SOME PROBLEMS ABOUT OPERATOR ALGEBRAS WITH SET-THEORETIC FLAVOR

ILIJAS FARAH

I state several problems on operator algebras—mostly C*-algebras—that may be of interest to set-theorists and metric model-theorists. This list being a companion piece to [35], I frequently use [35] as a reference for standard results. I am not repeating problems stated in Weaver's survey [78] unless there is something new to say about them.

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Terminology. The reader may want to consult [35] or some standard text on C*-algebras or functional analysis ([76], [56], [11]) for more information on the following notions. By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators on a complex Hilbert space H. Let A be a C*-algebra. A *representation* of A is a pair (π, H) where $\pi\colon A \to \mathcal{B}(H)$ is a *-homomorphism. An operator a in A is *positive* if $a = b^*b$ for some $b \in A$. This is equivalent to having $(\pi(a)\xi|\xi) \geq 0$ for every representation $\pi\colon A \to \mathcal{B}(H)$ and every $\xi \in H$. A continuous linear functional $\phi : A \to \mathbb{C}$ is *positive* if $\phi(a) \geq 0$ for all positive $a \in A$. It is a *state* if it is positive and of norm 1. The GNS construction associates a representation (π_ϕ, H_ϕ) to every state ϕ of A (see [35, Theorem 3.4]). This representation is *irreducible* (i.e., there are no nontrivial closed linear subspaces of H_ϕ under π_ϕ) if and only if ϕ is pure.

Let $\mathcal{S}(A)$ denote the space of all states of A. It is compact and convex in the weak*-topology. A state is *pure* if it is an extreme point of $\mathcal{S}(A)$. Let $\mathcal{P}(A)$ denote the space of all pure states of A.

A subalgebra A of B is a *masa* if it is a maximal (under the inclusion) abelian subalgebra of B.

1. Problems on pure states

The problems of this section were discussed in [35, §5] where we refer the reader for more details. The following is known as the *noncommutative Stone–Weierstrass*

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problem or the *general Stone–Weierstrass problem.* By 0 we denote the constantly zero function. Note that if A is a subalgebra of B and A is not a unital subalgebra, then some pure state on B restricts to the zero function on A. If A is a subalgebra of B we say that it *separates pure states of* B if for every pair of distinct pure statez ϕ and ψ of B there is $a \in A$ such that $\phi(a) \neq \psi(a)$. If A and B are commutative, then A separates pure states of B if and only if $A = B$. This is an easy consequence of the Gelfand–Naimark and Stone–Weierstrass theorems.

Problem 1.1. Is it true that if A is a subalgebra of B and A separates $\mathcal{P}(B) \cup \{0\}$ then $A = B$?

By the Gelfand–Naimark structure theorem for commutative C*-algebras, the case when B is commutative reduces to the Stone–Weierstrass theorem.

The following famous problem is equivalent to an arithmetic statement and therefore it is unlikely that set-theoretic methods can be of much use in attacking it. It is, however, a good problem and it naturally belongs to this section (also see Problem 6.1).

Problem 1.2 (Kadison-Singer)**.** Does every pure state of the atomic masa in $\mathcal{B}(H)$ extend uniquely to a pure state of $\mathcal{B}(H)$?

Let e_n, for $n \in \mathbb{N}$, denote an orthonormal basis of a Hilbert space H. Anderson proved that for an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the formula

$$\phi_{\mathcal{U}}(a) = \lim_{n \to \mathcal{U}} (ae_n | e_n)$$

defines a pure state. Such state is said to be *diagonalized* by (e_n). The Kadison–Singer problem asks whether $\phi_{\mathcal{U}}$ is the only pure state of $\mathcal{B}(H)$ that extends the restriction of $\phi_{\mathcal{U}}$ to the masa spanned by (e_n).

Conjecture 1.3 (Kadison–Singer, 1959)**.** *For every pure state ϕ of $\mathcal{B}(H)$ there is a masa $\mathcal{A}$ such that $\phi \restriction \mathcal{A}$ is multiplicative (i.e., pure).*

One could also consider the following stronger conjecture:

Conjecture 1.4. *For every pure state ϕ of $\mathcal{B}(H)$ there is an* atomic *masa $\mathcal{A}$ such that $\phi \restriction \mathcal{A}$ is multiplicative.*

Conjecture 1.5 (Anderson, 1981)**.** *Every pure state on $\mathcal{B}(H)$ is diagonalizable.*

These three conjectures were refuted in [2] using the Continuum Hypothesis, CH. In [35] it was shown that an assumption a bit weaker than CH suffices. I don't know whether any of these conjectures is consistent with ZFC, but I am inclined to conjecture none of them is.

In order to have a combinatorial reformulation of these problems we introduce definition (cf. a lemma of Weaver, [35, Lemma 6.41]).

Definition 1.6. A family $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))$ such that

(1) $p \in \mathcal{F}$ and $p \leq q$ implies $q \in \mathcal{F}$ and
(2) for all n and $p_1, \dots, p_n$ in $\mathcal{F}$ we have $\| \prod_{j=1}^n p_j \| = 1$

is called a *quantum filter*.

Condition (2) is weaker than stating that $\mathcal{F}$ is directed downwards. This is similar to the difference between centred sets and filters familiar from the theory of forcing. Also see a related result by Tristan Bice, [10].

We say that a quantum filter $\mathcal{F} \subset \mathcal{P}(\mathcal{C}(H))$ *lifts* if there is a commuting family $\mathbf{X} \subseteq \mathcal{P}(\mathcal{B}(H))$ that generates a filter $\mathbb{F}$ such that $\pi[\mathbb{F}] = \mathcal{F}$.

Question 1.7. *Does every maximal quantum filter $\mathcal{F}$ in $\mathcal{P}(\mathcal{C}(H))$ lift?*

A negative answer implies a negative answer to Conjectures 1.3–1.5 (see [35] for details).

Problem 1.8. Develop a theory of equivalence of pure states on $\mathcal{B}(H)$ analogous to the Rudin–Keisler equivalence of ultrafilters on $\mathbb{N}$. Is there an ordering analogous to the Rudin–Keisler ordering?

Since the selective ultrafilters are minimal in the Rudin–Keisler order, Problem 5.6 may be related to this.

1.1. **Naimark's problem.**

This section describes some possible lines of attack at the question whether a positive answer to Naimark's problem relatively consistent with ZFC? See [35, §5.1] for more information on Naimark's problem.

Problem 1.9. Is the following relatively consistent with ZFC?

Every C*-algebra A that has a unique irreducible representation up to the unitary equivalence is isomorphic to the algebra of compact operators on some Hilbert space.

The negation of this statement follows from Jensen's diamond principle (see [52, §2]) by [1]. As pointed out by N.C. Phillips, the counterexample can easily be chosen to be a nuclear C*-algebra.

A type I C*-algebra has a unique irreducible representation if and only if it is simple, if and only if it is isomorphic to an algebra of compact operators on some Hilbert space.

Assume A is counterexample to Naimark's problem. It is a non-type I C*-algebra with a unique irreducible representation. By a result of Glimm A has a separable subalgebra with a quotient isomorphic to the CAR algebra, $\bigotimes_{n=0}^{\infty} M_2(\mathbb{C})$. Since the CAR algebra has a family of $2^{\aleph_0}$ pure states separated by a countable family of projections, A has character density at least $2^{\aleph_0}$ (see [1]). Here *character density* of A is the smallest cardinality of a dense subset of A.

