THE RELATIVE COMMUTANT OF SEPARABLE C*-ALGEBRAS OF REAL RANK ZERO

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Abstract. We answer a question of E. Kirchberg (personal communication): does the relative commutant of a separable C*-algebra in its ultrapower depend on the choice of the ultrafilter?

All algebras and all subalgebras in this note are C*-algebras and C*-subalgebras, respectively, and all ultrafilters are nonprincipal ultrafilters on \( \mathbb{N} \). Our C*-terminology is standard (see e.g., [2]).

In the following \( \mathcal{U} \) ranges over nonprincipal ultrafilters on \( \mathbb{N} \). With \( A^\mathcal{U} \) denoting the (norm, also called C*-1) ultrapower of a C*-algebra \( A \) associated with \( \mathcal{U} \) we have

\[
F_\mathcal{U}(A) = A' \cap A^\mathcal{U},
\]

the relative commutant of \( A \) in its ultrapower. This invariant plays an important role in [8] and [7].

Theorem 1. For every separable infinite-dimensional C*-algebra \( A \) of real rank zero the following are equivalent.

1. \( F_\mathcal{U}(A) \cong F_\mathcal{V}(A) \) for any two nonprincipal ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \).
2. \( A^\mathcal{U} \cong A^\mathcal{V} \) for any two nonprincipal ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \).
3. The Continuum Hypothesis.

The equivalence of (3) and (2) in Theorem 1 for every infinite-dimensional C*-algebra \( A \) of cardinality \( 2^{\aleph_0} \) that has arbitrarily long finite chains in the Murray-von Neumann ordering of projections was proved in [6, Corollary 3.8], using the same Dow’s result from [4] used here.

We shall prove (1) implies (3) and (2) implies (3) in Corollary 10 below. The reverse implications are well-known consequences of countable saturatedness of ultrapowers associated with nonprincipal ultrafilters on \( \mathbb{N} \) (see [1, Proposition 7.6]). The implication from (3) to (1) holds for every separable C*-algebra \( A \) and the implication from (3) to (2) holds for every C*-algebra \( A \) of size \( 2^{\aleph_0} \). The point is that if \( A \) is separable then the isomorphism between diagonal copies of \( A \) extends to an isomorphism between
the ultrapowers. Countable saturation of \( A^U \) can be proved directly from its analogue, due to Keisler, in classical model theory. This also follows from the argument in [6, Theorem 3.2 and Remark 3.3].

While the Continuum Hypothesis implies that any two ultrapowers of \( B(H) \) associated with nonprincipal ultrafilters on \( \mathbb{N} \) are isomorphic, it does not imply that the relative commutants of \( B(H) \) in those ultrapowers are isomorphic. As a matter of fact, it implies the opposite (see [5]).

For a \( C^* \)-algebra \( A \) let \( \mathcal{P}(A) = \{ p : p \in A \text{ is a projection} \} \) ordered by \( p \leq q \) if and only if \( pq = p \). Our proof depends on the analysis of types of gaps in \( \mathcal{P}(A' \cap A^U) \) (see Definition 4). Gaps in \( \mathcal{P}(\mathbb{N}) / \text{Fin} \) and related quotient structures are well-studied; for example, analysis of such gaps is very important in the consistency proof of the statement ‘all Banach algebra automorphisms of \( C(X) \) into some Banach algebra are continuous’ (see [3]).

It was recently discovered that the gap-spectrum of \( \mathcal{P}(C(H)) \) (where \( C(H) \) is the Calkin algebra, \( B(H)/\mathcal{K}(H) \)) is much richer than the gap-structure of \( \mathcal{P}(\mathbb{N}) / \text{Fin} \) ([12]).

**Notational convention.** We denote elements of ultraproducts by boldface Roman letters such as \( \mathbf{p} \) and their representing sequences by \( p(n) \), for \( n \in \mathbb{N} \). We shall follow von Neumann’s convention and identify a natural number \( n \) with the set \( \{0, \ldots, n-1\} \). The symbol \( \omega \) is used for ultrafilters in the operator algebra literature and it is reserved for the least infinite ordinal in the set-theoretic literature. I will avoid using it in this note.

