A DICHOTOMY FOR THE MACKEY BOREL STRUCTURE

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Abstract. We prove that the equivalence of pure states of a separable C*-algebra is either smooth or it continuously reduces $[0,1]^{\mathbb{N}}/\ell_2$ and it therefore cannot be classified by countable structures. The latter was independently proved by Kerr–Li–Pichot by using different methods. We also give some remarks on a 1967 problem of Dixmier.

If E and F are Borel equivalence relations on Polish spaces X and Y, respectively, then we say that E is *Borel reducible* to F (in symbols, $E \leq_B F$) if there is a Borel-measurable map $f\colon X \to Y$ such that for all x and y in X we have xEy if and only if $f(x)Ff(y)$. A Borel equivalence relation E is *smooth* if it is Borel-reducible to the equality relation on some Polish space. Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ defined by xE_0y if and only if $x(n) = y(n)$ for all but finitely many n. The *Glimm–Effros* dichotomy ([8]) states that a Borel equivalence relation E is either smooth or $E_0 \leq_B E$.

One of the themes of the abstract classification theory is measuring relative complexity of classification problems from mathematics (see e.g., [12]). One can formalize the notion of 'effectively classifiable by countable structures' in terms of the relation $\leq_B$ and a natural Polish space of structures based on $\mathbb{N}$ in a natural way. In [10] Hjorth introduced the notion of turbulence for orbit equivalence relations and proved that an orbit equivalence relation given by a turbulent action cannot be effectively classified by countable structures.

The idea that there should be a small set $\mathcal{B}$ of Borel equivalence relations not classifiable by countable structures such that for every Borel equivalence relation E not classifiable by countable structures there is $F \in \mathcal{B}$ such that $F \leq_B E$ was put forward in [11] and, in a revised form, in [4]. In this note we prove a dichotomy for a class of Borel equivalence relations corresponding to the spectra of C*-algebras by showing that one of the standard turbulent orbit equivalence relations, $[0,1]^{\mathbb{N}}/\ell_2$, is Borel-reducible to every non-smooth spectrum.

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States. All undefined notions from the theory of C*-algebras and more details can be found in [2] or in [5]. Consider a separable C*-algebra A. Recall that a functional ϕ on A is *positive* if it sends every positive operator in A to a positive real number. A positive functional is a *state* if it is of norm ≤ 1. The states form a compact convex set, and the extreme points of this set are the *pure states*. The space of pure states on A, denoted by $\mathbb{P}(A)$, equipped with the weak*-topology, is a Polish space ([14, 4.3.2]).

A C*-algebra A is *unital* if it has the multiplicative identity. Otherwise, we define the *unitization* of A, $\tilde{A}$, the canonical unital C*-algebra that has A as a maximal ideal and such that the quotient $\tilde{A}/A$ is isomorphic to $\mathbb{C}$ (see [5, Lemma 2.3]). If u is a unitary in A (or $\tilde{A}$) then

$$(\operatorname{Ad} u)a = uau^*$$

defines an inner automorphism of A.

Two pure states ϕ and ψ are equivalent, $\phi \sim_A \psi$, if there exists a unitary u in A (or $\tilde{A}$) such that $\phi = \psi \circ \operatorname{Ad} u$.

Theorem 1. *Assume A is a separable C*-algebra. Then $\sim_A$ is either smooth or there is a continuous map*

$$\Phi\colon [0,1]^{\mathbb{N}} \to \hat{A}$$

such that $\alpha - \beta \in \ell_2$ if and only if $\Phi(\alpha) \sim_A \Phi(\beta)$.

Corollary 2. *Assume A is a separable C*-algebra. Then either $\sim_A$ is smooth or it cannot be classified by countable structures.*

Proof. By [10] it suffices to show that a turbulent orbit equivalence relation is Borel-reducible to $\sim_A$ if $\sim_A$ is not smooth. The equivalence relation $[0,1]^{\mathbb{N}}/\ell_2$ is well-known to be turbulent (e.g., [11]) and the conclusion follows by Theorem 1. □

This result was independently proved in [13, Theorem 2.8] by directly showing the turbulence. As pointed out in [13, §3], it implies an analogous result of Hjorth ([9]) on irreducible representations of discrete groups, as well as its strengthening to locally compact groups.

1. Proof of Theorem 1

Recall that the CAR (Canonical Anticommutation Relations) algebra (also know as the Fermion algebra, or M_{2^∞}) is defined as the infinite tensor product

$$M_{2^\infty} = \bigotimes_{n\in\mathbb{N}} M_2(\mathbb{C})$$

where $M_2(\mathbb{C})$ is the algebra of 2×2 matrices. Alternatively, one may think of M_{2^∞} as the direct limit of $2^n \times 2^n$ matrix algebras $M_{2^n}(\mathbb{C})$ for $n \in \mathbb{N}$.