The following problem may be more tractable than the consistency of a positive answer to the full Naimark's problem. However, I don't know whether its conclusion is equivalent to the positive answer to Naimark's problem.

Test Problem 1.10. Is the following relatively consistent with ZFC?

Every C*-algebra A of character density at most $2^{\aleph_0}$ that has a unique irreducible representation is isomorphic to the algebra of compact operators on some Hilbert space.

More generally, one could ask the following.

Question 1.11. *Assume there is a counterexample to Naimark's problem. What can be said about the minimal cardinal κ such that there exists a counterexample of cardinality κ?*

More precisely, this question asks whether this minimal cardinal is one of the 'standard' combinatorial cardinals (see e.g., [72]).

1.1.1. *Naimark's problem and forcing.* In this subsection I address some attempts at proving consistency of a positive answer to Naimark's problem. Before proceeding, I will only say that the only model of ZFC in which the answer to Naimark's problem is known is a model of $\diamondsuit$ on $\aleph_1$. Everything else is an open problem. In a forcing extension we identify a ground-model C*-algebra with its metric completion. The following lemma was inspired by the last section of [1].

Lemma 1.12. *If A is a counterexample to NP and forcing $\mathbb{P}$ adds a real then $\mathbb{P}$ forces that (the completion of) A has an irrep not equivalent to any ground-model irreducible representation.*

Therefore the assertion 'There are no counterexamples to NP' is relatively consistent with ZFC-Power set axiom. Just add class many reals (Cohen, random,…). I don't know a situation in which it is possible to destroy a counterexample to Naimark's problem by forcing which does not add reals. Here is a test problem.

Question 1.13. *Is the assertion that there are no counterexamples to Naimark's problem of cardinality $\mathfrak{c}$ relatively consistent with CH?*

1.1.2. *Beyond Naimark's problem.* A motivation for solving Naimark's problem comes from representation theory of C*-algebras (cf. §8.1). By a seminal result of Glimm ([39]), if a simple separable C*-algebra A has inequivalent irreducible representations, then it has $2^{\aleph_0}$ inequivalent irreducible representations. Until the Akemann–Weaver paper (from which the present discussion is borrowed) Naimark's problem was considered as a fundamental obstruction to the extension of Glimm's result to nonseparable algebras (cf. [64]).

Problem 1.14. Is the assertion that every simple C*-algebra with irreducible representations has at least $\mathfrak{c}$ many irreducible representations relatively consistent with ZFC? Does it follow from PFA? What is the 'right' extension of Glimm's result for not necessarily separable algebras?

The following lesser question goes in the opposite direction.

Question 1.15. *For what cardinals (finite or infinite) n there exists a* simple *C*-algebra with exactly n inequivalent irreps?*

It seems plausible that $\diamondsuit$ implies there is such algebra for each $n \in \mathbb{N}$.

2. Nuclearity

A C*-algebra A is *nuclear* (or *amenable*) if the identity map from A into itself can be arbitrarily well approximated by completely positive contractions through matrix algebras. See [11, IV.3.1.4], and [11, IV.3.1.5] for many equivalent definitions. The following question appears in [51, p. 365].

Question 2.1. *Assume A is a unital nuclear C*-algebra and ϕ and ψ are its pure states. Is there an automorphism α of A such that $\phi = \psi \circ \alpha$?*[1]

The answer is positive in case when A is separable (even without assuming nuclearity) and negative if the nuclearity of A is not assumed ([51]).

The following problem can be related.

[1] No. See [25].

Question 2.2. *Assume A is a simple nuclear C*-algebra faithfully represented on a Hilbert space H. Can the character density of A be larger than the character density of H?*[2]

Neither of the assumptions on A, nuclearity and simplicity, can be dropped. For example, the algebra $A = C(2^{2^{\aleph_0}})$ is abelian and therefore nuclear. Since the space $2^{2^{\aleph_0}}$ is separable A is faithfully represented on a separable Hilbert space. However, A is easily seen to have character density $2^{\aleph_0}$.

The last two questions in this section are the most important ones because of their relevance to Elliott's classification program. I can only outline them here and invite the reader to consult references for precise definitions and statement of problems.

All presently known nuclear C*-algebras are obtained from $\mathbb{C}$ by taking closure under simple operations such as tensoring with finite-dimensional matrix algebras and taking inductive limits. They belong to the so-called *bootstrap class*, see [11, IV.3.1.5, V.1.5.2]. The following is a central problem in the subject.

Question 2.3. *Does every nuclear C*-algebra belong to the bootstrap class?*

The following problem is closely related (see [11, V.1.5.8, V.1.5.12, and V.1.5.15] for the definitions).

Question 2.4. *Do all nuclear C*-algebras satisfy the UCT (Universal Coefficient Theorem)?*

A negative answer to these problems would be likely to provide an insight analogous to that given by Tsirelson's Banach space ([73].

3. Borel structures and classification

Theory of representations of C*-algebras has strongly descriptive set-theoretic flavor. For example, Chapter 3—that is, one quarter—of [6] is all descriptive set theory.

3.1. **Mackey Borel structure.** Let me first briefly describe representation theory of C*-algebras (also see §8.1). It is now considered somewhat old fashioned, because nobody has discovered any way to use it to good effect on simple C*-algebras not of type I.

The *spectrum* of a C*-algebra A, $\hat{A}$, is the space of equivalence classes of irreducible representations of A. Two irreducible representations π_1 and π_2 of A are *(unitarily) equivalent* if there is a unitary u in A (or the unitization of A, if A is not unital) such that $\pi_1 = \pi_2 \circ \operatorname{Ad} u$, where $\operatorname{Ad} u$ is the inner automorphism of A implemented by u via $\operatorname{Ad} u(a) = uau^*$. This equivalence relation, denoted below by E_A (this is not a standard notation) is clearly analytic.

Recall that the via the GNS construction irreducible representations correspond to pure states (see [35, §3] for definitions). If A is separable, then each irreducible representation of A has range in a separable Hilbert space, and therefore $\hat{A}$ can be considered as a quotient space of the direct sum of $\operatorname{Irr}(A, H_n)$: the space of irreducible representations of A on a Hilbert space H_n of dimension n for $n \in \mathbb{N} \cup \{\aleph_0\}$. Each $\operatorname{Irr}(A, H_n)$ is a Polish space with respect to the weakest topology making all functions $\operatorname{Irr}(A, H_n) \ni \pi \mapsto (\pi(a)\xi|\eta) \in \mathbb{C}$, for $a \in A$ and $\xi, \eta \in H_n$,

[2] Yes. See [25].

continuous. In other words, a net π_λ converges to π if and only if $\pi_\lambda(a)$ converges to $\pi(a)$ for all $a \in A$. Therefore $\hat{A}$ carries a Borel structure (known as the *Mackey Borel structure*) inherited from a Polish space. For *type I* C*-algebras (also called *GCR* or *postliminal*) this space is a standard Borel space. (All of these notions are explained in [6, §4].)

Problem 3.1 (Dixmier, 1967)**.** Is the Mackey Borel structure on the spectrum of a simple separable C*-algebra always the same when it is not standard?

G. Elliott proved (via a generalization of Glimm's result [35, Theorem 5.8]) that all nonstandard spectra of simple AF algebras are isomorphic ([19]). (A C*-algebra is an AF (approximately finite) algebra if it is a direct limit of finite-dimensional algebras. Elliott proved that separable unital AF algebras are classified by their K-theoretic invariants.)

