THE COMMUTANT OF $L(H)$ IN ITS ULTRAPOWER MAY OR MAY NOT BE TRIVIAL

ILIJAS FARAH, N. CHRISTOPHER PHILLIPS, AND JURIS STEPRĀNS

Abstract. Kirchberg asked in 2004 whether the commutant of $L(H)$ in its (norm) ultrapower is trivial. Assuming the Continnuum Hypothesis, we prove that the answer depends on the choice of the ultrafilter.

Let H be a separable infinite dimensional complex Hilbert space, fixed throughout. The purpose of this paper is to prove that, assuming the Continuum Hypothesis, the commutant of $L(H)$ (the algebra of bounded linear operators on H) in its ultrapower depends on the choice of the ultrafilter. This provides a somewhat surprising—and somewhat incomplete—answer to Question 2.22 of [11].

We follow the convention that $0 \in \mathbb{N}$. Let A be a C*-algebra, and let $\mathcal{V}$ be a nonprincipal ultrafilter on $\mathbb{N}$ (equivalently, a point in $\beta\mathbb{N} \setminus \mathbb{N}$). (In the operator algebra literature, nonprincipal ultrafilters are usually denoted ω; since in the set theory literature ω is reserved for the least infinite ordinal we suppress using this symbol altogether.) We denote by $\ell^\infty(A)$ the C*-algebra of all bounded functions from $\mathbb{N}$ to A, and we denote by $A^{\mathcal{V}}$ the ultrapower of A associated with $\mathcal{V}$ ([7], [2]). More precisely, $A^{\mathcal{V}}$ is the quotient of $\ell^\infty(A)$ by the (two-sided, norm-closed, selfadjoint) ideal

$$c_{\mathcal{V}}(A) = \big\{(a_n) \in \ell^\infty(A) \colon \lim_{n\to\mathcal{V}} \|a_n\| = 0\big\}.$$

(This algebra is denoted $A_{\mathcal{V}}$ in [11] and [12].) We identify A with the subalgebra of $A^{\mathcal{V}}$ consisting of all

$$(a, a, a, \ldots) + c_{\mathcal{V}}(A)$$

for $a \in A$, that is, the image in $A^{\mathcal{V}}$ of all constant sequences in $\ell^\infty(A)$. Following Kirchberg [11], we define the invariant $F_{\mathcal{V}}(A)$ of A to be the quotient of $A' \cap A^{\mathcal{V}}$ by the annihilator

$$\operatorname{Ann}\big(A, A^{\mathcal{V}}\big) = \{b \in A^{\mathcal{V}} \colon ba = 0 \text{ for all } a \in A\}$$

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If A is unital then the annihilator is trivial and

$$F_{\mathcal{V}}(A) = A' \cap A^{\mathcal{V}}.$$

The algebra $F_{\mathcal{V}}(A)$ for unital A, often denoted $A_{\mathcal{V}}$ in the literature (see [8]), has played an important role in the study of C*-algebras, particularly in the classification of separable nuclear unital purely infinite simple C*-algebras and group actions on them. (See [10], [12], [18], and [8].) The most important result, proved for example in [12], is that if A is a separable nuclear unital purely infinite simple C*-algebra, then $F_{\mathcal{V}}(A)$ is again a purely infinite simple C*-algebra. See [7] for further discussion of the uses of algebras of this type, and [11] for more recent applications of $F_{\mathcal{V}}(A)$.

The use of $F_{\mathcal{V}}(A)$ for purely infinite simple C*-algebras parallels an older use of its tracial analog for factors of type II_1. (In place of operator norm convergence, one uses convergence in the $L2$ norm derived from the trace in the definition of $c_{\mathcal{V}}(A)$.) For example, if M is the hyperfinite factor of type II_1, then the tracial analog of $c_{\mathcal{V}}(M)$ is again a factor of type II_1. (See Lemma XIV.4.5 and Theorems XIV.4.6 and XIV.4.18 of [22]; this is also in [3].) As one example, this fact is used to prove that outer actions of suitable groups on the hyperfinite factor of type II_1 have the Rokhlin property, in turn a key step in the classification of such actions. See [9] for the case of finite groups, and [17] for the case of countable amenable groups.

Assuming the Continuum Hypothesis, Ge and Hadwin proved ([7, Corollary 3.4]) that if A is separable, then the isomorphism class of $F_{\mathcal{V}}(A)$ is independent of the choice of the nonprincipal ultrafilter $\mathcal{V}$.

Let $K(H)$ denote the ideal of compact operators in $L(H)$. For the Calkin algebra $C(H) = L(H)/K(H)$, Kirchberg proved ([11, Corollary 2.21]) that $F_{\mathcal{V}}(C(H)) = \mathbb{C}$. This also implies $F_{\mathcal{V}}(L(H)) \subseteq \mathbb{C} + K(H)^{\mathcal{V}}$. In [11, Question 2.22], Kirchberg asked whether $F_{\mathcal{V}}(L(H)) = \mathbb{C}$. The answer to this question is somewhat surprising.

Theorem 1. *There is a nonprincipal ultrafilter $\mathcal{V}$ such that $F_{\mathcal{V}}(L(H)) \neq \mathbb{C}$.*

Proof. This follows from Theorem 3.3 and Theorem 4.1. □

Theorem 2. *Assume the Continuum Hypothesis. Then there is a nonprincipal ultrafilter $\mathcal{V}$ such that $F_{\mathcal{V}}(L(H)) = \mathbb{C}$.*

Proof. This follows from Corollary 2.4, where we prove $F_{\mathcal{V}}(L(H)) = \mathbb{C}$ for a selective ultrafilter $\mathcal{V}$, and from the fact that the Continuum Hypothesis implies selective ultrafilters exist (Proposition 1.4). □

We record a curious consequence of Theorem 2.

Corollary 3. *Assume the Continuum Hypothesis. Then the isomorphism type of $F_{\mathcal{V}}(L(H))$ depends on the choice of the ultrafilter $\mathcal{V}$.* □

We don't know whether, in the absence of the Continuum Hypothesis, $F_{\mathcal{V}}(A)$ can depend on the choice of $\mathcal{V}$ for some separable C*-algebra A, or whether some axiom beyond ZFC is needed for the conclusion of Corollary 3.

As noted above, the Continuum Hypothesis implies that the isomorphism type of $F_{\mathcal{V}}(A)$ does not depend on A for a separable A. The reason is that, under the Continuum Hypothesis, for any two nonprincipal ultrafilters $\mathcal{V}$ and $\mathcal{W}$ on $\mathbb{N}$ there is an isomorphism between $A^{\mathcal{V}}$ and $A^{\mathcal{W}}$ that sends A to itself ([7, Theorem 3.1]). The latter fact is an immediate consequence of the fact, provable in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), that the unit ball of $A^{\mathcal{V}}$ is *countably saturated* in the logic of metric structures. (See [2, Definition 7.5]—this is the case when $\kappa = \aleph_1$, the least uncountable cardinal—and [2, Proposition 7.6].) Since the Continuum Hypothesis implies that the ultrapowers are of size $\aleph_1$, a back-and-forth construction easily gives an isomorphism between $A^{\mathcal{V}}$ and $A^{\mathcal{W}}$ that sends the copy of A in one ultrapower to the copy of A in the other ultrapower.

