THE RELATIVE COMMUTANT OF SEPARABLE C*-ALGEBRAS OF REAL RANK ZERO

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Abstract. We answer a question of E. Kirchberg (personal communication): does the relative commutant of a separable C*-algebra in its ultrapower depend on the choice of the ultrafilter?

All algebras and all subalgebras in this note are C*-algebras and C*-subalgebras, respectively, and all ultrafilters are nonprincipal ultrafilters on $\mathbb{N}$. Our C*-terminology is standard (see e.g., [2]).

In the following $\mathcal{U}$ ranges over nonprincipal ultrafilters on $\mathbb{N}$. With $A^{\mathcal{U}}$ denoting the (norm, also called C*-) ultrapower of a C*-algebra A associated with $\mathcal{U}$ we have

$$F_{\mathcal{U}}(A) = A' \cap A^{\mathcal{U}},$$

the relative commutant of A in its ultrapower. This invariant plays an important role in [8] and [7].

Theorem 1. *For every separable infinite-dimensional C*-algebra A of real rank zero the following are equivalent.*

1. $F_{\mathcal{U}}(A) \cong F_{\mathcal{V}}(A)$ *for any two nonprincipal ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$ *on* $\mathbb{N}$.
2. $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ *for any two nonprincipal ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$ *on* $\mathbb{N}$.
3. *The Continuum Hypothesis.*

The equivalence of (3) and (2) in Theorem 1 for every infinite-dimensional C*-algebra A of cardinality $2^{\aleph_0}$ that has arbitrarily long finite chains in the Murray-von Neumann ordering of projections was proved in [6, Corollary 3.8], using the same Dow's result from [4] used here.

We shall prove (1) implies (3) and (2) implies (3) in Corollary 10 below. The reverse implications are well-known consequences of countable saturatedness of ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ (see [1, Proposition 7.6]). The implication from (3) to (1) holds for every separable C*-algebra A and the implication from (3) to (2) holds for every C*-algebra A of size $2^{\aleph_0}$. The point is that if A is separable then the isomorphism between diagonal copies of A extends to an isomorphism between

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the ultrapowers. Countable saturation of $A^{\mathcal{U}}$ can be proved directly from its analogue, due to Keisler, in classical model theory. This also follows from the argument in [6, Theorem 3.2 and Remark 3.3].

While the Continuum Hypothesis implies that any two ultrapowers of $\mathcal{B}(H)$ associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic, it does not imply that the relative commutants of $\mathcal{B}(H)$ in those ultrapowers are isomorphic. As a matter of fact, it implies the opposite (see [5]).

For a C*-algebra A let $\mathcal{P}(A) = \{p : p \in A \text{ is a projection}\}$ ordered by $p \leq q$ if and only if $pq = p$. Our proof depends on the analysis of types of gaps in $\mathcal{P}(A' \cap A^{\mathcal{U}})$ (see Definition 4). Gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and related quotient structures are well-studied; for example, analysis of such gaps is very important in the consistency proof of the statement 'all Banach algebra automorphisms of $C(X)$ into some Banach algebra are continuous' (see [3]). It was recently discovered that the gap-spectrum of $\mathcal{P}(\mathcal{C}(H))$ (where $\mathcal{C}(H)$ is the Calkin algebra, $\mathcal{B}(H)/\mathcal{K}(H)$)) is much richer than the gap-structure of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ ([12]).

Notational convention. We denote elements of ultraproducts by boldface Roman letters such as $\mathbf{p}$ and their representing sequences by $p(n)$, for $n \in \mathbb{N}$. We shall follow von Neumann's convention and identify a natural number n with the set $\{0, \dots, n-1\}$. The symbol ω is used for ultrafilters in the operator algebra literature and it is reserved for the least infinite ordinal in the set-theoretic literature. I will avoid using it in this note.

By $\sigma(a)$ we denote the spectrum of a normal operator a. Lemma 2 below is well-known. A sharper result can be found e.g., in [9, Lemma 2.5.4] but we include a proof for reader's convenience.