For a separable A let $X_A = \bigoplus_{n=1}^{\infty} \mathrm{Irr}(A, H_n)$. By the above, it is a Polish space. Note that Mackey Borel structures of A and B are isomorphic if and only if there is a Borel isomorphism $f\colon X_A \to X_B$ such that $\pi_1 \mathrm{E}_A \pi_2$ if and only if $f(\pi_1) \mathrm{E}_B f(\pi_2)$. Hence Problem 3.1 is rather close in spirit to the theory of Borel equivalence relations (see e.g., [47], [42]). The equivalence relation E_A is turbulent (see [47], [42]) if it is not smooth ([48] and [23], independently). A result of [40] easily implies that E_A is always F_σ.

N. Christopher Phillips suggested more general problems about the Mackey Borel structure of simple separable C*-algebras, motivated by his discussions with Masamichi Takesaki. There are two (related) kinds of questions: Can one do anything sensible, and, from the point of view of logic, how bad is the problem?

Nuclear C*-algebras are considered to be the 'tractable' ones and Elliott's program is usually restricted to nuclear algebras ([20], [63]). Exact C*-algebras are subalgebras of nuclear C*-algebras (at least in the separable case; see [63]). In $\mathcal{B}(H)$ there are operators a, b, c such that no algebra containing all three of them is nuclear (or exact); this is a consequence of a result of Junge and Pisier (see [15, §13]) that the Banach–Mazur distance on three-dimensional operator spaces is nonseparable (see the last section of [57]).

Problem 3.2. Does the complexity of the Mackey Borel structure of a simple separable C*-algebra increase as one goes from nuclear C*-algebras to exact ones to ones that are not even exact?

Problem 3.3. Assume A and B are C*-algebras and E_A is Borel-reducible to E_B. What does this fact imply about the relation between A and B?

3.2. **Classification of C*-algebras.** The following is rather a long-term project than a concrete problem. Measure the complexity of the classification problem for classes of separable C*-algebras (unital, simple, nuclear,... see [63]). How hard is the problem? Is it easier than the classification problem for non-selfadjoint algebras, such as triangular algebras (see [61])? By a recent result of Tørnquist and Sasyk ([65]), von Neumann factors cannot be effectively classified by countable structures. See also [48] and [23] for an application of turbulence to the spaces of representations of C*-algebras.

Here is a more concrete problem, essentially suggested by G. Elliott. Let X be the Polish space of all C*-subalgebras of $\mathcal{O}_2$. By a result on Kirchberg (see [15], [11, IV.3.4.18]) each separable exact C*-algebra is a subalgebra of $\mathcal{O}_2$ (cf. also Question 8.2).

Problem 3.4. What is the Borel complexity of the isomorphism relation of C*-subalgebras of $\mathcal{O}_2$?[3]

An answer to this may provide a strong anti-classification result for separable exact C*-algebras, modulo working out the following. The *Elliott invariant* of a C*-algebra A consists of two countable abelian groups $K_0(A)$ and $K_1(A)$, a distinguished subset of $K_0(A)$, a Choquet simplex, and a connecting map between the Choquet simplex and $K_0(A)$. See [63] for precise definition of the Elliott invariant. Each countable group can be coded by an element of $2^{\mathbb{N}^3}$ that represents the group operation. Such codes form a compact subspace of the compact metric space $2^{\mathbb{N}^3}$, and we associate a group with the isomorphism class of its code (see [8]). This gives a Borel space of all Elliott invariants.

Problem 3.5. Demonstrate that the map that associates the Elliott invariant to a subalgebra of $\mathcal{O}_2$ is Borel-measurable.[4]

Problem 3.4 and Problem 3.5 for subalgebras of a larger separable C*-algebra are also interesting. The Junge–Pisier result mentioned above implies that no separable C*-algebra is universal for all separable C*-algebras. However, there is a natural Borel space of separable C*-algebras (see [46]). The elements of this space are sequences of elements in $\mathcal{B}(H)$ and to each sequence we associate the C*-subalgebra of $\mathcal{B}(H)$ generated by it. In a preliminary work Andrew Toms and myself demonstrated that the function associating the Elliott invariant to an algebra is Borel.[5]

Remarkably, most of the existing classification results for classes of C*-algebras provide a complete description of the set of invariants and therefore provide the exact measure of complexity of the isomorphism problem.

The following questions are taken from [34]. In [34] it was shown that the bi-embeddability of AF algebras is above the complete K_σ Borel equivalence relation.

Problem 3.6. Is the bi-embeddability relation for nuclear simple separable C*-algebras a complete analytic equivalence relation? What about bi-embeddability of AF algebras?

The isomorphism of simple nuclear C*-algebras is Borel-reducible to an orbit equivalence relation of a Polish group action.

Question 3.7. *Is isomorphism of separable (simple) C*-algebras implemented by a Polish group action? In particular, is isomorphism of nuclear separable simple C*-algebras below a group action?*

Problem 3.8. What is the Borel cardinality of the isomorphism relation of larger classes of separable C*-algebras (simple or not), such as:

(i) nuclear C*-algebras;
(ii) exact C*-algebras;
(iii) arbitrary C*-algebras?

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[3]Asger Törnquist has observed that they are not classifiable by countable structures, and that even the classifiable simple nuclear separable C*-algebras are not classifiable by countable structures. See [34]

[4]Done. See [34] and [33]

[5]Not so preliminary anymore. See See [34] and [33].

Do these problems have strictly increasing Borel cardinality? Are all of them Borel reducible to the orbit equivalence relation of a Polish group action on a standard Borel space?

Problem 3.9. Determine whether the following C*-algebra constructions and/or subclasses are Borel:

(i) the maximum tensor product, and tensor products more generally;
(ii) crossed products, full and reduced;
(iii) groupoid C*-algebras;
(iv) $\mathcal{Z}$-stable C*-algebras;
(v) C*-algebras of finite or locally finite nuclear dimension;
(vi) approximately subhomogeneous (ASH) algebras;
(vii) the Thomsen semigroup of a C*-algebra.

Items (iv)–(vi) above are of particular interest to us as they are connected to Elliott's classification program (see [20]). Items (iv) and (v) are connected to the radius of comparison by the following conjecture of A. Toms and W. Winter (see [34] for definitions).

Conjecture 3.10. *Let A be a simple separable unital nuclear C*-algebra. The following are equivalent:*

(i) *A has finite nuclear dimension;*
(ii) *A is $\mathcal{Z}$-stable;*
(iii) *A has radius of comparison zero.*

4. Rigidity of the corona algebras

For more details on the problems from this section the reader may want to consult [27], where I proved that Todorcevic's Axiom, TA, implies all automorphisms of the Calkin algebra are inner. See also [21] for a broader context.

I shall now define *multiplier algebra*, $M(A)$, of a C*-algebra A (see [11, 1.7.3] for more detals). Fix a C*-algebra A. A representation $\pi\colon A \to \mathcal{B}(H)$ is *nondegenerate* if

$$(\forall b \in \mathcal{B}(H)) b(\pi[A]) = \{0\} \text{ if and only if } b = 0.$$

Fix a nondegenerate representation of A, identify A with $\pi(A)$ and let

$$M(A) = \{b \in \mathcal{B}(H) : bA \subseteq A \text{ and } Ab \subseteq A\}.$$

It is not difficult to show that $M(A)$ is a C*-subalgebra of $\mathcal{B}(H)$, and that A is an ideal of $M(A)$. A more remarkable fact is that the isomorphism type of $M(A)$ (and even the structure $(M(A), A)$ does not depend on the choice of the representation π (as long as π is nondegenerate).

