

# GRAPHS AND CCR ALGEBRAS

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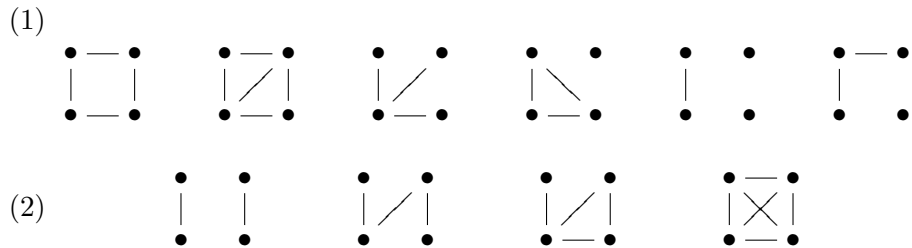
ABSTRACT. I introduce yet another way to associate a C\*-algebra to a graph and construct a simple nuclear C\*-algebra that has irreducible representations both on a separable and a nonseparable Hilbert space.

Kishimoto, Ozawa and Sakai have proved in [8] that the pure state space of every separable simple C\*-algebra is *homogeneous* in the sense that for every two pure states  $\phi$  and  $\psi$  there is an automorphism  $\alpha$  such that  $\phi \circ \alpha = \psi$ . They have shown that this fails for nonseparable algebras and asked whether the pure state space of every nuclear (not necessarily separable) C\*-algebra is homogeneous.

**Theorem 1.** *There is a simple nuclear C\*-algebra  $B$  that has irreducible representations both on a separable Hilbert space and on a nonseparable Hilbert space.*

**Corollary 2.** *There is a simple nuclear algebra whose pure state space is not homogeneous. This algebra moreover has a faithful representation on a separable Hilbert space.  $\square$*

As a curious side result, our construction gives a non-obvious equivalence relation on the class of all graphs. For example, among the graphs with four vertices there are three equivalence classes:



and the third one containing the null graph. I don't know whether there is a simple description of this relation or what is its computational complexity (see Question 3.4).

In §1 we prove Theorem 1 and in §2 we study some properties of the canonical commutation relation (CCR) algebras associated with graphs of

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which the algebra used in the proof of Theorem 1 is a special case. By  $|X|$  we denote the cardinality of the set  $X$ . We also use the following convention. If  $A$  and  $B$  are unital algebras then  $A \otimes B$  is identified with a subalgebra of  $B$ . Similarly, if  $A_i$ , for  $i \in X$ , are unital algebras and  $Y \subseteq X$  then  $\bigotimes_{i \in Y} A_i$  is considered as a subalgebra of  $\bigotimes_{i \in X} A_i$ . All the background can be found in [2] and in [13].

## 1. GRAPHS AND ALGEBRAS

Given a graph  $G = (V, E)$  let  $B(G)$  be the universal algebra generated by unitaries  $u_x$ , for  $x \in V$  that satisfy relations

$$\begin{aligned} u_x u_x^* &= 1 && \text{for all } x, \\ u_x^2 &= 1 && \text{for all } x, \\ u_x u_y &= u_y u_x && \text{if } x \text{ and } y \text{ are not adjacent,} \\ u_x u_y &= -u_y u_x && \text{if } x \text{ and } y \text{ are adjacent.} \end{aligned}$$

Recall that the *character density* of a  $C^*$  algebra is the minimal cardinality of its dense subset.

**Lemma 1.1.** *The algebra  $B(G)$  is well-defined for every graph  $G$ , and its character density is equal to  $|G| + \aleph_0$ .*

*Proof.* We first show that for every finite graph  $G$  there is a  $C^*$ -algebra generated by the unitaries  $u_x$ , for  $x \in V$ , satisfying the required relations.

Let  $n = |V|$  and let  $m = \binom{n}{2}$ , identified with the set of distinct pairs  $\{i, j\}$  of natural numbers in  $\{1, \dots, n\}$ . For each pair  $1 \leq i < j \leq n$  fix a two-dimensional complex Hilbert space  $H_{ij}$  and let  $H = \bigotimes_{1 \leq i < j \leq n} H_{i,j}$ .