Lemma 2. *For a self-adjoint a and a projection r, if $\|a - r\| < \varepsilon < 1$ then $\sigma(a) \subseteq (-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}) \cup (1-2\sqrt{\varepsilon}, 1+2\sqrt{\varepsilon})$. If in addition $\varepsilon < 1/16$ then there is a projection r' in $C^*(a)$ such that $\|r' - a\| < 2\sqrt{\varepsilon}$.*

Proof. Since $\|a\| < 1 + \varepsilon < 2$, we have $\|a^2 - a\| \leq \|a(a-r)\| + \|r(a-r)\| + \|a - r\| < 4\varepsilon$. Thus $|x(1-x)| < 4\varepsilon$ for all $x \in \sigma(a)$ and in turn $|x| < 2\sqrt{\varepsilon}$ or $|1 - x| < 2\sqrt{\varepsilon}$.

Now assume $\varepsilon < 1/16$. In this case $1/2 \notin \sigma(a)$. Define a continuous function f with domain $\sigma(a)$ as follows. Let $f(t) = 0$ for $-\infty < t < 1/2$ and $f(t) = 1$ for $1/2 \leq t < \infty$. Since $|f(t) - t| < 2\sqrt{\varepsilon}$ for all $t \in \sigma(a)$, $f(a)$ is a projection in $C^*(a)$ as required. □

A representing sequence $p(n)$ of a projection $\mathbf{p}$ in an ultrapower can be chosen so that each $p(n)$ is a projection (see [6, Proposition 2.5 (1)], this also follows immediately from [10, Lemma 4.2.2] or [9, Lemma 2.5.5]).

Lemma 3. *For projections $\mathbf{p}, \mathbf{q}$ in $A^{\mathcal{U}}$ the following are equivalent.*

(1) $\mathbf{p} \leq \mathbf{q}$,
(2) *There is a representing sequence $p'(i)$, for $i \in \mathbb{N}$, of $\mathbf{p}$ such that $p'(i) \leq q(i)$ for all i.*

(3) *There is a representing sequence* $q'(i)$, *for* $i \in \mathbb{N}$, *of* $\mathbf{q}$ *such that* $p(i) \leq q'(i)$ *for all* i.

Proof. Both (3) implies (1) and (2) implies (1) are trivial. We shall prove (1) implies (2). Assume $\mathbf{p} \leq \mathbf{q}$. For every $n \geq 1$ the set

$$X_n = \{j : \|q(j)p(j)q(j) - p(j)\| < 1/(4n)\}$$

belongs to $\mathcal{U}$. We may assume $\bigcap_n X_n = \emptyset$. Let $p'(j) = 0$ if $j \notin X_0$. If $j \in X_n \setminus X_{n+1}$ then Lemma 2, with $a(j) = q(j)p(j)q(j)$, implies there is a projection $p'(j) \in C^*(a(j))$ such that $\|p'(j) - a(j)\| < 1/(2\sqrt{n})$. Then $p'(j) \leq q(j)$ and $\|p'(j) - p(j)\| < 1/\sqrt{n}$ for all $j \in X_n$. Therefore $p'(j)$, for $j \in \mathbb{N}$, is a representing sequence of $\mathbf{p}$ as required.

In order to prove (1) implies (3) apply the above to $1 - \mathbf{p} \geq 1 - \mathbf{q}$ in the ultrapower of the unitization of A to find an appropriate representing sequence for $1 - \mathbf{q}$. □

By $\mathbb{N}^{\nearrow\mathbb{N}}$ we denote the set of all nondecreasing functions f from $\mathbb{N}$ to $\mathbb{N}$ such that $\lim_n f(n) = \infty$, ordered pointwise. Write $f \leq_{\mathcal{U}} g$ if $\{n : f(n) \leq g(n)\} \in \mathcal{U}$ and denote the quotient linear ordering by $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$.