Note that if A is unital, then $M(A) = A$: If $b \in M(A)$ then $b1 = b$ is in A. On the other hand, $M(A)$ always contains the unit. Unlike the unitization of A obtained by adding a unit in the minimal way, $M(A)$ is usually considered to be a noncommutative analogue of the Čech–Stone compactification. The former also the universality property akin to the universality property of the latter. Also, $M(\mathcal{K}(H)) = \mathcal{B}(H)$, $M(C_0(X)) = C(\beta X)$, and $M(A) = A$ for every unital C*-algebra A. Problem 4.1 below was suggested by G. Elliott.

Problem 4.1. Let A be a separable non-unital C*-algebra. Does each automorphism of the corona algebra $M(A)/A$ lift to a *-self-homomorphism of $M(A)$?

Such automorphisms are the 'trivial' automorphisms of the corona algebra $M(A)/A$. The answer is known to be independent of ZFC for many abelian C*-algebras A. In all known cases CH implies the existence of 'nontrivial' automorphisms while Todorcevic's Axiom, TA and Martin's Axiom, MA, imply all automorphisms are trivial. Recall that TA (also among the axioms known as OCA), is the following statement.

Todorcevic's Axiom, TA [70]**.** Assume (V, E) is a graph such that $E = \bigcup_{n=0}^{\infty} A_n \times B_n$ for some subsets A_n, B_n of V. Then one of the following applies.

(1) (V, E) has an uncountable *clique*: $Y \subseteq X$ such that any two vertices in Y are connected by an edge, or
(2) (V, E) is *countably chromatic*: there is a partition $V = \bigcup_{n=0}^{\infty} X_n$ so that no edge connects two vertices in the same X_n.

The known cases in which the statement 'all automorphisms of $M(A)/A$ are inner' include the following:

(1) When $A = C_0(\mathbb{N})$. Rudin constructed a nontrivial automorphism from CH. Shelah ([66]) proved the consistency of the statement 'all automorphisms of $M(A)/A$ are trivial.' Shelah–Steprāns and Velickovic proved that this follows from PFA and TA+MA, respectively.
(2) $A = C_0(X)$ for a countable locally compact X. Parovičenko proved a result that implies that under CH $M(A)/A \cong C(\beta\mathbb{N} \setminus \mathbb{N}^*)$, and therefore by Rudin's result $M(A)/A$ has nontrivial automorphisms. I proved that TA+MA implies all automorphisms of $\mathcal{C}(H)$ are trivial [21].
(3) $A = \mathcal{K}(H)$ for a separable H. Phillips and Weaver ([60]) constructed a nontrivial automorphism from CH. I proved that TA implies all automorphisms are trivial ([27]).

The first interesting case of Problem 4.1 is when A is a UHF algebra, for example $M_{2^\infty} \otimes \mathcal{K}(H)$.

There are two more ways in which one can try to extend [27].

Question 4.2. *Does the Proper Forcing Axiom (PFA) imply that all automorphisms of the Calkin algebra on any Hilbert space are inner?*[6]

Velickovic ([74]) proved that PFA implies all automorphisms of $\mathcal{P}(\kappa)/\mathrm{Fin}$ are trivial for all κ.

Question 4.3. (1) *Is the assertion that the Calkin algebra $\mathcal{C}(\aleph_1) = \mathcal{B}(\ell_2(\aleph_1))/\mathcal{K}(\ell_2(\aleph_1))$ has an outer automorphism relatively consistent with ZFC?*
(2) *Is the assertion that the Calkin algebra on $\ell_2(\kappa)$ for some (any) uncountable cardinal κ has an outer automorphism relatively consistent with ZFC?*
(3) *Assume $\mathcal{C}(\ell_2(\aleph_0))$ has an outer automorphism. Does $\mathcal{C}(\ell_2(\aleph_1))$ have an outer automorphism?*

In the above 'the Calkin algebra on $\ell_2(\kappa)$' stands for $\mathcal{B}(\ell_2(\kappa))/\mathcal{K}(\ell_2(\kappa))$, where $\mathcal{K}(\ell_2(\kappa))$ is the ideal of compact operators on $\ell_2(\kappa)$. However, unlike in the separable case, this is not the unique ideal on $\mathcal{B}(\ell_2(\kappa))$. For every infinite cardinal $\lambda \leq \kappa$, the closure of the family of operators of 'rank $< \lambda$' is a proper ideal. We shall denote this ideal by $\mathcal{I}(\kappa, \lambda)$, and the correspondind Calkin algebra by $\mathcal{C}(\kappa, \lambda)$. Hence $\mathcal{C}(\kappa, \aleph_0)$ is $\mathcal{C}(\kappa)$, or what is called the Calkin algebra above. In [30] it was proved that if

[6] Yes. See [26].

$2^\kappa = \kappa^+$ and κ is regular then $\mathcal{C}(\kappa, \kappa)$ has 2^{2^κ} automorphisms, and therefore an outer automorphism. The case $\kappa = \aleph_0$ is [60]. The proof of the general case is an extension of the proof of the Phillips–Weaver theorem given in [27]. A sufficient assumption in the uncountable case is that the club filter on κ is κ^+-generated.

Nothing else is known.

Question 4.4. *What else can be said about the automorphisms of $\mathcal{C}(\kappa, \lambda)$? Is it relatively consistent with ZFC that $\mathcal{C}(\aleph_1, \aleph_1)$ has no outer automorphisms? Is it relatively consistent with ZFC that $\mathcal{C}(\kappa, \kappa$ has no outer automorphisms for some (any) uncountable κ?*

For each $n \in \mathbb{N}$ fix an isomorphism $\Psi_n \colon \mathcal{C}(H) \otimes M_n(\mathbb{C}) \to \mathcal{C}(H)$. Each automorphism Φ of $\mathcal{C}(H)$ defines a unital *-endomorphism of $\mathcal{C}(H)$ by $\Psi_n(\Phi \otimes \mathrm{id}_n)$. Considering these as 'trivial' unital endomorphisms, one may ask the following question.

Question 4.5. *Assume TA. Are all unital *-homomorphisms of the Calkin algebra into itself of the form $(\operatorname{Ad} u) \otimes \mathrm{id}_n$ for some $n \in \mathbb{N}$?*

The obvious analogue of Question 4.5 for $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has a negative answer (except e.g., in a model in which the Axiom of Choice fails and all sets of reals have the property of Baire). Its proper analogue has a more involved formulation (see [21, §3.2], also [22]).

4.1. **Automorphisms of the Calkin algebra.**

All known outer automorphisms of the Calkin algebra ([60], [27, §2]) are pointwise inner. The following is what is left of the Brown–Douglas–Fillmore problem on automorphisms of the Calkin algebra ([14, 13]). Let $\dot{S}$ denote the image of the unilateral shift in the Calkin algebra.

Problem 4.6. Is it consistent with ZFC that there is an automorphism of the Calkin algebra with one of the following properties?

(1) Φ sends $\dot{S}$ to its adjoint?
(2) $\Phi(a) = b$ for some a and b that are not conjugate?
(3) The restriction of Φ to some separable algebra is not implemented by a unitary?