For  $k \leq n$  define the unitary  $u_k$  on  $H$  as

$$u_k = \bigotimes_{1 \leq i < j \leq n} u_{i,j,k}$$

where

$$u_{i,j,k} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } k \notin \{i, j\} \text{ or } i \text{ is not adjacent to } j \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } k = i \text{ and } i \text{ is adjacent to } j, \text{ and} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } k = j \text{ and } i \text{ is adjacent to } j. \end{cases}$$

Then each  $u_k$  is a self-adjoint unitary and clearly  $u_i$  and  $u_j$  commute if  $i$  is not adjacent to  $j$  and  $u_i$  and  $u_j$  anti-commute if  $i$  is adjacent to  $j$ . Therefore  $C^*(\{u_i : i \leq n\})$  realizes the defining relations for  $B(G)$ . If  $G$  is infinite, then clearly  $B(G)$  is the direct limit of  $B(G_0)$  where  $G_0$  ranges over all finite subgraphs of  $G$ . Therefore for every  $G$  there is a  $C^*$ -algebra that realizes the defining relations for  $G$ .

Since all the generators of  $B(G)$  are unitaries, by taking the direct sum of all representations one obtains  $B(G)$  for a finite  $G$ .

We claim that  $x \neq y$  implies  $\|u_x - u_y\| \geq \sqrt{2}$ . Since the matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  are  $\sqrt{2}$  apart from each other, this will follow from Lemma 1.2. Since the generating unitaries  $u_i$ , for  $i \in V$ , form a discrete generating set, the character density of  $B(G)$  is  $|G|$  if  $G$  is infinite and  $\aleph_0$  if  $G$  is finite.  $\square$

The following lemma is probably well-known but I could not find a reference (here  $\mathbb{T}$  denotes the unit circle in  $\mathbb{C}$ ).

**Lemma 1.2.** *In any  $C^*$ -algebra  $C \otimes D$  the following holds. If  $v$  and  $w$  are unitaries in  $D$  and  $a$  and  $b$  are in  $C$  then*

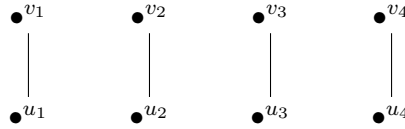
$$\|a \otimes v - b \otimes w\| \geq \inf_{\lambda \in \mathbb{T}} \|\lambda a - b\|.$$

*Proof.* Fix a representation of  $C \otimes D$  on  $H_1 \otimes H_2$ . Fix  $\varepsilon > 0$  and  $\lambda$  in the spectrum of  $w^*v$ . Pick a unit vector  $\eta$  in  $H_2$  such that  $\|w^*v\eta - \lambda\eta\| < \varepsilon$ . Now find a unit vector  $\xi$  in  $H_1$  such that  $\|(\lambda a - b)\xi\| > \|a - b\| - \varepsilon$ . Then

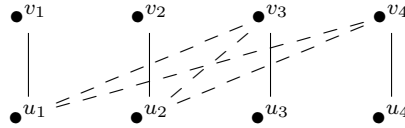
$$\begin{aligned} \|(a \otimes v - b \otimes w)(\xi \otimes \eta)\| &= \|(\lambda a \otimes v - b \otimes \lambda w)(\xi \otimes \eta)\| \\ &\geq \|((\lambda a - b) \otimes v)(\xi \otimes \eta)\| - \|(b \otimes (v - \lambda w))(\xi \otimes \eta)\| \\ &> \|\lambda a - b\| - \varepsilon(1 + \|b\|). \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, the conclusion follows.  $\square$

The algebra in (2) of Lemma 1.3 below, with  $n = 4$ , corresponds to



and the algebra in (3) of the same lemma, with  $l = 2$  and  $n = 2$ , corresponds to any graph of the form (the dashed line means that the vertices may or may not be adjacent)



The proof of Lemma 1.3 is implicit in [3] but we sketch it for the reader's convenience. A related result is proved in Lemma 2.2 below.

**Lemma 1.3.** *For a  $C^*$ -algebra  $A$  the following are equivalent.*

- (1)  *$A$  is isomorphic to  $M_{2^n}(\mathbb{C})$ .*

- (2)  $A$  is generated by self-adjoint unitaries  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  such that  $u_i$  and  $v_j$  commute if and only if  $i = j$  and  $u_i$  and  $v_j$  anti-commute if and only if  $i \neq j$ .
- (3)  $A$  is generated by self-adjoint unitaries  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  such that for some  $l \leq n$  we have
- (a) If  $j \leq l$  then  $u_i$  and  $v_j$  anti-commute if and only if  $i = j$ .
  - (b) If  $l < i$  then  $u_i$  and  $v_j$  anti-commute if and only if  $i = j$ .

*Proof.* The case  $n = 1$  is [3, Lemma 4.1], using

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the fact that  $A$  is a noncommutative  $C^*$ -algebra that is a 4-dimensional vector space over  $\mathbb{C}$  for the converse.