One curiosity deserves a mention here. By the countable saturatedness of ultrapowers, the Continuum Hypothesis implies that $L(H)^{\mathcal{U}}$ and $L(H)^{\mathcal{V}}$ are isomorphic for any two nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$. However, if $\mathcal{U}$ is selective and $\mathcal{V}$ is flat, then by Corollary 3 we have $F_{\mathcal{V}}(L(H)) \ncong F_{\mathcal{U}}(L(H))$ and therefore no isomorphism between $L(H)^{\mathcal{U}}$ and $L(H)^{\mathcal{V}}$ sends $L(H)$ to $L(H)$.

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1. Selective ultrafilters

Definition 1.1. Let M be a C*-algebra or a von Neumann algebra. We let $[a,b]$ denote the additive commutator, $ab - ba$. Following [11] we say that a norm-bounded sequence $(a_n)_{n\in\mathbb{N}}$ in A is *central* if for every $b \in M$ we have $\lim_{n\to\infty} \|[a_n, b]\| = 0$. A sequence $(a_n)_{n\in\mathbb{N}}$ is *trivial* if

$$\limsup_{n\to\infty} \inf_{\lambda\in\mathbb{C}} \|a_n - \lambda\| = 0.$$

(Note that such sequences are always central.) If $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$ then a norm-bounded sequence $(a_n)_{n\in\mathbb{N}}$ is a *$\mathcal{V}$-central sequence* if for every $b \in M$ we have $\lim_{n\to\mathcal{V}} \|[b, a_n]\| = 0$. A $\mathcal{V}$-central sequence $(a_n)_{n\in\mathbb{N}}$ is *trivial* if $\lim_{n\to\mathcal{V}} \inf_{\lambda\in\mathbb{C}} \|a_n - \lambda\| = 0$.

Note that we have defined norm central sequences and norm trivial central sequences, even in the case of a von Neumann algebra. These are not the same as the central sequences usually considered for a type II_1 factor.

Notation 1.2. If X is a set then $[X]2$ denotes the set of all two-element subsets of X and $[X]^\infty$ denotes the set of all infinite subsets of X. We consider the space $[\mathbb{N}]^\infty$ with the Polish topology inherited from the Cantor set (identified with the power set of $\mathbb{N}$).

Recall that a subset of a Polish space is *analytic* if it is a continuous image of a Borel subset of a Polish space. The following result is well-known, but we sketch a proof of the easy implications for convenience of the reader.

Theorem 1.3 (A. R. D. Mathias)**.** *The following are equivalent for an ultrafilter $\mathcal{V}$ on $\mathbb{N}$.*

(1) *If $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ are in $\mathcal{V}$ then there exists $X \in \mathcal{V}$ such $X \setminus \{0, 1, \dots, n\} \subseteq X_n$ for every $n \in X$.*
(2) *For every $g \colon \mathbb{N} \to \mathbb{N}$ there is $A \in \mathcal{V}$ such that $g \restriction A$ is either constant or injective.*
(3) *For every $E \subseteq [\mathbb{N}]2$ there is $A \in \mathcal{V}$ such that either $[A]2 \subseteq E$ or $[A]2 \cap E = \varnothing$.*
(4) *For every analytic $\mathbb{E} \subseteq [\mathbb{N}]^\infty$ there is $A \in \mathcal{V}$ such that either $[A]^\infty \subseteq \mathbb{E}$ or $[A]^\infty \cap \mathbb{E} = \varnothing$.*

Proof. We consider (4) implies (3). Let $E \subseteq [\mathbb{N}]2$. Let $\mathbb{E} \subseteq [\mathbb{N}]^\infty$ be the set of all $X \in [\mathbb{N}]^\infty$ such that the pair consisting of the least two elements of X is in E. Then $\mathbb{E}$ is analytic (and even open).

For (3) implies (2), given g let $\{m, n\} \in E$ if and only if $g(m) \neq g(n)$.

For (2) implies (1), we may assume $X_0 = \mathbb{N}$ and $\bigcap_{n=0}^\infty X_n = \varnothing$. Define $g(k) = n$ if $k \in X_n \setminus X_{n+1}$. Since $\mathcal{V}$ contains no set on which g is constant, (2) implies that there is $Y \in \mathcal{V}$ such that g is injective on Y.

Recursively find $0 = m_0 < m_1 < \cdots$ so that for all k and all $j \geq m_{k+1}$, we have $g(j) \geq m_k$. Let $g_1 \colon \mathbb{N} \to \mathbb{N}$ be defined by $g_1(j) = k$ if $m_k \leq j < m_{k+1}$. Let $Z \subseteq Y$ be a set in $\mathcal{V}$ on which g_1 is injective.

Assume for the moment that $X = Z \cap \bigcup_{k=0}^\infty [m_{2k}, \, m_{2k+1})$ is in $\mathcal{V}$. For $n \in X$ choose k such that $m_{2k} \leq n < m_{2k+1}$. Then $X \setminus \{0, 1, \dots, n\}$ is disjoint from $\{0, 1, \dots, m_{2k+2} - 1\}$. Therefore for $m \in X \setminus \{0, 1, \dots, n\}$ we have $g(m) \geq m_{2k+1} > n$. Hence $m \in X_n$ as required.

If $Z \cap \bigcup_{k=0}^\infty [m_{2k}, \, m_{2k+1}) \notin \mathcal{V}$, we instead set $X = Z \cap \bigcup_{k=0}^\infty [m_{2k+1}, \, m_{2k+2})$. This set is in $\mathcal{V}$ and an argument analogous to the above shows that $X \setminus \{0, 1, \dots, n\} \subseteq X_n$ for all $n \in X$.

The fact that (1) implies (4) is Mathias's theorem [15, Theorem 0.13]. □

An ultrafilter satisfying the conditions of Theorem 1.3 is said to be *selective* (or *Ramsey*, or even *happy*). More information on these remarkable objects can be found in [15] or [5]. (For example, see Section 5 and Theorem 4.9 of [5].)

It is well-known that the existence of selective ultrafilters can be deduced from the Continuum Hypothesis ([15, Proposition 0.11]). We sketch a proof of this fact for the convenience of the reader.

Proposition 1.4. *Assume the Continuum Hypothesis. Then there exists a nonprincipal selective ultrafilter.*

Proof. Let ω_1 be the first uncountable ordinal. Use the Continuum Hypothesis to enumerate all functions $g \colon \mathbb{N} \to \mathbb{N}$ as g_γ, for $\gamma < \omega_1$. We claim that there are infinite sets A_γ for $\gamma < \omega_1$ such that the following holds for $\gamma < \omega_1$:

(1) g_γ is either constant or injective on A_γ.
(2) $A_\gamma \setminus A_\delta$ is finite if $\delta < \gamma$.

We prove the claim by transfinite induction. If g_0 is constant on an infinite subset of $\mathbb{N}$, let A_0 be this set. Otherwise, the image of $\mathbb{N}$ under g_0 is infinite and we can find an infinite subset $A_0 \subseteq \mathbb{N}$ on which g_0 is injective.

Assume A_γ has been chosen. Use the argument above, with A_γ in place of $\mathbb{N}$ and $g_{\gamma+1}$ in place of g_0, to choose an infinite subset $A_{\gamma+1} \subseteq A_\gamma$ on which $g_{\gamma+1}$ is either constant or injective.