Since $\dot{S}$ and its adjoint have different Fredholm indices, the existence of Φ satisfying (1) clearly implies the existence of a Φ satisfying (2) which in turn implies the existence of a Φ satisfying (3). The Brown–Douglas–Fillmore classification of normal elements in the Calkin algebra up to the conjugacy implies that (1) is equivalent to (2) when a and b are required to be normal (in the Calkin algebra). It is also true that the general version of (2) is equivalent to (3).

A simple absoluteness argument shows that if there is a Φ as in (1), (2) or (3) above, then there is such a Φ in a forcing extension satisfying CH. Since Todorcevic's Axiom (TA) implies all automorphisms of the Calkin algebra are inner, one cannot construct Φ as above in ZFC. Since the existence of Φ satisfying any of these conditions is a Σ^2_1 statement, by Woodin's Σ^2_1-absoluteness theorem one may as well ask whether CH implies the existence of such Φ.

This supports a view that Problem 4.6 is model-theoretic in nature. The following is for readers familiar with the logic for metric structures ([9]). (Because of some technical reasons one needs to consider the unit ball of the Calkin algebra with appropriate operations instead of the Calkin algebra, as a model.)

Problem 4.7. Consider $\mathcal{C}(H)$ as a metric structure in the sense of [9]. Are the parameter-free 1-types of $\dot{S}$ and $\dot{S}^*$ equal?

Since an automorphism preserves type of an element, a negative answer to Problem 4.7 would imply a negative answer to Problem 4.6 (1). On the other hand, a positive answer to Problem 4.7 would imply that under CH there is an automorphism of $\mathcal{C}(H)$ sending $\dot{S}$ to its adjoint **if** we knew that $\mathcal{C}(H)$ was countably saturated (in the logic for metric structures). However, it isn't.

Problem 4.8. Is $\mathcal{C}(H)$ a countably homogeneous model?

One technical remark. If there is an automorphism of the Calkin algebra Φ such that $\|\Phi(\dot{S}) - \dot{S}^*\|$ is small enough, then there is an automorphism of the Calkin algebra that sends $\dot{S}$ to $\dot{S}^*$. This is because $\Phi(\dot{S})$ has the same spectrum as $\dot{S}$ (hence the same spectrum as $\dot{S}^*$), and being close to $\dot{S}^*$ it has the same Fredholm index as $\dot{S}^*$. Hence by Brown–Douglas–Fillmore results there is an automorphism sending $\Phi(\dot{S})$ to $\dot{S}^*$.

Fix a basis ξ_n of H and consider the *standard atomic masa* consisting of all operators a such that each ξ_n is an eigenvector of a. A result of Alperin, Covington, and Macpherson ([3]) easily implies that if an automorphism Φ of the Calkin algebra sends the atomic masa to itself (or to another atomic masa) then Φ cannot send $\dot{S}$ to its adjoint. On the other hand, the subalgebra of the Calkin algebra generated by $\dot{S}$ (so-called *Toeplitz extension*) has an automorphism that sends $\dot{S}$ to its adjoint. Note that the following test problem for (1) is a Σ^1_2 statement, and therefore absolute.

Problem 4.9. Is there a separable subalgebra A of the Calkin algebra that contains $\dot{S}$ and such that no homomorphism $\Phi\colon A \to \mathcal{C}(H)$ satisfies $\Phi(\dot{S}) = \dot{S}^*$?

Now we consider problems concerned with outer automorphisms that fail the property required in Problem 4.6 . Following [59], consider the following conditions on an automorphism Φ of the Calkin algebra.

(1) It is inner on every separable subalgebra.
(2) It is approximately inner, but not inner on at least some separable subalgebra.
(3) Trivial on $K_1(\mathcal{C}(H))$ but not approximately inner.
(4) Induces the map multiplication by -1 on $K_1(\mathcal{C}(H))$.

An automorphism of a C*-algebra is *approximately inner* if it is a pointwise limit of inner automorphsms. Here 'trivial on K_1' means that Φ preserves Fredholm index of unitaries in $\mathcal{C}(H)$ and 'induces the map multiplication by -1 on K_1' means that it multiplies Fredholm index of unitaries by -1 (see [62] for more on K-theory of C*-algebras).

Conditions (1)–(4) all imply that Φ is outer and they are ordered by the increasing degree of 'outerness.' All known examples of outer automorphisms satisfy (1). Nothing else is known about the relation between these four classes, except for the obvious fact that they are disjoint. Assuming TA all of these classes are empty, and the assertion that some of them is nonempty is a Σ^2_1 statement.

Question 4.10. *Assuming the Continuum Hypothesis, which of the classes (1)–(4) is nonempty?*

Problem 4.11 was also suggested by N. Christopher Phillips. Asymptotically inter automorphisms of a C*-algebra are defined in [60] and they play an important role in [60], where it was shown that CH implies the existence of an asymptotically inner, but not inner, automorphism.

Problem 4.11. Is it true that every approximately inner automorphism of $\mathcal{C}(H)$ is asymptotically inner?

While the status of Question 4.12 does not seem to have a direct bearing to Problem 4.6, I find it interesting in its own right.

Question 4.12. *Consider the subalgebra A of $\mathcal{C}(H)$ generated by the standard atomic masa and $\dot{S}$. Is there an automorphism of A that sends $\dot{S}$ to $\dot{S}^*$?*

A closely related problem asks whether there is an automorphism Φ of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ such that for every $X \subseteq \mathbb{N}$ we have $\Phi(X+1) = \Phi(X)-1$? Here $X+1 = \{n+1 : n \in X\}$ and $X-1 = \{n-1 : n \in X \setminus \{0\}\}$. Some information on the latter formulation can be found in [38].

Consider the operator S' on $\ell_2(\aleph_1)$ such that $S'(e_\xi) = e_{\xi+1}$ if ξ is finite and $S'(e_\xi) = e_\xi$ if ξ is infnite. Then (using π for the quotient map from $\mathcal{B}(\ell_2(\aleph_1))$ to $\mathcal{C}(\aleph_1)$ $\pi(S')$ is a unitary in $\mathcal{C}(\aleph_1)$. The following question is related with Question 4.3 (3) and with Problem 4.6.

Question 4.13. *Is the assertion that there is an automorphism of $\mathcal{C}(\aleph_1)$ that sends $\pi(S')$ to its adjoint relatively consistent with ZFC?*

The following question turns out to be more difficult than expected.

Question 4.14. *Does the Continuum Hypothesis imply that the corona of every separable non-unital C*-algebra has an outer automorphism?*

5. The Calkin algebra

Many known results about the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ translate into problems about the Calkin algebra. Finding the right analogues of statements about $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ in the Calkin algebra can be a challenging problem in itself. For example, the so-called 'Kasparov's technical theorem' (see e.g., [55, §8]), which is one of the basic tools in the KK-theory, is the analogue of the familiar fact that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is countably saturated. Pedersen's survey [55] well deserves further set-theoretic exploration.

5.1. **The structure of projections in the Calkin algebra.**

For more on the following problems see [35, §4.1].

Problem 5.1. Assume all maximal chains of projections in the Calkin algebra are order-isomorphic. Does CH hold?

D. Hadwin ([41]) conjectured that this problem has a positive solution and proved its converse. E. Wofsey ([79]) constructed a forcing extension in which not all maximal chains are isomorphic in which CH fails. There is a model of the negation of CH in which all maximal chains in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are isomorphic (essentially by [67]); it is unclear whether this model provides a counterexample to Hadwin's conjecture.