The following analogue of the Glimm–Effros dichotomy is an immediate consequence of [6] (Notably, the key combinatorial device in the proof of [8] comes from Glimm).

Proposition 3. *If A is a separable C*-algebra then exactly one of the following applies.*

(1) $\sim_A$ *is smooth.*
(2) $\sim_{M_{2^\infty}} \leq_B \sim_A$. □

We shall prove that $\sim_{M_{2^\infty}}$ is turbulent in the sense of Hjorth.

Lemma 4. *If ξ and η are unit vectors in H then*

(*) $\inf\{\|I-u\| : u \textit{ unitary in } \mathcal{B}(H) \textit{ and } (u\xi|\eta)=1\} = \sqrt{2(1-|(\xi|\eta)|)}$.

Proof. Let $t=(\xi|\eta)$. Let $\xi' = \frac{1}{\|\operatorname{proj}_{\mathbb{C}\eta}\xi\|}\operatorname{proj}_{\mathbb{C}\eta}\xi$. Then the square of the left-hand side of (*) is greater than or equal to"

$$\|\xi-\xi'\|^2 = \|\xi\|^2 + \|\xi'\|^2 - \frac{2}{(\xi|\eta)}(\xi|(\xi|\eta)\eta) = 2-2|t|.$$

For $\leq$ let ζ be the unit vector orthogonal to ξ such that

$$\eta = t\xi + \sqrt{1-t^2}\zeta$$

and let u be the unitary given by $\begin{pmatrix} t & -\sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{pmatrix}$ on the span of ξ and ζ and identity on its orthogonal complement. Then $u\xi=\eta$ and a straightforward computation gives $\|I-u\|^2 = 2-2t$ as required. □

If ξ is a unit vector in a Hilbert space then by ω_ξ we denote the vector state $a \mapsto (a\xi|\xi)$. If ξ_i is a unit vector in H_i for $1\leq i\leq m$ then $\xi = \bigotimes_{i=1}^m \xi_i$ is a unit vector in $H=\bigotimes_{i=1}^m \xi_i$ and ω_ξ is a vector state on $\mathcal{B}(H)$.

Lemma 5. *If H_i is a Hilbert space and ξ_i, η_i are unit vectors in H_i for $1\leq i\leq m$ then*

$$\inf\{\|I-u\| : u \textit{ unitary and } \omega_{\bigotimes_{i=1}^m \xi_i} = \omega_{\bigotimes_{i=1}^m \eta_i}\circ \operatorname{Ad} u\} = 2\sqrt{2(1-\textstyle\prod_{i=1}^m |(\xi_i|\eta_i)|)}.$$

Proof. The case when $m=1$ follows from Lemma 4 and the fact that $\omega_\xi = \omega_{\alpha\xi}$ when $|\alpha|=1$. Since $(\bigotimes_{i=1}^m \xi_i | \bigotimes_{i=1}^m \eta_i) = \prod_{i=1}^m (\xi_i|\eta_i)$, the general case is an immediate consequence of Lemma 4. □

Theorem 6. *There is a continuous map $\Phi\colon (-\frac{\pi}{2},\frac{\pi}{2})^{\mathbb{N}} \to \mathbb{P}(M_{2^\infty})$ such that for all $\vec{\alpha}$ and $\vec{\beta}$ in the domain we have*

$$\sum_n (\alpha_n-\beta_n)^2 < \infty \Leftrightarrow \Phi(\vec{\alpha}) \sim_{M_{2^\infty}} \Phi(\vec{\beta}).$$

Proof. Consider the standard representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. Then the pure states of $M_2(\mathbb{C})$ are of the form $\omega_{(\cos\alpha,\sin\alpha)}$ for $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Let $\Phi(\vec{\alpha}) = \bigotimes_{n=1}^\infty \omega_{(\cos\alpha_n,\sin\alpha_n)}$. This map is continuous: If $a\in M_{2^\infty}$ and $\varepsilon>0$, fix m and $a'\in M_{2^m}$ such that $\|a-a'\|<\varepsilon/2$. Then $\Phi(\vec{\alpha})(a')$ depends only on α_j for $j\leq m$, and in a continuous fashion.