Now assume γ is a limit ordinal and the sets A_δ have been chosen for $\delta < \gamma$. Re-enumerate the sets A_δ for $\delta < \gamma$ as A'_j for $j \in \mathbb{N}$. For $j \in \mathbb{N}$ let

$$\eta = \max\left(\left\{\delta < \gamma \colon A_\delta = A'_k \text{ for some } k < j\right\}\right).$$

Then the set $A_\eta \setminus \bigcap_{k<j} A'_k$ is finite, and in particular $\bigcap_{k<j} A'_k$ is infinite for every j. We can therefore choose a sequence $n_0 < n_1 < n_2 < \cdots$ so that $n_j \in \bigcap_{k<j} A'_k$ for all $j \in \mathbb{N}$. Set $B = \{n_j \colon j \in \mathbb{N}\}$. Then B is infinite and $B \setminus A_\delta$ is finite for all $\delta < \gamma$. Now $A_{\gamma+1} \subseteq B$ is chosen as above. □

For $\mathbf{a} \in A^{\mathcal{V}}$ a sequence $(a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ such that $\mathbf{a} = (a_n)_{n\in\mathbb{N}} + c_{\mathcal{V}}(A)$ is called a *representing sequence*.

Theorem 1.5. *Assume M is a separably acting von Neumann algebra and $\mathcal{V}$ is selective ultrafilter. Then for $\mathbf{a} \in M^{\mathcal{V}}$ the following are equivalent.*
(1) $\mathbf{a} \in M' \cap M^{\mathcal{V}}$.
(2) $\mathbf{a}$ *has a representing sequence that is a central sequence.*

Proof. We need only prove that (1) implies (2). Fix $\mathbf{a} \in M^{\mathcal{V}}$ and a representing sequence $(a_n)_{n\in\mathbb{N}}$.

Consider the closed unit ball $\mathbf{B}$ of M with the weak operator topology and consider the power set $\mathcal{P}(\mathbb{N})$ with the Cantor set topology. These are both Polish spaces. For each $\varepsilon > 0$ define $\Phi_\varepsilon \colon \mathbf{B} \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Phi_\varepsilon(b, X) = \{n \in X \colon \|[a_n, b]\| > \varepsilon\}$$

for $b \in \mathbf{B}$ and $X \subseteq \mathbb{N}$. We claim that Φ_ε is Borel measurable. First, since multiplication is separately weak operator continuous and norm closed balls in M are also weak operator closed, for each fixed n the set $\{b \in \mathbf{B} \colon \|[a_n, b]\| > \varepsilon\}$ is open in $\mathbf{B}$. Now let $U \subseteq \mathcal{P}(\mathbb{N})$ be a basic open set, that is, there is $N \in \mathbb{N}$ and a finite subset $S \subseteq \{0, 1, \ldots, N\}$ such that U consists of all $X \subseteq \mathbb{N}$ with $X \cap \{0, 1, \ldots, N\} = S$. Set

$$V = \left\{(b, X) \in \mathbf{B} \times \mathcal{P}(\mathbb{N}) \colon S \subseteq \{n \in \mathbb{N} \colon \|[a_n, b]\| > \varepsilon\} \cap \{0, 1, \ldots, N\}\right\}.$$

Define W similarly, using the set $T = \{0, 1, \ldots, N\} \setminus S$ in place of S. Then

$$\Phi_\varepsilon^{-1}(U) = V \cap \left[(\mathbf{B} \times \mathcal{P}(\mathbb{N})) \setminus W\right].$$

Also, V is open, since if $(b_0, X_0) \in V$, and (b, X) is sufficiently close to (b_0, X_0), then

$$X \cap \{0, 1, \ldots, N\} = X_0 \cap \{0, 1, \ldots, N\} = S$$

and we still have $\|[a_n, b]\| > \varepsilon$ for $n \in S$. Similarly W is open. So $\Phi_\varepsilon^{-1}(U)$ is Borel. This proves the claim.

Define $Z = \coprod_{j=1}^{\infty}[\mathbf{B} \times \mathcal{P}(\mathbb{N})]$, and define $\Phi\colon Z \to \mathcal{P}(\mathbb{N})$ by letting $\Phi = \Phi_{1/j}$ on the j component of the disjoint union. Since M acts on a separable Hilbert space, Z is a Polish space. Clearly Φ is Borel. The subset $[\mathbb{N}]^\infty \subseteq \mathcal{P}(\mathbb{N})$ is Borel, since its complement is countable. Therefore $\Phi^{-1}([\mathbb{N}]^\infty)$ is Borel. It follows that the set

$$\mathbb{E} = \Phi\big(\Phi^{-1}([\mathbb{N}]^\infty)\big) = [\mathbb{N}]^\infty \cap \bigcup_{j=1}^{\infty} \Phi_{1/j}[\mathbf{B} \times \mathcal{P}(\mathbb{N})]$$

is analytic. Since $\mathcal{V}$ is selective, there is $X \in \mathcal{V}$ such that $[X]^\infty \subseteq \mathbb{E}$ or $[X]^\infty \cap \mathbb{E} = \varnothing$.

Let us first assume the second possibility applies. Then for each $b \in \mathbf{B}$ and $j \in \mathbb{N}$ with $j > 0$, the set $\{n \in X\colon \|[b, a_n]\| > 1/j\}$ is not in $\mathbb{E}$, and therefore must be finite. Let $a_n' = a_n$ if $n \in X$ and $a_n' = 1$ if $n \notin X$. Then $\{n \in \mathbb{N}\colon \|[b, a_n']\| > \varepsilon\}$ is finite for all $b \in \mathbf{B}$ and $\varepsilon > 0$, so $(a_n)_{n\in\mathbb{N}}$ is a central sequence. It represents $\mathbf{a}$ since $X \in \mathcal{V}$.

Now assume there is $X \in \mathcal{V}$ such that $[X]^\infty \subseteq \mathbb{E}$. In particular, there are $b \in \mathbf{B}$ and $j \in \mathbb{N}$ such that $\|[a_n, b]\| \geq 1/j$ for all $n \in X$; therefore $\mathbf{a} \notin M' \cap M^{\mathcal{V}}$. □

Corollary 1.6. *Assume M is a separably acting von Neumann algebra. If $\mathcal{V}$ is selective then the following are equivalent.*

1. *There exists a nontrivial central sequence in M.*
2. *There exists a nontrivial $\mathcal{V}$-central sequence in M.*

Proof. Assume $(a_n)_{n\in\mathbb{N}}$ is a nontrivial central sequence. By passing to a subsequence, we may assume there is $\varepsilon > 0$ such that $\inf_{\lambda\in\mathbb{C}} \|a_n - \lambda\| \geq \varepsilon$ for all n. This sequence is clearly a nontrivial $\mathcal{V}$-central sequence for any nonprincipal ultrafilter $\mathcal{V}$. The converse implication is an immediate consequence of Theorem 1.5. □

Let M be a type II_1 factor with (unique) tracial state τ. Let $\|\cdot\|_2$ be the standard $L2$-norm on M, defined by $\|a\|_2 = \sqrt{\tau(a^*a)}$. A bounded sequence (a_n) in M is *tracially central* if $\lim_n \|[b, a_n]\|_2 = 0$ for every $b \in M$. (This, rather than that of Definition 1.1, is the usual definition of a central sequence in this context.) The *tracial ultrapower* of a II_1 factor is the ultrapower of the metric structure $(M, \|\cdot\|_2)$ in the sense of [2]. Analogues of Theorem 1.5 and Corollary 1.6 for tracial ultrapowers of II_1 factors could be proved by mimicking the above proofs. However, in this context the assumption that $\mathcal{V}$ is a selective ultrafilter can be dropped. In fact, for every nonprincipal ultrafilter $\mathcal{V}$ on $\mathbb{N}$, the commutant of M in $M^{\mathcal{V}}$ is trivial if and only if M has no nontrivial central sequences. By an observation of McDuff ([16, remark after Lemma 5]) this follows by a diagonalization argument from the fact that the metric $\|\cdot\|_2$ on a II_1 factor is separable. Similarly, the analogues of

Theorem 1.5 and Corollary 1.6 hold when M is a separable C*-algebra and $\mathcal{V}$ is any nonprincipal ultrafilter on $\mathbb{N}$.