Many cardinal invariants associated with $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (see e.g., [12]) have analogues in the poset of projections in the Calkin algebra. For example, let

$$\begin{aligned}\mathfrak{a}^* &= \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a maximal infinite family} \\ &\qquad\qquad \text{of almost orthogonal projections in } \mathcal{C}(H)\} \\ \mathfrak{t}^* &= \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a maximal tower of projections in } \mathcal{C}(H)\} \\ \mathfrak{b}^* &= \min\{\kappa \mid \text{ there is a } (\kappa, \aleph_0)\text{-gap in } \mathcal{P}(\mathcal{B}(H))\}.\end{aligned}$$

Problem 5.2. Is $\mathfrak{a} = \mathfrak{a}^*$? Is $\mathfrak{t} = \mathfrak{t}^*$?

Not much is known. It is consistent that $\mathfrak{a}^* = \mathfrak{t}^* = \aleph_1 < 2^{\aleph_0}$ (Wofsey) and MA implies $\mathfrak{a}^* = \mathfrak{t}^* = \mathfrak{c}$. Also, $\mathfrak{b} = \mathfrak{b}^*$ (Zamora–Avilés, see [80]).

Since $\mathcal{P}(\mathcal{B}(H))$ is a Polish space with respect to the strong topology, we say that a subset of $\mathcal{P}(\mathcal{B}(H))$ is *analytic* (*closed, Borel,...*) if it is analytic (closed, Borel,...) in this topology (see [45]).

Question 5.3 was originally asked by Anderson in [4]. For a discussion and some related problems see [68] where it was almost answered.

Question 5.3. *Does every masa in the Calkin algebra that is generated by its projections lift to a masa in $\mathcal{B}(H)$?*

5.2. **Set-theoretic questions.** Here I collect some problems that will be probably be of more interest to set-theorists than to operator-algebraists.

Problem 5.4. Is there an analogue of 's analytic P-ideal theorem ([69]) for projections in $\mathcal{B}(H)$?[7]

Using [71] one can prove that the natural embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into the poset of projections of the Calkin algebra preserves all gaps (see [80]). It is not clear whether there is such an embedding for some other analytic ideal, although the poset of projections of the Calkin algebra contains gaps of very different nature from those that can be found in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ ([80]).

Question 5.5. *For which analytic ideals $\mathcal{I}$ on $\mathbb{N}$ is there an embedding of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into the post of projections on the Calkin algebra that preserves all gaps?*

Recall that for projections p and q one writes $p \leq q$ iff $pq = p$. Consider $\mathbb{P} = \mathcal{P}(\mathcal{C}(H)) \setminus \{0\}$ as a forcing notion. This is a quantized version of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, and it shares some of its properties. For example, it is σ-closed. Assuming CH¡ two forcing notions are isomorphic. It is well-known that the $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$-generic filter is a selective ultrafilter. The following is motivated by the importance of selective ultrafilters.

Problem 5.6. What are the combinatorial properties of the $\mathbb{P}$-generic ultrafilter? Does the quantization of the Mathias forcing share some of its remarkable properties?

A closely related problem.

Problem 5.7. Are there quantized versions of Ramsey, Galvin–Prikry and Silver theorems?

[7] Yes for order-ideals of positive operators. See [80]).

6. Tensor products

If A, B are C*-algebras there may be more than one way of defining a C*-algebra norm on the algebraic tensor product $A \odot B$. However, there are always the minimal and maximal tensor product norms on $A \odot B$, giving rise to C*-algebras $A \otimes_{\min} B$ and $A \otimes_{\max} B$. A C*-algebra A is *nuclear* if $A \otimes_{\min} B = A \otimes_{\max} B$ for every C*-algebra B. By a result of Junge and Pisier, $\mathcal{B}(H) \otimes_{\max} \mathcal{B}(H) \neq \mathcal{B}(H) \otimes_{\min} \mathcal{B}(H)$ (see [15, §13]).

In [75] Wassermann studied the question whether $C \otimes D$ is a masa in a tensor product $A \otimes B$ whenever C is a masa in A and D is a masa in D. The following problems are taken from [75, §4].

Problem 6.1 (Wassermann)**.** Assume C_1 and C_2 are masas in $\mathcal{B}(H)$.

(1) Is the relative commutant of $C_1 \otimes_{\min} 1$ in $\mathcal{B}(H) \otimes_{\max} \mathcal{B}(H)$ equal to $C_1 \otimes_{\min} \mathbb{C} \cdot \mathcal{B}(H)$?
(2) Is $C_1 \otimes_{\max} C_2$ a masa in $\mathcal{B}(H) \otimes_{\max} \mathcal{B}(H)$?

If C_1 has the *extension property*, stating that every pure state on C_1 has a unique extension to a pure state of $\mathcal{B}(H)$, then the answer to (2) above is positive by a result from [75]. Therefore the answer in the case when C_1 and C_2 are atomic masas would follow from a positive solution to the Kadison–Singer problem, Problem 1.2.

A subalgebra B of a C*-algebra A is *hereditary* if $bAb \subseteq B$ for every self-adjoint $b \in B$. An element a of a C*-algebra A is *fulll* if it is not contained in a proper (closed, two-sided, self-adjoint) ideal of A.

Problem 6.2 (Kirchberg)**.** Consider the quotient map

$$\pi \colon \mathcal{B}(H) \otimes_{\max} \mathcal{B}(H) \to \mathcal{B}(H) \otimes_{\min} \mathcal{B}(H).$$

Is there $a \in \ker(\pi)$ such that in the hereditary subalgebra generated by a there are no full projections?

Problem 6.3. If A and D are unital C*-algebras and A_0 is a unital subalgebra of A, is

$$A_0' \cap (A \otimes D) = (A_0' \cap A) \otimes D?$$

The inclusion $\supseteq$ always holds. The above question has a positive answer if D is nuclear or if A is LM (see [36, §2]; also cf. the 'slice map property,' e.g., [15]).

7. Ultrapowers

The most famous open problem about ultrapowers of operator algebras is the following *Connes Embedding Problem.*

Problem 7.1. Is every separable II_1 factor isomorphic to a subfactor of an ultrapower of the hyperfinite II_1 factor with respect to an ultrafilter on $\mathbb{N}$?

Just like its C*-algebraic version, Problem 7.2, this problem is absolute and it is therefore unlikely that the answer is independent from ZFC. On way to see this is by noting that a model-theoretic reformulation of the problem is 'is the universal theory of every II_1 factor equal to the universal theory of the hyperfinite II_1 factor?' Here the universal theory is taken in the logic of continuous structures ([9]).

The following problems come from [50] and a conversation with Eberhard Kirchberg. Ultrapowers of C*-algebras are defined as ultrapowers of metric structures ([37], [9]). Since this note was written with set theorists in mind, we do not denote an ultrafilter by ω and we write $\prod_{\mathcal{U}} A$ (instead of A^ω or $A^{\mathcal{U}}$) for the ultrapower of A. Recall that the *Cuntz algebra* $\mathcal{O}_2$ is the unital C*-algebra generated by two isometries s and t such that $s^*s = t^*t = 1$ and $ss^* + tt^* = 1$. Cuntz proved that these relations determine $\mathcal{O}_2$ up to the isomorphism (see [63]).

The following is a C*-algebraic relative of the famous Connes's Embedding Problem (see e.g., [54]).

Problem 7.2 (Kirchberg)**.** Does every separable C*-algebra embed into an ultrapower of $\mathcal{O}_2$ with respect to an ultrafilter on $\mathbb{N}$?