Following [4], for an ultrafilter $\mathcal{U}$ we write $\kappa(\mathcal{U})$ for the *coinitiality* of $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$, i.e., the minimal cardinality of $X \subseteq \mathbb{N}^{\nearrow\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\nearrow\mathbb{N}}$ there is $f \in X$ such that $f \leq_{\mathcal{U}} g$. (It is not difficult to see that this is equal to $\kappa(\mathcal{U})$ as defined in [4, Definition 1.3].)

Definition 4. Let λ be a cardinal. An $(\aleph_0, \lambda)$-*gap* in a partially ordered set $\mathbb{P}$ is a pair consisting of a $\leq_{\mathbb{P}}$-increasing family $\mathbf{a}_m$, for $m \in \mathbb{N}$, and a $\leq_{\mathbb{P}}$-decreasing family $\mathbf{b}_\gamma$, for $\gamma < \lambda$, such that $\mathbf{a}_m \leq_{\mathbb{P}} \mathbf{b}_\gamma$ for all m and γ but there is no $\mathbf{c} \in \mathbb{P}$ such that $\mathbf{a}_m \leq_{\mathbb{P}} \mathbf{c}$ for all m and $\mathbf{c} \leq_{\mathbb{P}} \mathbf{b}_\gamma$ for all γ.

Assume $r^0(n) \leq r^1(n) \leq \cdots \leq r^{l(n)-1}(n)$ are projections in A and $\lim_{n\to\infty} l(n) = \infty$. For $h\colon \mathbb{N} \to \mathbb{N}$ define $\mathbf{r}^h$ via its representing sequence (let $r^i(n) = r^{l(n)-1}(n)$ for $i \geq l(n)$)

$$r^h(n) = r^{h(n)}(n).$$

Let $\mathbf{p}_m = \mathbf{r}^{\bar{m}}$, where $\bar{m}(j) = m$ for all j.

Lemma 5. *With notation from the previous paragraph, for every projection* $\mathbf{s}$ *in* $A^{\mathcal{U}}$ *such that* $\mathbf{p}_m \leq \mathbf{s}$ *for all* m *there is* $h\colon \mathbb{N} \to \mathbb{N}$ *such that* $\mathbf{p}_m \leq \mathbf{r}^h$ *for all* m *and* $\mathbf{r}^h \leq \mathbf{s}$.

Proof. Since $\mathbf{p}_m \leq \mathbf{s}$, for each $m \in \mathbb{N}$ the set

$$X_m = \{i : \|r^m(i)s(i) - r^m(i)\| < 1/m\}$$

belongs to $\mathcal{U}$. Since the value of $\|r^m(i)s(i) - r^m(i)\|$ is increasing in m we have $X_m \supseteq X_{m+1}$. We may assume $\bigcap_m X_m = \emptyset$. Define $h\colon \mathbb{N} \to \mathbb{N}$ by letting $h(i) = 0$ for $i \notin X_0$ and for $i \in X_m \setminus X_{m+1}$ let $h(i) = m$.

For each m and $i \in X_m$ we have $h(i) \geq m$ and therefore $\mathbf{r}^h \geq \mathbf{p}_m$. Also, $i \in X_m$ implies $\|r^h(i)s(i) - r^h(i)\| < 1/m$ hence $\mathbf{r}^h \leq \mathbf{s}$. □

The proof of Proposition 6 was inspired by Alan Dow's [4, Proposition 1.4]. Dow's result was independently proved by Saharon Shelah and can be found in [11].[1]

By $A_{\leq 1}$ we denote the unit ball of a C*-algebra A.

Proposition 6. *Assume A is a separable C*-algebra and there are finite self-adjoint sets $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq A_{\leq 1}$ whose union is dense in $A_{\leq 1}$ and such that for each n there is a $\leq$-increasing chain $\mathcal{C}_n$ of projections in $B_n = F'_n \cap A$ of length at least n.*

Then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every cardinal λ there is an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^{\mathcal{U}})$ if and only if $\kappa(\mathcal{U}) = \lambda$.