On the other hand, Theorem 1 and Theorem 2.1 below imply that some assumption on $\mathcal{V}$ in Theorem 1.5 and Corollary 1.6 is necessary in the case when $M = L(H)$.

2. No nontrivial central sequences

Let M be a von Neumann algebra with center $\mathcal{Z}(M)$. In [21, Proposition 3.1], David Sherman proved that a sequence $(a_n)_{n\in\mathbb{N}}$ in M is norm central if and only if $\lim_{n\to\infty} \inf_{z\in\mathcal{Z}(M)} \|a_n - z\| = 0$. In particular, if M is a factor then it has no nontrivial central sequences. Note that we are referring to **norm** central sequences, even if M is a II_1-factor. For the reader's convenience we provide a self-contained proof of the special case of Sherman's result needed in the proof of Theorem 2.

Theorem 2.1. *There are no nontrivial central sequences in $L(H)$.*

The proof of Theorem 2.1 will be given after two lemmas.

Lemma 2.2. *Let $c \in L(H)$ be selfadjoint. Suppose that $\delta = \inf_{\lambda\in\mathbb{C}} \|c-\lambda\| > 0$. Then for every $\varepsilon > 0$ there is a rank one partial isometry $s \in L(H)$ such that $\|[s, c]\| > 2\delta - \varepsilon$.*

Proof. We denote the spectrum of c by $\mathrm{sp}(c)$. Set $\lambda_1 = \inf(\mathrm{sp}(c))$ and $\lambda_2 = \sup(\mathrm{sp}(c))$. By considering the isomorphism $C^*(c, 1) \cong C(\mathrm{sp}(c))$, we see that $\delta = \frac{1}{2}(\lambda_2 - \lambda_1)$. Set

$$\varepsilon_0 = \frac{\varepsilon}{4\|c\| + 2}.$$

Choose $\varepsilon_1 > 0$ so small that if $\alpha \in \mathbb{R}$ satisfies $|\alpha-1| < \varepsilon_1\delta^{-1}$, then $|\alpha^{-1}-1| < \varepsilon_0$. We also require $\varepsilon_1 \leq \min(\varepsilon_0, \delta\varepsilon_0)$.

Choose $\xi_1, \eta \in H$ with

$$\|\xi_1\| = \|\eta\| = 1, \quad \|c\xi_1 - \lambda_1\xi_1\| < \varepsilon_1, \quad \text{and} \quad \|c\eta - \lambda_2\eta\| < \varepsilon_1.$$

Set

$$\mu = \eta - \langle \eta, \xi_1\rangle \xi_1 \quad \text{and} \quad \xi_2 = \|\mu\|^{-1}\mu.$$

Then $\|\xi_2\| = 1$ and $\langle \xi_1, \xi_2\rangle = 0$.

We need to estimate $\|c\xi_2 - \lambda_2\xi_2\|$. We begin as follows, using $\langle \xi_1, c\eta\rangle = \langle c\xi_1, \eta\rangle$ at the second step:

$$\begin{aligned} |(\lambda_2 - \lambda_1)\langle \xi_1, \eta\rangle| &= |\langle \xi_1, \lambda_2\eta\rangle - \langle \lambda_1\xi_1, \eta\rangle| \\ &= |\langle \xi_1,\, \lambda_2\eta - c\eta\rangle + \langle c\xi_1 - \lambda_1\xi_1,\, \eta\rangle| \\ &\leq \|\xi_1\| \cdot \|\lambda_2\eta - c\eta\| + \|\lambda_1\xi_1 - c\xi_1\| \cdot \|\eta\| < 2\varepsilon_1. \end{aligned}$$

It follows that

$$\|\mu - \eta\| = |\langle \xi_1, \eta\rangle| < \frac{2\varepsilon_1}{\lambda_2 - \lambda_1} = \varepsilon_1\delta^{-1},$$

so $\big|1 - \|\mu\|\big| < \varepsilon_1 \delta^{-1}$, and

$$\|\xi_2 - \eta\| \leq \big\| \|\mu\|^{-1}\mu - \mu \big\| + \|\mu - \eta\| < \varepsilon_0 + \varepsilon_1 \delta^{-1} \leq 2\varepsilon_0.$$

Therefore

$$\|c\xi_2 - \lambda_2 \xi_2\| < 2\varepsilon_0 \|c\| + \|c\eta - \lambda_2 \eta\| + 2\varepsilon_0 |\lambda_2| \leq (4\|c\| + 1)\varepsilon_0.$$

Now let $s \in L(H)$ be the partial isometry such that $s\xi_1 = \xi_2$ and $s\xi = 0$ whenever $\langle \xi, \xi_1 \rangle = 0$. Then

$$\|sc\xi_1 - \lambda_1 \xi_2\| \leq \|s\| \cdot \|c\xi_1 - \lambda_1 \xi_1\| < \varepsilon_1 \leq \varepsilon_0.$$

So

$$\begin{aligned} \|[s, c]\| &\geq \|sc\xi_1 - cs\xi_1\| = \|sc\xi_1 - c\xi_2\| \\ &\geq (\lambda_2 - \lambda_1) - \|\lambda_1 \xi_2 - sc\xi_1\| - \|\lambda_2 \xi_2 - c\xi_2\| \\ &> 2\delta - \varepsilon_0 - (4\|c\| + 1)\varepsilon_0 \geq 2\delta - \varepsilon. \end{aligned}$$

This completes the proof. □

Lemma 2.3. *Let A be a unital C*-algebra, let $a \in A$, and let $p \in A$ be a nonzero projection. Then*

$$\inf_{\lambda \in \mathbb{C}} \|a - \lambda\| \geq \tfrac{1}{2}(\|a\| - \|pap\|).$$

Proof. Let $\lambda \in \mathbb{C}$. We have

$$\|a - \lambda\| \geq \|a\| - |\lambda|.$$

Also, using $p \neq 0$ at the third step,

$$\|a - \lambda\| \geq \|p(a - \lambda)p\| = \|pap - \lambda p\| \geq |\lambda| - \|pap\|.$$

Therefore

$$\|a - \lambda\| \geq \tfrac{1}{2}(\|a\| - |\lambda|) + \tfrac{1}{2}(|\lambda| - \|pap\|) = \tfrac{1}{2}(\|a\| - \|pap\|),$$

as desired. □

Proof of Theorem 2.1. We need only prove that there are no nontrivial selfadjoint central sequences. Let $(a_j)_{j \in \mathbb{N}}$ be a norm bounded sequence of selfadjoint elements of $L(H)$ (not necessarily central) which is not trivial in the sense of Definition 1.1. We prove that $(a_j)_{j \in \mathbb{N}}$ is not central. It suffices to find a subsequence which is not central. By passing to a subsequence, we may assume

$$\inf_{j \in \mathbb{N}} \inf_{\lambda \in \mathbb{C}} \|a_j - \lambda\| > 0.$$

In particular, our sequence $(a_j)_{j \in \mathbb{N}}$ has no subsequence which converges in norm to any element of $\mathbb{C} \cdot 1$.