Every exact C*-algebra embeds into $\mathcal{O}_2$ (Kirchberg) and every separable C*-algebra trivially embeds into $\mathcal{B}(H)$ (and into $\mathcal{C}(H)$). Phillips ([57]) used a result of Junge–Pisier to show that there is no separable C*-algebra such that every separable C*-algebra embeds into it.

There is a simple separable C*-algebra A such that every separable C*-algebra embeds into an ultrapower of A. One takes A to be a separable elementary submodel of $\mathcal{C}(H)$ in the logic for metric structures ([9]). If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then the ultrapower $\prod_{\mathcal{U}} A$ is countably saturated and elementarily equivalent to the Calkin algebra (again in logic for metric structures [9]). Since every separable algebra embeds into the Calkin algebra, the conclusion follows.

If $\phi_j \colon A \to \prod_{\mathcal{U}} \mathcal{O}_2$ are *-homomorphisms write $\phi_1 \leq \phi_2$ if there is a partial isometry u in A such that $u^*\phi_2 u = \phi_1$.

Problem 7.3 (Kirchberg)**.** Assume A is separable and A embeds into $\prod_{\mathcal{U}} \mathcal{O}_2$. Is there a $\leq$-maximal embedding ϕ of A into $\prod_{\mathcal{U}} \mathcal{O}_2$?

A positive answer implies that A has the *local lifting property* (see [15, 13.1]).

Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Identify a C*-algebra A with its image via the diagonal embedding into the ultrapower $\prod_{\mathcal{U}} A$ and consider the *relative commutant* $A' \cap \prod_{\mathcal{U}} A$ and the *anihilator* $\operatorname{Ann}(A, \prod_{\mathcal{U}} A)$:

$$A' \cap \prod_{\mathcal{U}} A = \{b \in \prod_{\mathcal{U}} A \mid ab = ba \text{ for all } a \in A\}.$$

$$\operatorname{Ann}(A, \prod_{\mathcal{U}} A) = \{b \in \prod_{\mathcal{U}} A \mid ab = ba = 0\}.$$

Kirchberg ([50]) defines the invariant

$$F(A) = (A' \cap \prod_{\mathcal{U}} A) / \operatorname{Ann}(A, \prod_{\mathcal{U}} A).$$

Note that if A is unital then $\operatorname{Ann}(A, \prod_{\mathcal{U}} A) = \{0\}$ and $F(A)$ coincides with the relative commutant of A.

Problem 7.4 (Kirchberg)**.** Assume A is separable. Does $F(A)$ depend on the choice of the ultrafilter $\mathcal{U}$?[8]

The following is what remains of another question of Kirchberg.

Problem 7.5. Can one prove in ZFC that for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have $\mathcal{B}(H)' \cap \prod_{\mathcal{U}} \mathcal{B}(H) = \mathbb{C}I$?

[8] This problem has been solved. For infinite-dimensional C*-algebras the Continuum Hypothesis is equivalent to the assertion that the isomorphism type of $F(A)$ does not depend on the choice of the ultrafilter. See [24] for the case of C*-algebras of real rank zero and an upcoming paper by Farah–Hart–Sherman for the general C*-algebra case and the analogous problem for II_1 factors.

In [31] it was proved that selective ultrafilters have the property from Problem 7.5 and that there is (in ZFC) an ultrafilter $\mathcal{U}$ for which $\mathcal{B}(H)' \cap \prod_{\mathcal{U}} \mathcal{B}(H) \neq \mathbb{C}I$. Since CH implies the existence of selective ultrafilters, it implies that $F(\mathcal{B}(H))$ does depend on the choice of the ultrafilter $\mathcal{U}$ (cf. Problem 7.4). In [32] it was proved that p-points have the property from Problem 7.5.

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is *flat* if there are $h_n \colon \mathbb{N} \searrow [0,1]$ such that

(1) $h_n(0) = 1$,
(2) $\lim_j h_n(j) = 0$,
(3) $(\forall f \colon \mathbb{N} \nearrow \mathbb{N}) \lim_{n \to \mathcal{U}} \sup_{j \in \mathbb{N}} |h_n(j) - h_n(f(j))| = 0$.

Question 7.6. *Is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ flat if and only if $\mathcal{B}(H)' \cap \mathcal{B}(H)^{\mathcal{U}} \neq \mathbb{C}I$?*

The direct implication was proved in [31]. It is not clear whether a flat ultrafilter can be constructed in ZFC.

The present state of ignorance is reflected in the following question.

Question 7.7. *Can $\mathcal{B}(H)' \cap \prod_{\mathcal{U}} \mathcal{B}(H)$ be nonabelian?*

8. Nonseparable C*-algebras

The first systematic study of nonseparable C*-algebras ([36]) resulted in a number of surprising results and embarrassing open problems.

Problem 8.1 (Farah–Katsura)**.** Assume A is a tensor product of algebras of the form $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, $\kappa < \kappa'$ are cardinals and $\bigotimes_{\kappa'} M_2(\mathbb{C})$ unitally embeds into $A \otimes \bigotimes_{\kappa} M_2(\mathbb{C})$. Can we conclude that there is a unital embedding of $\bigotimes_{\kappa} M_2(\mathbb{C})$ into A?

Such A would have to be nonseparable, and the simplest open case is whether $\bigotimes_{\aleph_1} M_2(\mathbb{C})$ unitally embeds into $\bigotimes_{\aleph_0} M_2(\mathbb{C}) \otimes \bigotimes_{\aleph_1} M_3(\mathbb{C})$. See the last section of [36] for more information.

Question 8.2 (Kirchberg)**.** *Is every exact C*-algebra a subalgebra of a nuclear C*-algebra?*

As pointed out in §3.2, the answer is positive for separable algebras since every separable exact C*-algebra is a subalgebra of $\mathcal{O}_2$. The following related problem came up in an engaging conversation during the March 2010 Oberwolfach meeting.

Question 8.3. *Is there a universal nuclear C*-algebra of character density $\aleph_1$? More generally, for which cardinals κ is there a universal nuclear C*-algebra of character density κ?*

Similar question can be asked for exact algebras. As pointed out earlier, for $\kappa = \aleph_0$ the universal algebra is $\mathcal{O}_2$. Nate Brown pointed out that every separable C*-algebra is a quotient of the free group C*-algebra $C^*(\mathbb{F}_\infty)$, hence $C^*(\mathbb{F}_\infty)$ is a surjectively universal separable C*-algebra. Similarly, the free group C*-algebra with κ generators is surjectively universal for C*-algebras of character density κ for every infinite cardinal κ.

The following question was suggested by N.C. Phillips (who also characterized it as 'minor').

Question 8.4. *Suppose a C*-algebra has an approximate identity consisting of projections. Does it have an increasing approximate identity (on some index set, possibly different) consisting of projections?*

The answer is positive in the separable case. A curious fact (that a short reflection shows to be relevant to the above) pointed out by N.C. Phillips is that there is a real rank 0 C*-algebra in which the set of projections is not directed—i.e., there are two projections p and q such that there is no projection r such that $r \geq p$ and $r \geq q$. A simple example of (provided by Takeshi Katsura) an algebra in which the set of projections is not directed was given by the algebra $\{f \in M_2(C([0,1]) \mid f(0) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\}$. If p and q are two projections of rank two such that $p(1) \neq q(1)$, then any projection r such that $r \geq p$ and $r \geq q$ would have to have rank 2.

