ADDENDUM TO "ALL AUTOMORPHISMS OF THE CALKIN ALGEBRA ARE INNER"

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Abstract. The proof of my recent result that all automorphisms of the Calkin algebra are inner can be simplified by using a simple observation. Moreover, Martin's Axiom can be removed from the list of its assumptions.

Notation and terminology are taken from [1]. In particular, H is a separable infinite dimensional complex Hilbert space and $\mathcal{B}(H)$ is its algebra of bounded linear operators. Also, $\mathcal{C}(H)$ is the Calkin algebra and $\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$ is the quotient map. We fix Φ, an automorphism of $\mathcal{C}(H)$. If $\mathcal{D} \subseteq \mathcal{B}(H)$ we say Φ is *inner on* $\mathcal{D}$ if there is an inner automorphism Φ' of $\mathcal{C}(H)$ such that the restrictions of Φ and Φ' to $\pi[\mathcal{D}]$ coincide.

Lemma 0.1. *Assume $\mathcal{D}_1$ and $\mathcal{D}_2$ are subsets of $\mathcal{B}(H)$ such that for some partial isometry u we have $u\mathcal{D}_2u^* \subseteq \mathcal{D}_1$ and $P = u^*u$ satisfies $P\mathcal{D}_2P = \mathcal{D}_2$. If Φ is inner on $\mathcal{D}_1$, then it is inner on $\mathcal{D}_2$.*

Proof. Fix v such that $a \mapsto vav^*$ is a representation of Φ on $\mathcal{D}_1$ and w such that $\pi(w) = \Phi(\pi(u))$. If $b \in \mathcal{D}_2$ then $ubu^* \in \mathcal{D}_1$ and $u^*ubu^*u = b$. If Ψ is any representation of Φ, then we have (writing $c \sim_{\mathcal{K}} d$ for '$c - d$ is compact')

$$\Psi(b) \sim_{\mathcal{K}} \Psi(u^*)\Psi(ubu^*)\Psi(u) \sim_{\mathcal{K}} w^*vubu^*v^*w.$$

Therefore w^*vu witnesses Φ is inner on $\mathcal{D}_2$. □

An analogue of Lemma 0.1 fails for automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. For example, in [2] it was proved that a weakening of the Continuum Hypothesis implies the existence of a nontrivial automorphism whose ideal of trivialities is a maximal ideal.

As in [1], an orthonormal basis (e_n) for H is fixed and $\vec{E}$ stands for a partition of $\mathbb{N}$ into finite intervals. By $\mathcal{D}[\vec{E}]$ we denote the von Neumann algebra of all operators in $\mathcal{B}(H)$ for which each $\overline{\operatorname{span}}\{e_i \mid i \in E_n\}$ is invariant. For $M \subseteq \mathbb{N}$ by $\mathcal{D}_M[\vec{E}]$ we denote the ideal of $\mathcal{D}[\vec{E}]$ consisting of all a identically equal to 0 on each E_i for $i \notin M$.

Lemma 0.2. *Assume $\dim(E_n)$ is a nondecreasing sequence. If Φ is inner on $\mathcal{D}_M[\vec{E}]$ for some infinite M, then it is inner on $\mathcal{D}[\vec{E}]$.*

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Proof. By Lemma 0.1, it will suffice to find a partial isometry u such that $u\mathcal{D}[\vec{E}]u^* \subseteq \mathcal{D}_M[\vec{E}]$ and $u^*u = I$. If (m_j) is an increasing enumeration of M, then $j \leq m_j$ by our assumption. Let $u_j \colon E_j \to E_{m_j}$ be a partial isometry. Then $u = \sum_j u_j$ is as required. □

As a consequence, clauses (a) and (b) of [1, Theorem 6.2] are equivalent and it suffices to prove the easier (a). Moreover, the proof of (a) can be considerably simplified by ignoring all even-numbered intervals. (For instance, the use of Lemma 5.4 at the bottom of page 29 is now unnecessary, since one need not worry about Λ_{2i} at all.) Also, [1, Proposition 8.2] is an immediate consequence of Lemma 0.2 and [1, Proposition 7.1]. Since MA was not used elsewhere in the proof, we have the following strengthening of the main result of [1].

Theorem 0.3. *OCA_∞ implies that all automorphisms of the Calkin algebra are inner.* □

It is not known whether OCA_∞ implies that all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ (or any other analytic quotient) are inner.

References

1. I. Farah, *All automorphisms of the Calkin algebra are inner*, 2007, preprint, available at http://www.math.yorku.ca/∼ifarah.
2. S. Shelah and J. Steprāns, *Non-trivial homeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ without the continuum hypothesis*, Fundamenta Mathematicae **132** (1989), 135–141.

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