We will find $\varepsilon > 0$, an element $b \in L(H)$, and a subsequence $(a_{m(j)})_{j \in \mathbb{N}}$ of $(a_j)_{j \in \mathbb{N}}$, such that $\|[a_{m(j)}, b]\| > \varepsilon$ for all $j \in \mathbb{N}$. This will show that $(a_j)_{j \in \mathbb{N}}$ is not central, and prove the theorem.

Let $(r_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of finite rank projections such that $\lim_{n \to \infty} r_n = 1$ in the strong operator topology, and with $r_0 = 0$.

First suppose that there are $n \in \mathbb{N}$, a number $\delta > 0$, and a subsequence $(a_{l(j)})_{j \in \mathbb{N}}$ of $(a_j)_{j \in \mathbb{N}}$, such that

$$\inf_{j \in \mathbb{N}} \inf_{\lambda \in \mathbb{C}} \| r_n a_{l(j)} r_n - \lambda r_n \| \geq \delta.$$

Since r_n has finite rank, there is a further subsequence $(a_{m(j)})_{j \in \mathbb{N}}$ of $(a_{l(j)})_{j \in \mathbb{N}}$ such that $c = \lim_{j \to \infty} r_n a_{m(j)} r_n$ exists. Then also $\inf_{\lambda \in \mathbb{C}} \| c - \lambda r_n \| \geq \delta$. Lemma 2.2 provides $s \in r_n L(H) r_n$ such that $\| [s, c] \| > \frac{3}{2} \delta$. Then

$$\| [s, \, a_{m(j)}] \| = \| [s, \, r_n a_{m(j)} r_n] \| > \delta$$

for all sufficiently large j. Dropping initial terms of the subsequence $(a_{m(j)})_{j \in \mathbb{N}}$, we obtain the required subsequence with $b = s$ and $\varepsilon = \delta$.

Accordingly, we may now assume that

$$\lim_{j \to \infty} \inf_{\lambda \in \mathbb{C}} \| r_n a_j r_n - \lambda r_n \| = 0$$

for all $n \in \mathbb{N}$.

Set $M = \sup_{j \in \mathbb{N}} \| a_j \|$. Then

$$\lim_{j \to \infty} \inf_{\lambda \in [-M, \, M]} \| r_n a_j r_n - \lambda r_n \| = 0$$

for all $n \in \mathbb{N}$. In particular, there are numbers $\lambda_{0,j} \in [-M, \, M]$ such that

$$\lim_{j \to \infty} \| r_0 a_j r_0 - \lambda_{0,j} r_0 \| = 0.$$

By compactness, there are $\lambda_0 \in [-M, \, M]$ and a strictly increasing sequence $(l_0(j))_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} \lambda_{0, l_0(j)} = \lambda_0$. Then

$$\lim_{j \to \infty} \| r_0 a_j r_0 - \lambda_0 r_0 \| = 0.$$

Similarly, there are $\lambda_{1,j} \in [-M, \, M]$ such that

$$\lim_{j \to \infty} \| r_1 a_{l_0(j)} r_1 - \lambda_{1,j} r_1 \| = 0,$$

and then there are $\lambda_1 \in [-M, \, M]$ and a subsequence $(l_1(j))_{j \in \mathbb{N}}$ of $(l_0(j))_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} \| r_1 a_{l_1(j)} r_1 - \lambda_1 r_1 \| = 0.$$

We proceed inductively, obtaining numbers $\lambda_n \in [-M, \, M]$ and subsequences $(l_n(j))_{j \in \mathbb{N}}$ of $(l_{n-1}(j))_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} \| r_n a_{l_n(j)} r_n - \lambda_n r_n \| = 0.$$

Setting $l(j) = l_j(j)$, we then get

$$\lim_{j \to \infty} \| r_n a_{l(j)} r_n - \lambda_n r_n \| = 0$$

for all $n \in \mathbb{N}$.

Clearly $\lambda_0 = \lambda_1 = \cdots$. Subtracting this common value from each a_j does not change the conditions on $(a_j)_{j \in \mathbb{N}}$ or the existence of the required subsequence, so without loss of generality $\lim_{j \to \infty} \| r_n a_{l(j)} r_n \| = 0$ for all n.

Suppose now that there is $n \in \mathbb{N}$ such that $\|r_n a_{l(j)}(1 - r_n)\|$ does not converge to 0 as $j \to \infty$. Then there is $\rho > 0$ and a subsequence $(a_{m(j)})_{j \in \mathbb{N}}$ of $(a_{l(j)})_{j \in \mathbb{N}}$ such that $\|r_n a_{m(j)}(1 - r_n)\| > \rho$ for all $j \in \mathbb{N}$. With $b = r_n$, and using $r_n(1 - r_n) = 0$ at the second step, we have

$$\|[b, a_{m(j)}]\| \geq \|r_n [b, a_{m(j)}](1 - r_n)\| = \|r_n a_{m(j)}(1 - r_n)\| > \rho$$

for all $j \in \mathbb{N}$. Thus, we have a subsequence $(a_{m(j)})_{j \in \mathbb{N}}$ of the required type with $b = r_n$ and $\varepsilon = \rho$.

Since $\|r_n a_j (1 - r_n)\| = \|(1 - r_n) a_j r_n\|$, we may assume without loss of generality that

$$\lim_{j \to \infty} \|r_n a_j (1 - r_n)\| = \lim_{j \to \infty} \|(1 - r_n) a_j r_n\| = 0$$

for all $n \in \mathbb{N}$. Combining these with $\lim_{j \to \infty} \|r_n a_j r_n\| = 0$ for all n, we have reduced to consideration of the case

$$\lim_{j \to \infty} \|a_j - (1 - r_n) a_j (1 - r_n)\| = 0$$

for all $n \in \mathbb{N}$.

Since by assumption $(a_j)_{j \in \mathbb{N}}$ does not converge to 0 in norm, there is $\delta > 0$ and a subsequence $(a_{l(j)})_{j \in \mathbb{N}}$ of $(a_j)_{j \in \mathbb{N}}$ such that $\inf_{j \in \mathbb{N}} \|a_{l(j)}\| \geq \delta$. Passing to this subsequence, without loss of generality $\inf_{j \in \mathbb{N}} \|a_j\| \geq \delta$. We now construct recursively a subsequence $(a_{m(j)})_{j \in \mathbb{N}}$ of $(a_j)_{j \in \mathbb{N}}$, and an increasing sequence $(n(j))_{j \in \mathbb{N}}$ with $n(0) = 0$, such that the elements

$$y_j = \big(r_{n(j+1)} - r_{n(j)}\big) a_{m(j)} \big(r_{n(j+1)} - r_{n(j)}\big) \tag{1}$$

satisfy

$$\inf_{\lambda \in \mathbb{C}} \big\|y_j - \lambda(r_{n(j+1)} - r_{n(j)})\big\| > \tfrac{1}{4}\delta$$

for all $j \in \mathbb{N}$. We repeatedly use the observation that $\lim_{n \to \infty} \|r_n x r_n\| = \|x\|$ for all $x \in L(H)$.