8.1. **Representation theory.** A representation $\pi\colon A \to \mathcal{B}(H)$ is *factorial* if there is a vector $\xi \in H$ whose orbit $\pi[A]\xi$ is dense in H (cf. §1.1.2). The existence of such vector is equivalent to stating that the weak closure of $\pi[A]$ is a *factor*, i.e., a von Neumann algebra such that its intersection with its commutant is equal to $\mathcal{C}$.

Answering a question of Dixmier, Weaver used transfinite induction to construct a C*-algebra A which is *prime* (i.e., every two ideals of A have nontrivial intersection) but A has no irreducible representation ([77]; see also [18] and [44] for simpler constructions). Known examples of such algebras are nonseparable and don't have factorial representations. Nonseparability is necessary by a result of Dixmier. The following was suggested by Bruce Blackadar.

Question 8.5. *Is there a C*-algebra that has a factorial representation but no irreducible representation?*

By a result of Glimm, for a separable C*-algebra A the following conditions are equivalent.

(1) A is not of type I,
(2) A has the CAR algebra as a subquotient,
(3) A has type II factorial representation,
(4) A has type III factorial representation,
(5) A has continuum many nonequivalent irreducible representations.

Glimm's result was extended to the nonseparable case, mostly by a work of Sakai. However, two prominent problems remain. First, it is not clear whether a non-type I algebra necessarily has two irreducible representations. This is Naimark's problem, and as pointed out in §1.1 a negative answer is relatively consistent with ZFC. S. Wassermann pointed out the following problem.

Question 8.6. *Does every non-type I C*-algebra necessarily have a type II factorial representation?*

By a result of Anderson ([5]), the answer is positive in case of $\mathcal{B}(H)$. Combinatorics used in this proof is related to the Kadison–Singer problem.

9. Other problems

A *derivation* is a map $\delta\colon A \to B$ such that

$$\delta(ab) = a\delta(b) + \delta(a)b$$

(A and B are assumed to be C*-algebras). If A is a subalgebra of B and $x \in B$ then $\delta_x \colon A \to B$ defined by

$$\delta_x(a) = ax - xa$$

is a derivation. Such derivations are *inner*. I would like to thank Julien Giol for turning my attention to the following prominent open problem.

Problem 9.1 (The derivation problem)**.** If A is a C*-subalgebra of $\mathcal{B}(H)$, is every derivation of A into $\mathcal{B}(H)$ inner?

A positive answer to this problem is known to be equivalent to a positive answer to some other outstanding open problems. For more information see e.g., [49] and the survey [17].

Question 9.2 arose in a recent work by N.C. Phillips and S. Wassermann who gave a positive answer under some additional assumptions. Although a positive answer seems unlikely, no conterexample is known.

Question 9.2. *Assume $A = \varinjlim A_\lambda$ and D_λ is a masa in A_λ for each λ. Is $D = \varinjlim D_\lambda$ necessarily a masa in A?*

The following problem was communicated to me by N.C. Phillips. Even though the positive answer is, being a $\mathbf{\Pi^1_2}$ statement, absolute, it may be interesting to set theorists.

Problem 9.3. Is every separable simple C*-algebra A generated by a single element? What if in addition we assume A is nuclear?

Every separable unital C*-algebra A is contained in a singly generated separable unital C*-algebra. By [53], both $A \otimes \mathcal{K}(H)$ and $A \otimes M_{2^\infty}$ are singly generated whenever A is separable.

Problem 9.4. Is there a separable, purely infinite C*-algebra not isomorphic to its opposite algebra?

A positive solution would follow from a confirmation of Conjecture 9.5 below. If X is a countable elementary submodel of H_θ for $\theta \geq \mathfrak{c}^+$ and $M \in X$ is a nonseparable C*-algebra, let M_X denote the norm-closure of $X \cap M$ in M.

Conjecture 9.5. *Assume X is a countable elementary submodel of $H_{\mathfrak{c}^+}$ and separably acting factors M, N belong to X. Then $M_X \cong N_X$ if and only if $M \cong N$.*

Alternatively, instead of M_X and N_X one may consider separable elementary submodels (in the sense of the logic for metric structures) of M and N. However, note that two separable elementary submodels of the same algebra are not necessarily isomorphic. Conjecture 9.5 is true in the case when M and N are II_1 factors. This follows from an argument of [58] using uniqueness of the trace on II_1 factors. Phillips used this and Connes's II_1 factor not isomorphic to itself to show that there is a simple unital separable stably finite C*-algebra not isomorphic to its opposite algebra. A confirmation of Conjecture 9.5, together with Connes's III_λ factor not isomorphic to its opposite, would give a positive answer to Problem 9.4. It would also give a variety of new examples of simple separable (non-exact) C*-algebras.

9.1. **Operator systems.** The following problem is not exactly about C*-algebras but it certainly has a strong set-theoretic flavor (see the last section of [7]). An *operator system* is a self-adjoint vector subspace S of a unital C*-algebra that contains the unit. A *boundary representation* of S (as defined in [7]) is an irreducible representation of $C^*(S)$ such that its restriction to S has a unique extension to $C^*(S)$. An operator system S has *sufficiently many boundary representations* if for every $a \in S$ we have $\|a\| = \sup \|\pi(a)\|$ where π ranges over all boundary representations of S.

Problem 9.6 (Arveson)**.** Does every operator system have sufficiently many boundary representations?

This question was originally motivated by Arveson's notion of *noncommutative Choquet boundary* of an operator system. The separable case of Problem 9.6 was proved in [7]. Some parts of this proof (e.g., [7, Theorem 2.5]) work in the general case with obvious modifications. However, one of the key parts of the proof, the use of disintegration theory (see §9.2), is closely tied to the separable case. I don't know whether the separability of S is necessary in [7, Lemma 8.3].

9.2. **Von Neumann algebras.** Basic terminology on von Neumann algebras can be found in [43] and in [11]. There is a rich interaction and rapidly growing interaction between descriptive set theory and the theory of von Neumann algebras, such as applications of Popa superrigidity to the structure of countable Borel equivalence relations. I will mention some other problems.

Question 9.7 was suggested by S. Popa, but I should point out that it was a test problem for a problem solved in [16]. The latter paper warrants additional attention from continuous model theorists.

Question 9.7. *Are all tracial ultraproducrts of matrix algebras $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, isomorphic?*

The answer is clearly negative for C*-algebra ultraproducts. It is also negative if CH fails ([28]), and the question is equivalent to the question whether metric theories of matrix algebras converge (see [9] and [29]).

The following question about ultraproducts of II_1 factors also comes from Sorin Popa.

Question 9.8. *Assume M is a separable II_1 factor and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ such that $(M' \cap M^{\mathcal{U}})' = M$. Is M necessarily isomorphic to the hyperfinite II_1 factor R?*

The fact that $(R' \cap R^{\mathcal{U}})' = R$ is an unpublished result of Popa. Question 9.8 is a test question to the following, more difficult, conjecture of Popa.

Conjecture 9.9. *Assume M is a separable II_1 factor not isomorphic to R. Is it true that there exists a subfactor N of M such that $(N' \cap M)^{\mathcal{U}} = N' \cap M^{\mathcal{U}}$?*

Every separable von Neumann algebra with separable predual can be decomposed as an integral of factors. Dan Voiculescu has pointed out that it is not known whether the separability assumption is necessary.

Problem 9.10. Is there analoguos disintegration theory for von Neumann algebras that don't necessarily have a separable predual?

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 35, Belgrade, Serbia

URL: `http://www.math.yorku.ca/~ifarah`

E-mail address: `ifarah@mathstat.yorku.ca`