By \( \sigma(a) \) we denote the spectrum of a normal operator \( a \). Lemma 2 below is well-known. A sharper result can be found e.g., in [9, Lemma 2.5.4] but we include a proof for reader’s convenience.

**Lemma 2.** For a self-adjoint \( a \) and a projection \( r \), if \( \|a-r\|<\varepsilon<1 \) then \( \sigma(a) \subseteq (-2\sqrt{\varepsilon},2\sqrt{\varepsilon}) \cup (1-2\sqrt{\varepsilon},1+2\sqrt{\varepsilon}) \). If in addition \( \varepsilon<1/16 \) then there is a projection \( r' \) in \( C^*(a) \) such that \( \|r'-a\|<2\sqrt{\varepsilon} \).

**Proof.** Since \( \|a\|<1+a<2 \), we have \( \|a^2-a\| \leq \|a(a-r)\| + \|r(a-r)\| + \|a-r\| < 4\varepsilon \). Thus \( |x(1-x)| < 4\varepsilon \) for all \( x \in \sigma(a) \) and in turn \( |x| < 2\sqrt{\varepsilon} \) or \( |1-x| < 2\sqrt{\varepsilon} \).

Now assume \( \varepsilon<1/16 \). In this case \( 1/2 \notin \sigma(a) \). Define a continuous function \( f \) with domain \( \sigma(a) \) as follows. Let \( f(t) = 0 \) for \( -\infty < t < 1/2 \) and \( f(t) = 1 \) for \( 1/2 < t < \infty \). Since \( |f(t) - t| < 2\sqrt{\varepsilon} \) for all \( t \in \sigma(a) \), \( f(a) \) is a projection in \( C^*(a) \) as required. \( \square \)

A representing sequence \( p(n) \) of a projection \( \mathbf{p} \) in an ultrapower can be chosen so that each \( p(n) \) is a projection (see [6, Proposition 2.5 (1)]), this also follows immediately from [10, Lemma 4.2.2] or [9, Lemma 2.5.5]).

**Lemma 3.** For projections \( \mathbf{p}, \mathbf{q} \) in \( A^U \) the following are equivalent.

1. \( \mathbf{p} \leq \mathbf{q} \).
2. There is a representing sequence \( p'(i) \), for \( i \in \mathbb{N} \), of \( \mathbf{p} \) such that \( p'(i) \leq q(i) \) for all \( i \).
(3) There is a representing sequence \( q'(i) \), for \( i \in \mathbb{N} \), of \( q \) such that \( p(i) \leq q'(i) \) for all \( i \).

**Proof.** Both (3) implies (1) and (2) implies (1) are trivial. We shall prove (1) implies (2). Assume \( p \leq q \). For every \( n \geq 1 \) the set
\[
X_n = \{ j : \|q(j)p(j)q(j) - p(j)\| < 1/(4n) \}
\]
belongs to \( \mathcal{U} \). We may assume \( \bigcap_n X_n = \emptyset \). Let \( p'(j) = 0 \) if \( j \notin X_0 \). If \( j \in X_n \setminus X_{n+1} \) then Lemma 2, with \( a(j) = q(j)p(j)q(j) \), implies there is a projection \( p'(j) \in C^*(a(j)) \) such that \( \|p'(j) - a(j)\| < 1/(2\sqrt{n}) \). Then \( p'(j) \leq q(j) \) and \( \|p'(j) - p(j)\| < 1/\sqrt{n} \) for all \( j \in X_n \). Therefore \( p'(j) \), for \( j \in \mathbb{N} \), is a representing sequence of \( p \) as required.