Proof. First we prove the converse implication. Assume g_γ, for $\gamma < \lambda = \kappa(\mathcal{U})$, is a $\leq_{\mathcal{U}}$-decreasing and $\leq_{\mathcal{U}}$-unbounded below chain of functions in $\mathbb{N}^{\nearrow\mathbb{N}}$. Let $0 = r^0(n) \leq r^1(n) \leq \dots \leq r^{n-1}(n)$ be an enumeration of $\mathcal{C}_n$.

Claim 7. *For all f, g in $\mathbb{N}^{\nearrow\mathbb{N}}$ the following are equivalent.*

(1) $f \leq_{\mathcal{U}} g$,
(2) $\mathbf{r}^f \leq \mathbf{r}^g$,

Proof. Assume $f \leq_{\mathcal{U}} g$. Then $X = \{j : f(j) \leq g(j)\} \in \mathcal{U}$ and $\mathbf{r}^f(j) \leq \mathbf{r}^g(j)$ for all $j \in X$ hence (2) follows. If $f \not\leq_{\mathcal{U}} g$ then $X = \{j : f(j) > g(j)\} \in \mathcal{U}$ and for all $j \in X$ we have $\|r^f(i)r^g(i) - r^g(i)\| = 1$, hence $\mathbf{r}^f \not\leq \mathbf{r}^g$. □

Let $\mathbf{q}_\gamma = \mathbf{r}^{g_\gamma}$, for $\gamma < \lambda$. By Claim 7 we have

$$\mathbf{p}_m \leq \mathbf{p}_{m+1} \leq \mathbf{q}_\delta \leq \mathbf{q}_\gamma$$

for all m and all $\gamma < \delta < \lambda$. All of $\mathbf{p}_m$ and $\mathbf{q}_\gamma$ belong to $A' \cap A^{\mathcal{U}}$.

We shall show that this family forms a gap in $\mathcal{P}(A^{\mathcal{U}})$ (and therefore it forms a gap in $\mathcal{P}(A' \cap A^{\mathcal{U}})$. Assume $\mathbf{s} \in A^{\mathcal{U}}$ is such that $\mathbf{s} \leq \mathbf{q}_\gamma$ for all γ. By Lemma 5 there is h such that $\mathbf{p}_m \leq \mathbf{r}^h \leq \mathbf{s}$ for all m. By Claim 7 we have $h \leq_{\mathcal{U}} g_\gamma$ for all γ and $\bar{m} \leq_{\mathcal{U}} h$ for all m, a contradiction.

In order to prove the direct implication, assume that $\mathbf{p}_m$, $\mathbf{q}_\gamma$ form an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^{\mathcal{U}})$. By successively using Lemma 3 for $m = 1, 2, \dots$ find representing sequences $p_m(i)_{i \in \mathbb{N}}$, for $\mathbf{p}_m$ such that $p_m(i) \leq p_{m+1}(i)$ for all i. Choose an increasing sequence $0 = m_0 < m_1 < m_2 < \dots$ such that the following holds for all k.

(*) for all $j < m_k$ and all $a \in F_{m_k}$, if $l \geq m_{k+1}$ then $\|[p_j(l), a]\| < 1/k$.

For $n \in \mathbb{N}$ and i such that for some k we have $i < m_k$ and $m_{k+1} \leq n$ let $r^i(n) = p_i(n)$. Thus we have projections

$$r^0(n) \leq r^1(n) \leq \dots \leq r^{m_k}(n)$$

whenever $n \geq m_{k+1}$. For $h \colon \mathbb{N} \to \mathbb{N}$ define $\mathbf{r}^h$ as in the paragraph before Lemma 5, by its representing sequence (let $r^i(n) = r^{m_k}(n)$ if $i \geq m_k$)

$$r^h(n) = r^{h(n)}(n).$$

[1] I could not find it, but it should be somewhere in Chapter VI.