Fix  $n > 1$ . Note that  $M_{2^n}(\mathbb{C})$  is isomorphic to  $\bigotimes_{i=1}^n M_2(\mathbb{C})$ . Using the convention stated before the lemma, identify the unitaries  $u_i$  and  $v_i$  generating the  $i$ -th copy of  $M_2(\mathbb{C})$  with elements of  $M_{2^n}(\mathbb{C})$ . Then  $u_i, v_i$ , for  $1 \leq i \leq n$  are as in (2)

To see that (2) implies (1), assume  $A$  is generated by  $u_i, v_i$ , for  $1 \leq i \leq n$ , as in the statement of the lemma. Then  $A_i = C^*(u_i, v_i)$  is a subalgebra of  $A$  isomorphic to  $M_2(\mathbb{C})$ . These subalgebras are commuting and they generate  $A$ , and therefore (1) follows.

Since (3) is a special case of (2) (with either  $l = 1$  or  $l = n$ ) it remains to prove (3) implies (2). For  $l < j \leq n$  define

$$K(j) = \{i \leq l : v_j u_i = -u_i v_j\}.$$

For all  $m \leq n$  we have that  $w_j = v_j \prod_{i \in K(j)} v_i$  commutes with  $u_m$  if  $m \neq j$  and anticommutes with  $u_m$  if  $m = j$ .

Let  $w_j = v_j$  for  $j \leq l$ . Since for  $l < j \leq n$  we have  $v_j = w_j \prod_{i \in K(j)} w_i$ ,  $A$  is generated by  $w_1, \dots, w_n$  and  $u_1, \dots, u_n$  and they satisfy (2).  $\square$

For a set  $Y$  identify the power-set of  $Y$  with  $2^Y$  and consider it with the product topology. If  $\mathbb{A} \subseteq 2^Y$  then let  $G(Y, \mathbb{A})$  denote the bipartite graph with the set of vertices  $Y \cup \mathbb{A}$  such that  $i \in Y$  and  $x \in \mathbb{A}$  are adjacent if and only if  $i \in x$ .

**Lemma 1.4.** *Assume  $\mathbb{A} \subseteq 2^Y$ . Then the  $C^*$ -algebra  $B = B(G(Y, \mathbb{A}))$  has a representation on a Hilbert space of density  $|Y|$ . If  $\mathbb{A}$  is dense in  $2^Y$  then this representation can be chosen to be irreducible.*

*Proof.* We shall denote the generating unitaries by  $u_i$ ,  $i \in Y$  and  $v_x$ ,  $x \in \mathbb{A}$ .

For each pair  $i \in Y$ ,  $x \in \mathbb{A}$  let  $H_{i,x}$  be the two-dimensional complex Hilbert space and let  $\zeta_{i,x}$  denote the vector  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  in  $H_{i,x}$ . We shall represent  $B$  on  $H = \bigotimes_{i \in Y} (H_i, \zeta_i)$ . (Recall that this is the closure of the linear span of elementary tensors of the form  $\bigotimes_i \xi_i$  such that  $\xi_i = \zeta_i$  for all but finitely

many pairs  $i$ .) Since the character density of  $H$  is equal to  $|Y|$ , this will prove the claim. For  $i \in Y$  let  $u_i \in \mathcal{B}(H)$  be defined by

$$u_i = \bigotimes_{j \in Y} u_{ij}$$

where  $u_{ii} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $u_{ij}$  is the identity matrix whenever  $i \neq j$ . For  $x \in \mathbb{A}$  let  $v_x \in \mathcal{B}(H)$  be defined by (using the convention that the omitted terms are equal to the identity matrix)

$$v_x = \bigotimes_{i \in x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , every elementary tensor of the form  $\bigotimes_i \xi_i$  such that  $\xi_i = \zeta_i$  for all but finitely many  $i$  is sent to an elementary tensor of this form. Since  $H$  is the closed linear span of such vectors,  $v_x$  is an operator on  $H$ .

It is clear that  $v_x$  and  $u_i$  commute if  $i \notin x$  and that  $v_x$  and  $u_i$  anticommute if  $i \in x$ . Since  $B$  was assumed to be simple, it is isomorphic to the algebra  $C^*(\{u_i : i \in Y\} \cup \{v_x : x \in \mathbb{A}\})$ .