In order to prove (1) implies (3) apply the above to \( 1 - p \geq 1 - q \) in the ultrapower of the unitization of \( A \) to find an appropriate representing sequence for \( 1 - q \). \( \square \)

By \( \mathbb{N}^{\mathbb{N}} \) we denote the set of all nondecreasing functions \( f \) from \( \mathbb{N} \) to \( \mathbb{N} \) such that \( \lim_n f(n) = \infty \), ordered pointwise. Write \( f \leq_U g \) if \( \{ n : f(n) \leq g(n) \} \in \mathcal{U} \) and denote the quotient linear ordering by \( \mathbb{N}^{\mathbb{N}}/\mathcal{U} \).

Following [4], for an ultrafilter \( \mathcal{U} \) we write \( \kappa(\mathcal{U}) \) for the **coinitiality** of \( \mathbb{N}^{\mathbb{N}}/\mathcal{U} \), i.e., the minimal cardinality of \( X \subseteq \mathbb{N}^{\mathbb{N}}/\mathcal{U} \) such that for every \( g \in \mathbb{N}^{\mathbb{N}}/\mathcal{U} \) there is \( f \in X \) such that \( f \leq_U g \). (It is not difficult to see that this is equal to \( \kappa(\mathcal{U}) \) as defined in [4, Definition 1.3].)

**Definition 4.** Let \( \lambda \) be a cardinal. An \( (\mathbb{N}_0, \lambda) \)-**gap** in a partially ordered set \( \mathbb{P} \) is a pair consisting of a \( \leq_{\mathbb{P}} \)-increasing family \( a_m \), for \( m \in \mathbb{N} \), and a \( \leq_{\mathbb{P}} \)-decreasing family \( b_\gamma \), for \( \gamma < \lambda \), such that \( a_m \leq_{\mathbb{P}} b_\gamma \) for all \( m \) and \( \gamma \) but there is no \( c \in \mathbb{P} \) such that \( a_m \leq_{\mathbb{P}} c \) for all \( m \) and \( c \leq_{\mathbb{P}} b_\gamma \) for all \( \gamma \).

Assume \( r^0(n) \leq r^1(n) \leq \cdots \leq r^{l(n)-1}(n) \) are projections in \( A \) and \( \lim_{n \to \infty} l(n) = \infty \). For \( h : \mathbb{N} \to \mathbb{N} \) define \( r^h \) via its representing sequence (let \( r^i(n) = r^{l(n)-1}(n) \) for \( i \geq l(n) \))
\[
r^h(n) = r^{h(n)}(n).
\]
Let \( p_m = r^m \), where \( m(j) = m \) for all \( j \).

**Lemma 5.** With notation from the previous paragraph, for every projection \( s \) in \( A^\mathcal{U} \) such that \( p_m \leq s \) for all \( m \) there is \( h : \mathbb{N} \to \mathbb{N} \) such that \( p_m \leq r^h \) for all \( m \) and \( r^h \leq s \).

**Proof.** Since \( p_m \leq s \), for each \( m \in \mathbb{N} \) the set
\[
X_m = \{ i : \|r^m(i)s(i) - r^m(i)\| < 1/m \}
\]
belongs to \( \mathcal{U} \). Since the value of \( \|r^m(i)s(i) - r^m(i)\| \) is increasing in \( m \) we have \( X_m \supseteq X_{m+1} \). We may assume \( \bigcap_m X_m = \emptyset \). Define \( h : \mathbb{N} \to \mathbb{N} \) by letting \( h(i) = 0 \) for \( i \notin X_0 \) and for \( i \in X_m \setminus X_{m+1} \) let \( h(i) = m \).

For each \( m \) and \( i \in X_m \) we have \( h(i) \geq m \) and therefore \( r^h \geq p_m \). Also, \( i \in X_m \) implies \( \|r^h(i)s(i) - r^h(i)\| < 1/m \) hence \( r^h \leq s \). \( \square \)
The proof of Proposition 6 was inspired by Alan Dow’s [4, Proposition 1.4]. Dow’s result was independently proved by Saharon Shelah and can be found in [11].