Begin by choosing $n(0) = 0$, so that $r_{n(0)} = 0$. Choose $m(0) \in \mathbb{N}$ such that $\|r_1 a_{m(0)} r_1\| < \frac{1}{4}\delta$. Now use $\|a_{m(0)}\| \geq \delta$ to choose $n(1)$ so large that the element $y_0 = r_{n(1)} a_{m(0)} r_{n(1)}$ satisfies $\|y_0\| > \frac{3}{4}\delta$. Apply Lemma 2.3 with $A = r_{n(1)} L(H) r_{n(1)}$, with $a = y_0$, and with $p = r_1$, to get

$$\inf_{\lambda \in \mathbb{C}} \|y_0 - \lambda r_{n(1)}\| > \tfrac{1}{2}\left(\tfrac{3}{4}\delta - \tfrac{1}{4}\delta\right) = \tfrac{1}{4}\delta.$$

Given $m(j)$ and $n(j+1)$, choose $m(j+1) > m(j)$ so large that, with

$$x = \big(1 - r_{n(j+1)+1}\big) a_{m(j+1)} \big(1 - r_{n(j+1)+1}\big),$$

we have $\|a_{m(j+1)} - x\| < \frac{1}{6}\delta$. Then choose $n(j+2)$ so large that

$$\|r_{n(j+2)} x r_{n(j+2)}\| > \|x\| - \tfrac{1}{6}\delta.$$

Set

$$y_{j+1} = \big(r_{n(j+2)} - r_{n(j+1)}\big) a_{m(j+1)} \big(r_{n(j+2)} - r_{n(j+1)}\big).$$

Then

$$\begin{aligned}\|y_{j+1}\| &\geq \big\|\big(r_{n(j+2)} - r_{n(j+1)+1}\big)a_{m(j+1)}\big(r_{n(j+2)} - r_{n(j+1)+1}\big)\big\| \\ &= \|r_{n(j+2)}xr_{n(j+2)}\| > \|x\| - \tfrac{1}{6}\delta > \|a_{m(j+1)}\| - \tfrac{1}{6}\delta - \tfrac{1}{6}\delta \geq \tfrac{2}{3}\delta.\end{aligned}$$

Also,

$$\big\|r_{n(j+1)+1}y_{j+1}r_{n(j+1)+1}\big\| \leq \big\|r_{n(j+1)+1}(a_{m(j+1)} - x)r_{n(j+1)+1}\big\| < \tfrac{1}{6}\delta.$$

Apply Lemma 2.3 with

$$A = \big(r_{n(j+2)} - r_{n(j+1)}\big)L(H)\big(r_{n(j+2)} - r_{n(j+1)}\big),$$

with $a = y_{j+1}$, and with $p = r_{n(j+1)+1} - r_{n(j+1)}$, to get

$$\inf_{\lambda\in\mathbb{C}} \big\|y_{j+1} - \lambda(r_{n(j+2)} - r_{n(j+1)})\big\| > \tfrac{1}{2}\left(\tfrac{2}{3}\delta - \tfrac{1}{6}\delta\right) = \tfrac{1}{4}\delta,$$

as desired. This completes the construction of $(a_{m(j)})_{j\in\mathbb{N}}$ and $(n(j))_{j\in\mathbb{N}}$.

Now let y_j be as in (1) for $j \in \mathbb{N}$. Lemma 2.2 provides

$$s_j \in \big(r_{n(j+1)} - r_{n(j)}\big)L(H)\big(r_{n(j+1)} - r_{n(j)}\big)$$

such that $\|[s_j, y_j]\| > \frac{1}{4}\delta$ and $\|s_j\| = 1$. The series $s = \sum_{j=0}^{\infty} s_j$ converges in the strong operator topology, and for $j \in \mathbb{N}$ we have

$$\|[s, a_{m(j)}]\| \geq \big\|\big(r_{n(j+1)} - r_{n(j)}\big)[s, a_{m(j)}]\big(r_{n(j+1)} - r_{n(j)}\big)\big\| = \|[s_j, y_j]\| > \tfrac{1}{4}\delta.$$

Thus, the subsequence $(a_{m(j)})_{j\in\mathbb{N}}$ satisfies the required condition with $b = s$ and $\varepsilon = \frac{1}{4}\delta$. □

The following is an immediate consequence of Corollary 1.6 and Theorem 2.1.

Corollary 2.4. *If $\mathcal{V}$ is a selective ultrafilter then $F_{\mathcal{V}}(L(H)) = \mathbb{C}$.* □

3. Flat ultrafilters

Notation 3.1. By $f\colon \mathbb{N} \nearrow \mathbb{N}$ we mean that f is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$ such that $f(0) > 0$.

For such f and nonincreasing $h\colon \mathbb{N} \to [0, 1]$ the assertion $\|f - h \circ f\|_\infty \leq \varepsilon$ is equivalent to stating that the variation of h on any interval of the form $\mathbb{N} \cap [j, f(j)]$ is at most ε.

Definition 3.2. An ultrafilter $\mathcal{V}$ on $\mathbb{N}$ is *flat* if there are nonincreasing functions $h_n\colon \mathbb{N} \to [0, 1]$, for $n \in \mathbb{N}$, such that:

(1) $h_n(0) = 1$ for all $n \in \mathbb{N}$.
(2) $\lim_{j\to\infty} h_n(j) = 0$ for all $n \in \mathbb{N}$.
(3) For every $f\colon \mathbb{N} \nearrow \mathbb{N}$, we have $\lim_{n\to\mathcal{V}} \|h_n - h_n \circ f\|_\infty = 0$.

Theorem 3.3. *Flat ultrafilters exist.*

We need a lemma.

Lemma 3.4. *Let $f \colon \mathbb{N} \nearrow \mathbb{N}$. Let $n \in \mathbb{N}$ with $n > 0$, let $m_0 = 0$, and suppose $m_{l+1} \geq f(m_l)$ for $0 < l \leq n$. Set*

$$h = \sum_{l=0}^{n} \frac{n-l}{n} \chi_{\mathbb{N} \cap [m_l, \, m_{l+1})}.$$

Then $\|h - h \circ f\|_\infty \leq 1/n$.

Proof. Fix $j \in \mathbb{N}$. If $j \geq m_{n+1}$ then $h(j) = 0 = h \circ f(j)$. Otherwise there is l such that $m_l \leq j < m_{l+1}$. Then $f(j) < f(m_{l+1}) \leq m_{l+2}$ (writing $m_{n+2} = \infty$). Since h is nonincreasing,

$$\frac{n-l}{n} = h(j) \geq h \circ f(j) \geq \frac{n-l-1}{n}.$$

The required estimate is now clear. □

Proof of Theorem 3.3. Let $\mathbb{F}$ be the countable set of all nonincreasing functions $h \colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$ that are eventually zero and such that $h(0) = 1$. We start by constructing an ultrafilter $\mathcal{V}$ on $\mathbb{F}$. For $f \colon \mathbb{N} \nearrow \mathbb{N}$ and $\varepsilon > 0$ let

$$X_{f,\varepsilon} = \{h \in \mathbb{F} \colon \|h - h \circ f\|_\infty \leq \varepsilon\}.$$

By Lemma 3.4 this set is infinite. On the other hand,

$$X_{f,\varepsilon} \cap X_{g,\delta} \supseteq X_{\max(f,g), \, \min(\varepsilon,\delta)}.$$

Therefore the collection of all $X_{f,\varepsilon}$, for $f \colon \mathbb{N} \nearrow \mathbb{N}$ and $\varepsilon > 0$, has the finite intersection property. Let $\mathcal{W}$ be any ultrafilter which extends this collection.