Recall that for $0 < t_j < 1$ we have $\prod_{j=1}^{\infty} t_j > 0$ if and only if $\sum_{j=1}^{\infty}(1-t_j) < \infty$. Therefore

$$\sum_{n=1}^{\infty}(\alpha_n - \beta_n)^2 < \infty \Leftrightarrow \sum_{n=1}^{\infty} \sin^2\left(\frac{\alpha_n - \beta_n}{2}\right) < \infty \Leftrightarrow \prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) > 1.$$

Assume $\prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) > 0$. In the n-th copy of M_2 in $M_{2^\infty} = \bigotimes_{n=1}^{\infty} M_2$ pick a unitary u_n such that

$$\|1 - u_n\| < \sqrt{2(1 - |\cos(\alpha_n - \beta_n)|)}$$

and $u_n(\cos\alpha_n, \sin\alpha_n) = (\cos\beta_n, \sin\beta_n)$. Note that

$$((\cos\alpha_n, \sin\alpha_n)|(\cos\beta_n, \sin\beta_n)) = \cos(\alpha_n - \beta_n).$$

Let $v_n = \bigotimes_{j=1}^{n} u_j$. Then v_n for $n \in \mathbb{N}$ form a Cauchy sequence, because $v_m - v_{m+n} = v_m(1 - \bigotimes_{j=m+1}^{n} u_j)$ and therefore

$$\|v_m - v_n\| < \sqrt{2(1 - \prod_{j=m}^{\infty} \cos(\alpha_j - \beta_j))}.$$

Let $v \in M_{2^\infty}$ be the limit of this Cauchy sequence. Since for each m and $a \in M_{2^m}$ we have $\Phi(\vec{\alpha})(a) = \Phi(\vec{\beta})(v_n a v_n^*)$ for any $n \geq m$, we have $\Phi(\vec{\alpha}) = \Phi(\vec{\beta}) \circ \operatorname{Ad} v$.

Now assume $\Phi(\vec{\alpha}) \sim_{M_{2^\infty}} \Phi(\vec{\beta})$ and, for the sake of obtaining a contradiction, that $\prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) = 0$. There is m and a unitary $u \in M_{2^m}$ such that

$$\|\Phi(\vec{\alpha}) - \Phi(\vec{\beta}) \circ \operatorname{Ad} u\| < \frac{1}{2}.$$

(by e.g., [6]). However, we can find $n > m$ large enough so that with $\xi_n = \bigotimes_{j=m}^{n}(\cos\alpha_j, \sin\alpha_j)$ and $\eta_n = \bigotimes_{j=m}^{n}(\cos\beta_j, \sin\beta_j)$ the quantity

$$(\xi_n|\eta_n) = \prod_{j=m}^{n} \cos(\alpha_j - \beta_j)$$

is as close to zero as desired. Then $\|\omega_{\xi_n} - \omega_{\eta_n}\|$ is as close to 2 as desired, since $a_n = \operatorname{proj}_{\mathbb{C}\xi_n} - \operatorname{proj}_{\mathbb{C}\eta_n}$ has norm close to 1 and $\omega_{\xi_n}(a_n)$ is close to 1 while $\omega_{\eta_n}(a_n)$ is close to -1. □

Proof of Theorem 1. Assume $\sim_A$ is not smooth. The conclusion follows by Glimm's Proposition 3 and Theorem 6. □

2. Concluding remarks

We note that the class of equivalence relations corresponding to spectra of C*-algebras is restrictive in another sense. The following proposition was probably well-known (cf. [9, Corollary 1.3]).

Proposition 7. *If A is a separable C*-algebra then the relation $\phi \sim_A \psi$ on $\mathbb{P}(A)$ is F_σ.*

Proof. By replacing A with its unitization if necessary we may assume A is unital. Fix a countable dense set $\mathcal{U}$ in the unitary group of A and a countable dense set $\mathcal{D}$ in $A_{\leq 1}$. We claim that

$$\phi \sim_A \psi \Leftrightarrow (\exists u \in \mathcal{U})(\forall a \in \mathcal{D})|\phi(a) - \psi(uau^*)| < 1.$$

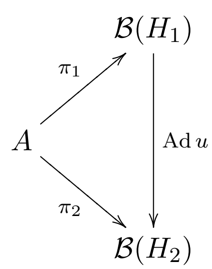
Assume $\phi \sim_A \psi$ and fix v such that $\phi = \psi \circ \operatorname{Ad} v$. If $u \in \mathcal{U}$ is such that $\|v - u\| < 1/2$ then

$$|\psi(uau^* - vav^*)| = |\psi((u - v)au^* - va(u^* - v^*))| < 1$$

for all $a \in A_{\leq 1}$.

Now assume $u \in \mathcal{U}$ is such that $|\phi(a) - \psi(uau^*)| < 1$ for all $a \in \mathcal{D}$. Then $\|\phi - \psi \circ \operatorname{Ad} u\| < 2$ and by [7] we have $\phi \sim_A \psi$. □

For a Hilbert space H by $\mathcal{B}(H)$ we denote the algebra of its bounded linear operators. Let $\pi_1 \colon A \to \mathcal{B}(H_1)$ and $\pi_2 \colon A \to \mathcal{B}(H_2)$ be representations of A. We say π_1 and π_2 are *(unitarily) equivalent* and write $\pi_1 \sim \pi_2$ if there is a Hilbert space isomorphism $u \colon H_1 \to H_2$ such that the diagram



$$\operatorname{Ad} u(a) = uau^*$$

commutes.

