \leq i$ we have $\|P_\zeta Q_i\|_n \geq 1/2$. Hence we have P_ξ ($\xi < \omega_1$) satisfying (1), (2), (3), and by Lemma 2 the conclusion follows. □

References

1. N. Weaver, *Set theory and C^*-algebras*, Bull. Symb. Logic (to appear).

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