Let $k \colon \mathbb{N} \to \mathbb{F}$ be a bijection, and set $\mathcal{V} = \{A \subseteq \mathbb{N} \colon k(A) \in \mathcal{W}\}$, which is an ultrafilter on $\mathbb{N}$. We claim that $\mathcal{V}$ is flat. The functions h_n required in the definition are given by $h_n = k(n)$ for $n \in \mathbb{N}$. Conditions (1) and (2) in Definition 3.2 are immediate. For Condition (3), let $f \colon \mathbb{N} \nearrow \mathbb{N}$ and let $\varepsilon > 0$. Then $Y = k^{-1}(X_{f,\varepsilon}) \in \mathcal{V}$, and for $n \in Y$ we have $h_n \in X_{f,\varepsilon}$ by construction, so that $\|h - h \circ f\|_\infty \leq \varepsilon$. This proves (3) in Definition 3.2. □

4. Nontrivial relative commutants

The present section is devoted to the proof of the following result.

Theorem 4.1. *If $\mathcal{V}$ is a flat ultrafilter then $F_{\mathcal{V}}(L(H)) \neq \mathbb{C}$.*

Notation 4.2. Fix an orthonormal basis $(\xi_n)_{n \in \mathbb{N}}$ for our separable infinite-dimensional complex Hilbert space H, and let e_n be the orthogonal projection onto $\mathbb{C}\xi_n$. Let $\mathbb{D}$ be the set of all nonincreasing functions $h \colon \mathbb{N} \to [0,1]$ such that $h(0) = 1$ and $\lim_{n \to \infty} h(n) = 0$. For $h \in \mathbb{D}$ define a compact operator a_h (with $\|a_h\| = 1$ since $h(0) = 1$) by

$$a_h = \sum_{j=0}^{\infty} h(j) e_j.$$

Notation 4.3. Let $\vec{E} = (E_n)_{n\in\mathbb{N}}$ be a family of closed orthogonal subspaces of H such that $H = \bigoplus_{n=0}^{\infty} E_n$. Let $\mathcal{D}(\vec{E})$ be the von Neumann algebra

$$\{a \in L(H)\colon aE_n \subseteq E_n \text{ for all } n \in \mathbb{N}\}.$$

For $f\colon \mathbb{N} \nearrow \mathbb{N}$ (as in Notation 3.1), and with $(\xi_n)_{n\in\mathbb{N}}$ as in Notation 4.2, let f^n be the composite $f \circ f \circ \cdots \circ f$ (with n terms), and take $f0$ to be the constant function with value 0. Define $\vec{E}^f$ by

$$\vec{E}^f_n = \operatorname{span}\{\xi_j\colon f^n(0) \leq j < f^{n+1}(0)\},$$

and set $\mathcal{D}(f) = \mathcal{D}(\vec{E}^f)$.

Lemma 4.4. *Adopt Notation 4.2 and Notation 4.3. If $h \in D$, $f\colon \mathbb{N} \nearrow \mathbb{N}$, and $\|h - h\circ f\|_\infty \leq \varepsilon$, then for every $b \in \mathcal{D}(f)$ we have $\|[a_h, b]\| \leq 2\varepsilon\|b\|$.*

Proof. Let q_n be the orthogonal projection onto E^f_n. We can write

$$a_h = \sum_{n=0}^{\infty} q_n a_h q_n.$$

Define

$$y = \sum_{n=0}^{\infty} h(f^n(0))q_n.$$

(Both series converge in norm because $\lim_{n\to\infty} h(n) = 0$.) For any $n \in \mathbb{N}$ and for $f^n(0) \leq k < f^{n+1}(0)$, we have

$$h(f^n(0)) \geq h(k) \geq h(f^{n+1}(0)) \geq h(f^n(0)) - \varepsilon,$$

so $\big\|q_n a_h q_n - h(f^n(0))q_n\big\| \leq \varepsilon$. Therefore $\|a_h - y\| \leq \varepsilon$. Since y is a central element of $\mathcal{D}(f)$, the conclusion follows. □

Lemma 4.5. *Let A be a unital C*-algebra, let $e, f \in A$ be orthogonal projections, and let $a \in A$. Then*

$$\|eae + eaf + fae\| \leq 2\|a\|.$$

Proof. We have

$$eae + eaf + fae = (e + f)a(e + f) - faf,$$

and $\|(e + f)a(e + f)\|$, $\|faf\| \leq \|a\|$. □

Examples using 2×2 matrices show that it is not possible to replace the constant 2 in Lemma 4.5 by 1, even if a is selfadjoint.

The use of ‘stratification’ of $L(H)$ into von Neumann algebras $\mathcal{D}(g_l)$ as given in Lemma 4.7 below resembles the use in [4, Lemma 3.1], and the following lemma is a minor improvement to [4, Lemma 1.3].

Lemma 4.6. *Let $F = \{a_1, a_2, \ldots, a_k\} \subset L(H)$ be finite, and let $\delta > 0$. Then there exist $g_0, g_1\colon \mathbb{N} \nearrow \mathbb{N}$ and decompositions $a_j = a_j^{(0)} + a_j^{(1)} + c_j$ for $j = 1, 2, \ldots, k$, such that for $j = 1, 2, \ldots, k$ we have:*

(1) $a_j^{(0)} \in \mathcal{D}(g_0)$.

(2) $a_j^{(1)} \in \mathcal{D}(g_1)$.
(3) $\|a_j^{(0)}\|, \|a_j^{(1)}\| \leq 2\|a_j\|$.
(4) c_j *is compact.*
(5) $\|c_j\| < \delta$.

Proof. Let p_n be the orthogonal projection onto $\operatorname{span}(\{\xi_0, \xi_1, \ldots, \xi_{n-1}\})$. Thus $p_0 = 0$. Also choose $\rho_0, \rho_1, \ldots > 0$ such that $2\sum_{n=0}^{\infty} \rho_{n+1} \leq \delta$.

We claim that there is a strictly increasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(0) = 0$ and such that for every $n \in \mathbb{N}$ and every $a \in F$, we have

$$\big\|(1 - p_{f(n+1)})ap_{f(n)}\big\| < \rho_n \quad \text{and} \quad \big\|p_{f(n)}a(1 - p_{f(n+1)})\big\| < \rho_n. \tag{2}$$

(For $n = 0$ the condition is vacuous because $p_0 = 0$.) We construct f recursively. Start by taking $f(0) = 0$. Given $f(n)$, use compactness of $p_{f(n)}a$ and $ap_{f(n)}$, finiteness of F, and the fact that $(p_m)_{m \in \mathbb{N}}$ is an approximate identity for $K(H)$, to choose $m > f(n)$ such that

$$\big\|(1 - p_m)ap_{f(n)}\big\| < \rho_n \quad \text{and} \quad \big\|p_{f(n)}a(1 - p_m)\big\| < \rho_n$$

for all $a \in F$. Then set $f(n+1) = m$. This proves the claim.

For $n \in \mathbb{N}$, we now set $q_n = p_{f(n+1)} - p_{f(n)}$. Since $p_{f(0)} = 0$, the series $\sum_{n=0}^{\infty} q_n$ converges to 1 in the strong operator topology.

