ORTHONORMAL BASES OF HILBERT SPACES

ILIJAS FARAH

Assume H is a Hilbert space and K is a dense linear (not necessarily closed) subspace. The question whether K necessarily contains an orthonormal basis for H even when H is nonseparable was mentioned by Bruce Blackadar in an informal conversation during the Canadian Mathematical Society meeting in Ottawa in December 2008 and this note provides a negative answer. Note that the Gram–Schmidt process gives a positive answer when H is separable.

I will use $\aleph_1$ to denote both the first uncountable ordinal and the first uncountable cardinal and I will use $\mathfrak{c} = 2^{\aleph_0}$ to denote both the cardinality of the continuum and the least ordinal of this cardinality. All bases are orthonormal.

For cardinals $\lambda < \theta$ consider $\ell^2(\lambda)$ as a subspace of $\ell^2(\theta)$ consisting of vectors supported on the first λ coordinates. Let p_λ denote the projection of $\ell^2(\theta)$ to $\ell^2(\lambda)$.

Lemma 1. *Assume $\lambda < \theta$ are infinite cardinals such that θ is regular and x_γ, for $\gamma < \theta$, is an orthonormal family in $\ell^2(\theta)$. Then there is $\gamma_0 < \theta$ such that x_γ is orthogonal to $\ell^2(\lambda)$ for all $\gamma \geq \gamma_0$.*

Proof. For $\alpha \leq \theta$ let $X(\alpha)$ denote the closed linear span of x_γ for $\gamma < \alpha$. Let e_ξ, for $\xi < \lambda$, be the standard basis for $\ell^2(\lambda)$. Let $\alpha(\xi) < \kappa$ be the minimal ordinal such that the projection of e_ξ to $X(\theta)$ is in $X(\alpha(\xi))$. Since $\theta > \lambda$ we have $\alpha(\xi) < \theta$ and by the regularity of θ we have that $\gamma_0 = \sup_{\xi<\lambda} \alpha(\xi) < \theta$ is as required. □

Lemma 2. *Assume $\lambda < \theta$ are infinite cardinals such that θ is regular and $\lambda^{\aleph_0} \geq \theta$. Then there is a dense linear subspace K of $\ell^2(\theta)$ such that the kernel of the restriction of p_λ to K is $\{0\}$. Such K does not contain an orthonormal family of size greater than λ.*

Proof. Let z_γ, for $\gamma < \theta$, be a dense subset of $\ell^2(\theta)$. We shall find $y_{\gamma,m}$, for $\gamma < \theta$ and $m \in \mathbb{N}$, such that $\|y_{\gamma,m} - z_\gamma\| \leq 1/m$ for all γ and m and $p_\lambda(y_{\gamma,m})$, for $\gamma < \theta$ and $m \in \mathbb{N}$, are linearly independent.

Fix a Hamel basis $\mathbf{B}$ for $\ell^2(\lambda)$ considered as a vector space over $\mathbb{C}$. We have that $|\mathbf{B}| = \lambda^{\aleph_0} \geq \theta$. Assume $y_{\gamma,m}$ have been constructed for all $\gamma < \alpha$ and all m. Let F be the minimal subset of $\mathbf{B}$ such that $\{p_\lambda(z_\alpha)\} \cup \{y_{\gamma,m} : \gamma < \alpha, m \in \mathbb{N}\}$ is included in the linear span of F. Then $|F| \leq |\gamma| + \aleph_0 < \theta \leq |\mathbf{B}|$. Fix distinct vectors t_m, for $m \in \mathbb{N}$, in $\mathbf{B} \setminus F$ and let $y_{\alpha,m} = z_\alpha + \frac{1}{m} t_m$. (We are assuming t_m are unit vectors, but this is not required from $y_{\alpha,m}$.) Then $\|y_{\alpha,m} - z_\alpha\| = \frac{1}{m}$ and and $y_{\gamma,m}$, for $\gamma \leq \alpha$ and $m \in \mathbb{N}$, are linearly independent.