A representation of A on some Hilbert space H is *irreducible* if there are no nontrivial closed subspaces of H invariant under the image of A. The spectrum of A, denoted by $\hat{A}$, is the space of all equivalence classes of irreducible representations of A. The GNS construction associates a representation π_ϕ of A to each state ϕ of A (see e.g., [5, Theorem 3.9]). Moreover, ϕ is pure if and only if π_ϕ is irreducible ([5, Theorem 3.12]) and for pure states ϕ_1 and ϕ_2 we have that ϕ_1 and ϕ_2 are equivalent if and only if π_{ϕ_1} and π_{ϕ_2} are equivalent ([5, Proposition 3.20]).

Fix a separable C*-algebra A. Let $\operatorname{Irr}(A, H_n)$ denote the space of irreducible representations of A on a Hilbert space H_n of dimension n for $n \in \mathbb{N} \cup \{\aleph_0\}$. Each $\operatorname{Irr}(A, H_n)$ is a Polish space with respect to the weakest topology making all functions $\operatorname{Irr}(A, H_n) \ni \pi \mapsto (\pi(a)\xi|\eta) \in \mathbb{C}$, for $a \in A$ and $\xi, \eta \in H_n$, continuous. In other words, a net π_λ converges to π if and only if $\pi_\lambda(a)$ converges to $\pi(a)$ for all $a \in A$. Since A is separable, each irreducible representation of A has range in a separable Hilbert space, and therefore $\hat{A}$ can be considered as a quotient space of the direct sum of $\operatorname{Irr}(A, H_n)$ for $n \in \mathbb{N} \cup \{\aleph_0\}$. Therefore $\hat{A}$ carries a Borel structure (known as the *Mackey Borel structure*) inherited from a Polish space. For *type I*

C*-algebras (also called *GCR* or *postliminal*) this space is a standard Borel space. (All of these notions are explained in [1, §4].)

Since pure states correspond to irreducible representations, we can identify the Mackey Borel structure of A with a σ-algebra of sets in $\hat{A}$. It is easy to check that this σ-algebra consists exactly of those sets whose preimages in $\mathbb{P}(A)$ are Borel subsets in $\mathbb{P}(A)$.

Glimm proved ([6], [14, §6.8]) that the Mackey Borel structure of a C*-algebra A is *smooth* (i.e., isomorphic to a standard Borel space) if and only if A is a type I C*-algebra. Proposition 3 is a consequence of this result.

Problem 8 (Dixmier, 1967)**.** Is the Mackey Borel structure on the spectrum of a simple separable C*-algebra always the same when it is not standard?

G. Elliott generalized Glimm's result and proved that the Mackey Borel structures of simple AF algebras are isomorphic ([3]). (A C*-algebra is an AF (approximately finite) algebra if it is a direct limit of finite-dimensional algebras.) One reformulation of Elliott's result is that for any two simple separable AF algebras A and B there is a Borel isomorphism $F\colon \mathbb{P}(A) \to \mathbb{P}(B)$ such that $\phi \sim_A \psi$ if and only if $F(\phi) \sim_B F(\psi)$ (see [3, §6]). Also, [3, Theorem 2] implies that if A is a simple separable AF algebra and B is a non-Type I simple separable algebra we have $\sim_A \leq_B \sim_B$.

With this definition the quotient structure $\mathrm{Borel}(\mathbb{P}(A))/\sim_A$ is isomorphic to the Mackey Borel structure of A. Note that $\sim_A$ is smooth exactly when the Mackey Borel structure of A is smooth.

Note that Mackey Borel structures of A and B of separable C*-algebras are isomorphic if and only if there is a Borel isomorphism $f\colon \hat{X} \to \hat{X}$ such that $\pi_1 \sim_A \pi_2$ if and only if $f(\pi_1) \sim_B f(\pi_2)$. Hence Problem 8 is rather close in spirit to the theory of Borel equivalence relations.

N. Christopher Phillips suggested more general problems about the Mackey Borel structure of simple separable C*-algebras, motivated by his discussions with Masamichi Takesaki. There are two (related) kinds of questions: Can one do anything sensible, and, from the point of view of logic, how bad is the problem?

Problem 9. Does the complexity of the Mackey Borel structure of a simple separable C*-algebra increase as one goes from nuclear C*-algebras to exact ones to ones that are not even exact?

For definitions of nuclear and exact C*-algebras see e.g., [2].

Problem 10. Assume A and B are C*-algebras and $\sim_A$ is Borel-reducible to $\sim_B$. What does this fact imply about the relation between A and B?

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