Claim 8. *If $h\colon \mathbb{N} \to \mathbb{N}$ then $\mathbf{r}^h \in A' \cap A^{\mathcal{U}}$.*

Proof. Fix any b in the unit ball of A and $\varepsilon > 0$. If $k > 1/\varepsilon$ and there is $b' \in F_{2k}$ satisfying $\|b - b'\| < \varepsilon/2$ then for $i > n_{2k}$ in Y we have that $\|[p_j(i), b']\| < \varepsilon/2$ and therefore $\|[\mathbf{r}^h(i), b]\| < \varepsilon$ for $\mathcal{U}$-many i. □

Using Lemma 5 for each $\mathbf{q}_\gamma$ find h_γ such that $\mathbf{r}^\gamma = \mathbf{r}^{h_\gamma}$ satisfies $\mathbf{p}_i \leq \mathbf{r}^\gamma \leq \mathbf{q}_\gamma$ for all i. Since $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$ is a linear ordering and λ is a regular cardinal, we can find a cofinal subset $\mathcal{Z}$ of λ such that for $\gamma < \delta$ in $\mathcal{Z}$ we have $\mathbf{r}^\delta \leq \mathbf{r}^\gamma$. By reenumerating we may assume $\mathcal{Z} = \lambda$ and then $\mathbf{r}^\gamma$, for $\gamma \in \mathcal{Z}$, together with $\mathbf{p}_i$, for $i \in \mathbb{N}$, form an $(\aleph_0, \lambda)$-gap. However, $\mathbf{r}^\delta \leq \mathbf{r}^\gamma$ is equivalent to $h_\delta \leq_{\mathcal{U}} h_\gamma$, and therefore h_γ, for $\gamma < \lambda$, form a $\leq_{\mathcal{U}}$-decreasing and $\leq_{\mathcal{U}}$-unbounded below sequence in $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$, and therefore $\lambda = \kappa(\mathcal{U})$. □

The proof of Proposition 6 can be modified (by removing some of its parts) to a proof of the following.

Proposition 9. *Assume A is a separable C*-algebra and $\mathcal{P}(A)$ has arbitrarily long finite chains. Then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every cardinal λ there is an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A^{\mathcal{U}})$ if and only if $\kappa(\mathcal{U}) = \lambda$.* □

Corollary 10. *Assume the Continuum Hypothesis fails. If A is an infinite-dimensional separable C*-algebra of real rank zero then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $F_{\mathcal{U}}(A) \not\cong F_{\mathcal{V}}(A)$ and $A^{\mathcal{U}} \not\cong A^{\mathcal{V}}$.*

Proof. By [4, Theorem 2.2] we can find $\mathcal{U}$ and $\mathcal{V}$ so that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$ (here $\aleph_1$ and $\aleph_2$ are the least two uncountable cardinals; all that matters for us is that they are both less or equal than $2^{\aleph_0}$ and different). Therefore $\mathcal{P}(A' \cap A^{\mathcal{U}})$ has an $(\aleph_0, \aleph_1)$-gap while $\mathcal{P}(A' \cap A^{\mathcal{V}})$ does not, and $A' \cap A^{\mathcal{U}}$ and $A' \cap A^{\mathcal{V}}$ cannot be isomorphic.

It remains to prove that if A is an infinite-dimensional C*-algebra of real rank zero then $\mathcal{P}(A)$ has an infinite chain of projections. We may assume A is unital. Recursively find a decreasing sequence r_n for $n \in \mathbb{N}$ in $\mathcal{P}(A)$ so that $r_n A r_n$ is infinite-dimensional for all n. Assume r_n has been chosen. Since A has real rank zero, in $r_n A r_n$ we can fix a projection $q \notin \{0, r_n\}$. If qA_nq is infinite-dimensional then let $r_{n+1} = q$. Otherwise, let $r_{n+1} = r_n - q$ and note that $r_{n+1}Ar_{n+1}$ is infinite-dimensional. □

It is likely that Theorem 1 and Corollary 10 can be extended to all infinite-dimensional separable C*-algebras (possibly by considering the Cuntz ordering of positive elements instead of $\mathcal{P}(A)$).

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