For $j = 1, 2, \ldots, k$, define, with convergence in the strong operator topology,

$$a_j^{(0)} = \sum_{n=0}^{\infty} \big(q_{2n}a_jq_{2n} + q_{2n}a_jq_{2n+1} + q_{2n+1}a_jq_{2n}\big)$$

and

$$a_j^{(1)} = \sum_{n=0}^{\infty} \big(q_{2n+1}a_jq_{2n+1} + q_{2n+1}a_jq_{2n+2} + q_{2n+2}a_jq_{2n+1}\big).$$

The nth term in the series for $a_j^{(0)}$ is in $(q_{2n} + q_{2n+1})L(H)(q_{2n} + q_{2n+1})$, and the nth term in the series for $a_j^{(1)}$ is in $(q_{2n+1} + q_{2n+2})L(H)(q_{2n+1} + q_{2n+2})$. Accordingly, if for $j \in \mathbb{N}$ we set $g_0(j) = f(2j + 2)$ and $g_1(j) = f(2j + 1)$, then $g_0, g_1 \colon \mathbb{N} \nearrow \mathbb{N}$ and parts (1) and (2) are satisfied. Part (3) follows from Lemma 4.5.

The estimates (2) give, for every $n \in \mathbb{N}$,

$$\big\|q_na_j(1 - p_{f(n+2)})\big\| = \big\|q_np_{f(n+1)}a_j(1 - p_{f(n+2)})\big\| < \rho_{n+1},$$

and similarly

$$\big\|(1 - p_{f(n+2)})a_jq_n\big\| < \rho_{n+1}.$$

Therefore the series

$$\sum_{n=0}^{\infty} \big[q_na_j(1 - p_{f(n+2)}) + (1 - p_{f(n+2)})a_jq_n\big]$$

converges in norm to a compact operator c_j with

$$\|c_j\| < 2\sum_{n=0}^{\infty}\rho_{n+1} \leq \delta.$$

This is parts (4) and (5). Also, $a_j^{(0)} + a_j^{(1)} + c_j = a_j$ is clear. □

Lemma 4.7. *Let $\mathcal{V}$ be an arbitrary ultrafilter on $\mathbb{N}$. For $\mathbf{a} \in L(H)^{\mathcal{V}}$ the following are equivalent:*

(1) $\mathbf{a} \in L(H)' \cap L(H)^{\mathcal{V}}$.
(2) $\mathbf{a} \in \bigcap_{f\colon \mathbb{N}\nearrow\mathbb{N}} \left[\mathcal{D}(f)' \cap L(H)^{\mathcal{V}}\right]$.

Proof. The implication from (1) to (2) is trivial.

Assume (2) and fix $\mathbf{b} \in L(H)$. Fix $\delta > 0$. By Lemma 4.6 we can find $g_0, g_1\colon \mathbb{N} \nearrow \mathbb{N}$ and a decomposition $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1 + \mathbf{c}$ such that $\mathbf{b}_j \in \mathcal{D}(g_j)$ for $j = 0, 1$ and $\|\mathbf{c}\| \leq \frac{1}{2}\delta$. Thus $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}_0 + \mathbf{b}_1 + \mathbf{c}] = [\mathbf{a}, \mathbf{c}]$ and therefore $\|[\mathbf{a}, \mathbf{b}]\| \leq \delta\|\mathbf{a}\|$. Since $\mathbf{b} \in L(H)$ and $\delta > 0$ were arbitrary, $\mathbf{a} \in L(H)' \cap L(H)^{\mathcal{V}}$. □

Proof of Theorem 4.1. Fix a sequence $(h_n)_{n\in\mathbb{N}}$ of functions witnessing the flatness of $\mathcal{V}$. Let $a_n = a_{h_n} = \sum_{j=0}^{\infty} h_n(j)e_n$, as in Notation 4.2. Fix $f\colon \mathbb{N} \nearrow \mathbb{N}$. Since $\lim_{n\to\mathcal{V}} \|h_n - h_n \circ f\|_\infty = 0$, by Lemma 4.4 the sequence $(a_n)_{n\in\mathbb{N}}$ is a representing sequence of an element $\mathbf{a}$ of $\mathcal{D}(f)' \cap L(H)^{\mathcal{V}}$. Since $f\colon \mathbb{N} \nearrow \mathbb{N}$ was arbitrary, by Lemma 4.7 we have $\mathbf{a} \in L(H)' \cap L(H)^{\mathcal{V}}$.

Therefore $(a_n)_{n\in\mathbb{N}}$ is a $\mathcal{V}$-central sequence. Since each a_n is compact and has norm one, this sequence is nontrivial. □

5. Concluding remarks

The following is what remains of Kirchberg's question.

Question 5.1. *Does there exist a nonprincipal ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that $F_{\mathcal{V}}(L(H)) = \mathbb{C}$?*

By our Theorem 2, the Continuum Hypothesis implies a positive answer, but the question is whether such an ultrafilter can be constructed in ZFC. A 'typical' statement independent from ZFC is decided by the Continuum Hypothesis or a strengthening such as Jensen's diamond principle in one way and by Martin's Axiom or a strengthening such as the Proper Forcing Axiom in another way. (See [14, Chapter II] for an introduction to Martin's Axiom.) An example in theory of operator algebras is the statement 'the Calkin algebra has an outer automorphism,' which follows from the Continuum Hypothesis ([19]) and is incompatible with a consequence of the Proper Forcing Axiom ([4]). This, however, is not the case with Question 5.1. It is well-known that (a rather weak form of) Martin's Axiom implies the existence of selective ultrafilters, and therefore the existence of $\mathcal{U}$ such that $F_{\mathcal{U}}(L(H)) = \mathbb{C}$. A closer look at the proof of Proposition 1.4 reveals that it goes through when the Continuum Hypothesis is weakened to the assertion that for every family $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ such that the intersection of any finitely many sets in $\mathcal{F}$ is infinite, and such that $|\mathcal{F}| < 2^{\aleph_0}$, there is $B \in [\mathbb{N}]^\infty$ such that $B \setminus A$ is finite for all $A \in \mathcal{F}$. This assertion (known as $\mathfrak{p} = 2^{\aleph_0}$) is an easy consequence of Martin's Axiom. (See [1, Section 7].)

By a result of Kunen ([13]), if ZFC is consistent then so is the theory 'ZFC + there are no selective ultrafilters'. However, in Kunen's model there exists an ultrafilter $\mathcal{V}$ such that $F_\mathcal{V}(L(H)) = \mathbb{C}$. An ultrafilter $\mathcal{V}$ is a *P-point* if for every $g\colon \mathbb{N} \to \mathbb{N}$ there is $A \in \mathcal{V}$ such that g is either constant or finite-to-one on A. In [6] it is proved that if $\mathcal{V}$ is a P-point then $F_\mathcal{V}(L(H)) = \mathbb{C}$. While P-points exist in Kunen's model, Shelah has proved that if ZFC is consistent then so is ZFC + 'there are no P-points'. (See [20].)

We could not resolve the following question.

Question 5.2. *If $\mathcal{V}$ is an ultrafilter such that $F_\mathcal{V}(L(H)) \neq \mathbb{C}$, does it follow that $\mathcal{V}$ is flat?*

As pointed out in the introduction, tools from the logic of metric structures ([2]) are very relevant to the study of ultrapowers of C*-algebras. For example, it would be interesting to reformulate some of the results of [11] using the language of model theory. In particular, can the notion of σ-sub-Stonean ([11, Definition 1.4]) be replaced with the notion of $\aleph_1$-saturated ([2, Definition 7.5], the case when $\kappa = \aleph_1$, the least uncountable cardinal)?

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, tel: 416-736-2100 x66093, fax: 416-736-5757
URL: http://www.math.yorku.ca/∼ifarah
E-mail address: ifarah@mathstat.yorku.ca

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222, U.S.A.
URL: http://www.uoregon.edu/∼ncp/
E-mail address: ncp@uoregon.edu

Fields Institute, 222 College Street, Toronto, ON, Canada
E-mail address: steprans@fields.utoronto.ca