By $A_{\leq 1}$ we denote the unit ball of a C*-algebra $A$.

**Proposition 6.** Assume $A$ is a separable C*-algebra and there are finite self-adjoint sets $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq A_{\leq 1}$ whose union is dense in $A_{\leq 1}$ and such that for each $n$ there is a $\leq$-increasing chain $C_n$ of projections in $B_n = F_n' \cap A$ of length at least $n$.

Then for every nonprincipal ultrafilter $U$ on $\mathbb{N}$ and every cardinal $\lambda$ there is an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^U)$ if and only if $\kappa(U) = \lambda$.

**Proof.** First we prove the converse implication. Assume $g_\gamma$, for $\gamma < \lambda = \kappa(U)$, is a $\leq_U$-decreasing and $\leq_U$-unbounded below chain of functions in $\mathbb{N}/\mathbb{N}$. Let $0 = r^0(n) \leq r^1(n) \leq \cdots \leq r^{n-1}(n)$ be an enumeration of $C_n$.

**Claim 7.** For all $f, g$ in $\mathbb{N}/\mathbb{N}$ the following are equivalent.

1. $f \leq_U g$,
2. $r^f \leq r^g$.

**Proof.** Assume $f \leq_U g$. Then $X = \{ j : f(j) \leq g(j) \} \in U$ and $r^f(j) \leq r^g(j)$ for all $j \in X$ hence (2) follows. If $f \not\leq_U g$ then $X = \{ j : f(j) > g(j) \} \in U$ and for all $j \in X$ we have $\| r^f(i)r^g(i) - r^f(i) \| = 1$, hence $r^f \not\leq r^g$. □

Let $q_\gamma = r^{g_\gamma}$, for $\gamma < \lambda$. By Claim 7 we have

$$p_m \leq p_{m+1} \leq q_\delta \leq q_\gamma$$

for all $m$ and all $\gamma < \delta < \lambda$. All of $p_m$ and $q_\gamma$ belong to $A' \cap A^U$.

We shall show that this family forms a gap in $\mathcal{P}(A^U)$ (and therefore it forms a gap in $\mathcal{P}(A' \cap A^U)$). Assume $s \in A^U$ is such that $s \leq q_\gamma$ for all $\gamma$. By Lemma 5 there is $h$ such that $p_m \leq r^h \leq s$ for all $m$. By Claim 7 we have $h \leq_U g_\gamma$ for all $\gamma$ and $m \leq_U h$ for all $m$, a contradiction.

In order to prove the direct implication, assume that $p_m$, $q_\gamma$ form an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^U)$. By successively using Lemma 3 for $m = 1, 2, \ldots$ find representing sequences $p_m(i)_{i \in \mathbb{N}}$, for $p_m$ such that $p_m(i) \leq p_{m+1}(i)$ for all $i$. Choose an increasing sequence $0 = m_0 < m_1 < m_2 < \ldots$ such that the following holds for all $k$.

(*) for all $j < m_k$ and all $a \in F_{m_k}$, if $l \geq m_{k+1}$ then $\| [p_j(l), a] \| < 1/k$.

For $n \in \mathbb{N}$ and $i$ such that for some $k$ we have $i < m_k$ and $m_{k+1} \leq n$ let $r^i(n) = p_i(n)$. Thus we have projections

$$r^0(n) \leq r^1(n) \leq \cdots \leq r^{m_k}(n)$$

whenever $n \geq m_{k+1}$. For $h : \mathbb{N} \to \mathbb{N}$ define $r^h$ as in the paragraph before Lemma 5, by its representing sequence (let $r^i(n) = r^{m_k}(n)$ if $i \geq m_k$)

$$r^h(n) = r^{h(n)}(n).$$

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1 I could not find it, but it should be somewhere in Chapter VI.
Claim 8. If $h: \mathbb{N} \to \mathbb{N}$ then $r^h \in A' \cap A^U$.