This describes the recursive construction. The linear span K of $\{y_{\gamma,m} : \gamma < \theta, m \in \mathbb{N}\}$ is dense and for $x \in K$ we have $p_\lambda(x) = 0$ if and only if $x = 0$. Lemma 1 implies that K cannot contain an orthonormal family of size greater than λ. □

Date: August 13, 2009.
Partially supported by NSERC.
Filename: `2008l09-basis.tex`.

Proposition 3. *Every nonseparable Hilbert space H contains a dense subspace that contains no basis for H.*

Proof. We may assume $H = \ell^2(\theta)$ for some uncountable cardinal θ. In the case when $\theta \leq 2^{\aleph_0}$ the existence of K is guaranteed by the case $\lambda = \aleph_0$ of Lemma 2.

We may therefore assume $\theta > 2^{\aleph_0}$ and write $H = \ell^2(\mathfrak{c}) \oplus \ell^2(\theta)$. Let H_0 be a separable subspace of $\ell^2(\mathfrak{c})$ and let K be a dense subspace of $\ell^2(\mathfrak{c})$ as in Lemma 2, so that the projection p_0 of $\ell^2(\mathfrak{c})$ to H_0 satisfies $\ker(p_0) \cap K = \{0\}$.

The dense subspace $K_1 = K \oplus \ell^2(\theta)$ of H contains no basis for H. Assume the contrary and let η_γ, for $\gamma < \theta$, be such a basis. Write q_0 for the projection of H to H_0 and $q_\mathfrak{c}$ for the projection of H to $\ell^2(\mathfrak{c})$. By Lemma 1 the set $X = \{\gamma : q_0(\eta_\gamma) \neq 0\}$ is countable. On the other hand, since the vectors $\{q_\mathfrak{c}(\eta_\gamma) : \gamma < \theta\}$ span $\ell^2(\mathfrak{c})$ the set $\{\gamma < \theta : q_\mathfrak{c}(\eta_\gamma) \neq 0\}$ is uncountable. Therefore for some γ we have $q_0(\eta_\gamma) = 0$ and $q_\mathfrak{c}(\eta_\gamma) \neq 0$. Since $p_0(q_\mathfrak{c}(\eta_\gamma)) = q_0(\eta_\gamma)$ this contradicts the choice of K. □

I shall end by providing an explanation why the subspace K of $\ell^2(\theta)$ constructed in the proof of Proposition 3 has a much stronger property when $\theta \leq 2^{\aleph_0}$ than when, for example, $\theta = (2^{\aleph_0})^+$.

Proposition 4. *Assume θ is a regular cardinal. The following are equivalent.*

(1) *For all cardinals $\lambda < \theta$ we have $\lambda^{\aleph_0} < \theta$.*
(2) *If Y is a linear subspace of some Hilbert space such that $|Y| = \theta$ then Y contains an orthonormal family of size θ.*

Proof. By Lemma 2, (2) implies (1). Now we assume (1) and prove (2). We may assume Y is a subspace of $\ell^2(\theta)$. Let y_γ, $\gamma < \theta$, be distinct vectors in Y. For each γ let X_γ be the support of y_γ. Applying the generalized Δ-system lemma ([1, Theorem 1.6], with $\kappa = \aleph_1$) to X_γ, for $\gamma < \theta$, we find $X \subseteq \theta$ and $I_1 \subseteq \theta$ of cardinality θ such that $X_\beta \cap X_\gamma = X$ for all $\beta \neq \gamma$ in I_1. Let p denote the projection of $\ell^2(\theta)$ to $\ell^2(X)$. Since X is at most countable and $\theta > 2^{\aleph_0}$ is regular we can find $y \in \ell^2(X)$ and $I_2 \subseteq I_1$ of cardinality θ such that $p(y_\gamma) = y$ for all $\gamma \in I_2$. Then $z_\gamma = y_\gamma - y$ for $\gamma \in I_2$ clearly form an orthonormal family of size θ. □

I would like to thank Justin Moore for pointing out to a few typos.

References

[1] K. Kunen, *Set theory: An introduction to independence proofs*, North–Holland, 1980.

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 34, Belgrade, Serbia

URL: http://www.math.yorku.ca/∼ifarah

E-mail address: ifarah@mathstat.yorku.ca