Proof. Fix any $b$ in the unit ball of $A$ and $\varepsilon > 0$. If $k > 1/\varepsilon$ and there is $b' \in F_{2k}$ satisfying $\|b - b'\| < \varepsilon/2$ then for $i > n_{2k}$ in $Y$ we have that $\|p_{\gamma}(i), b')\| < \varepsilon/2$ and therefore $\|r^h(i), b\| < \varepsilon$ for $U$-many $i$. □

Using Lemma 5 for each $q$, find $h_{\gamma}$ such that $r^\gamma = r^{h_{\gamma}}$ satisfies $p_i q_s \leq r^\gamma$ for all $i$. Since $\mathbb{N}/\mathcal{U}$ is a linear ordering and $\lambda$ is a regular cardinal, we can find a cofinal subset $Z$ of $\lambda$ such that for $\gamma < \delta$ in $\mathcal{Z}$ we have $r^\delta \leq r^\gamma$. By reenumerating we may assume $\mathcal{Z} = \lambda$ and then $r^\gamma$, for $\gamma \in \mathcal{Z}$, together with $p_i$, for $i \in \mathbb{N}$, form an $(\aleph_\lambda, \lambda)$-gap. However, $r^\delta \leq r^\gamma$ is equivalent to $h_{\delta} \leq_{\mathcal{U}} h_{\gamma}$, and therefore $h_{\gamma}$, for $\gamma < \lambda$, form a $\leq_{\mathcal{U}}$-decreasing and $\leq_{\mathcal{U}}$-unbounded below sequence in $\mathbb{N}/\mathcal{U}$, and therefore $\lambda = \kappa(\mathcal{U})$. □

The proof of Proposition 6 can be modified (by removing some of its parts) to a proof of the following.

Proposition 9. Assume $A$ is a separable C*-algebra and $\mathcal{P}(A)$ has arbitrarily long finite chains. Then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every cardinal $\lambda$ there is an $(\aleph_\lambda, \lambda)$-gap in $\mathcal{P}(A^U)$ if and only if $\kappa(\mathcal{U}) = \lambda$. □

Corollary 10. Assume the Continuum Hypothesis fails. If $A$ is an infinite-dimensional separable C*-algebra of real rank zero then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $F_\mathcal{U}(A) \not\cong \mathcal{F}_\mathcal{V}(A)$ and $A^\mathcal{U} \not\cong A^\mathcal{V}$.

Proof. By [4, Theorem 2.2] we can find $\mathcal{U}$ and $\mathcal{V}$ so that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$ (here $\aleph_1$ and $\aleph_2$ are the least two uncountable cardinals; all that matters for us is that they are both less or equal than $2^{\aleph_0}$ and different). Therefore $\mathcal{P}(A' \cap A^\mathcal{U})$ has an $(\aleph_0, \aleph_1)$-gap while $\mathcal{P}(A' \cap A^\mathcal{V})$ does not, and $A' \cap A^\mathcal{U}$ and $A' \cap A^\mathcal{V}$ cannot be isomorphic.

It remains to prove that if $A$ is an infinite-dimensional C*-algebra of real rank zero then $\mathcal{P}(A)$ has an infinite chain of projections. We may assume $A$ is unital. Recursively find a decreasing sequence $r_n$ for $n \in \mathbb{N}$ in $\mathcal{P}(A)$ so that $r_n A r_n$ is infinite-dimensional for all $n$. Assume $r_n$ has been chosen. Since $A$ has real rank zero, in $r_n A r_n$ we can fix a projection $q \notin \{0, r_n\}$. If $q A r_n q$ is infinite-dimensional then let $r_{n+1} = q$. Otherwise, let $r_{n+1} = r_n - q$ and note that $r_{n+1} A r_{n+1}$ is infinite-dimensional. □

It is likely that Theorem 1 and Corollary 10 can be extended to all infinite-dimensional separable C*-algebras (possibly by considering the Cuntz ordering of positive elements instead of $\mathcal{P}(A)$).

References


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