Now assume  $\mathbb{A}$  is dense in  $2^Y$ . For  $F \subseteq Y$  write  $H_F$  for  $\bigotimes_{i \in F} H_i$ . Fix a finite  $F \subseteq Y$  and write

$$\zeta = \bigotimes_{i \in Y \setminus F} \zeta_i.$$

Therefore  $\xi \in H_F$  implies  $\xi \otimes \zeta \in H_Y$ . For every  $x \subseteq Y$  and every  $\xi \in H_F$  we have  $v_x(\xi \otimes \zeta) = (v_{x \cap F} \xi) \otimes \zeta$ . Since  $\mathbb{A}$  is dense in  $2^Y$ , Lemma 1.3 implies  $C^*(\{u_i : i \in F\} \cup \{v_{x \cap F} : x \in \mathbb{A}\}) = \mathcal{B}(H_F)$ . Therefore for any two unit vectors  $\xi \otimes \zeta$  and  $\eta \otimes \zeta$  there is  $a \in B$  such that  $a\xi = \eta$ . Since  $H_Y$  is the direct limit of  $H_F$  for  $F \subseteq Y$  finite, we conclude that  $H_Y$  has no nontrivial closed  $B$ -invariant subspace.  $\square$

**Definition 1.5.** A family  $\mathbb{A}$  of subsets of  $Y$  is *independent* if for all finite disjoint subsets  $F$  and  $G$  of  $\mathbb{A}$  we have that  $\bigcap F \setminus \bigcup G$  is nonempty.

It is not difficult to see that if  $\mathbb{A}$  is infinite then this is equivalent to requiring such intersections to always be infinite. The proof of this fact is included in the proof of Lemma 1.7.

A full matrix algebra is an algebra of the form  $M_n(\mathbb{C})$ . Following [3] we say that an algebra is AM (approximately matricial) if it is a direct limit of full matrix algebras. The following lemma will be generalized in Lemma 2.4.

**Lemma 1.6.** *Assume  $\mathbb{A}$  is infinite, independent, and dense in  $2^Y$ . Then  $B = B(G(Y, \mathbb{A}))$  is simple, nuclear, unital and it has the unique trace.*

*Proof.* We shall denote the generating unaries by  $u_i$ ,  $i \in Y$  and  $v_x$ ,  $x \in \mathbb{A}$ . It suffices to prove that  $B$  is AM, since every AM algebra is simple, nuclear, unital, and has the unique trace ([3]).

Let  $\Lambda_0$  be the set of all pairs  $(F, G)$  such that  $F \subseteq Y$  is finite,  $G \subseteq \mathbb{A}$  is finite ordered by the coordinatewise inclusion. With

$$D(F, G) = C^*(\{u_i : i \in F\} \cup \{v_x : X \in G\})$$

we have that  $B = \varinjlim_{\Lambda_0} D(F, G)$ . Now let  $\Lambda$  be the set of all  $(F, G) \in \Lambda_0$  such that for some  $1 \leq l \leq n \in \mathbb{N}$  we have the following.

- (1)  $F = \{x(1), \dots, x(n)\}$  and  $G = \{k(1), \dots, k(n)\}$ ,
- (2) If  $j \leq l$  then  $k(i) \in x(j)$  if and only if  $i = j$ ,
- (3) If  $l < i$  then  $k(i) \in x(j)$  if and only if  $i = j$ .

Lemma 1.3 implies that  $D(F, G)$  is isomorphic to  $M_{2^n}(\mathbb{C})$  (with  $n$  as above) and it therefore suffices to prove that  $\Lambda$  is cofinal in  $\Lambda_0$ .

Fix  $(F, G) \in \Lambda_0$ . We may assume  $|F| = |G| = l$  and enumerate them as  $F = \{x(i) : l < i \leq 2l\}$  and  $G = \{k(i) : i \leq l\}$ . Since  $\mathbb{A}$  is independent, for each  $j$  such that  $l < j \leq 2l$  we can pick

$$k(j) \in x(j) \setminus \left( \bigcup_{l < i \leq 2l} x(i) \cup G \right).$$

By the density of  $\mathbb{A}$  for each  $j \leq l$  pick  $x(j) \in \mathbb{A}$  such that

$$x(j) \cap \{k(1), \dots, k(2l)\} = \{k(j)\}.$$

Let  $F' = \{x(1), \dots, x(2l)\}$  and  $G' = \{k(1), \dots, k(2l)\}$  and  $n = 2l$  we see that  $(F', G')$  is in  $\Lambda$ , concluding the proof.  $\square$

For a family  $\mathbb{A}$  of subsets of  $Y$  consider the dual family  $\hat{\mathbb{A}} = \{z(i) : i \in Y\}$  of subsets of  $\mathbb{A}$  defined by

$$x \in z(i) \text{ if and only if } i \in x.$$

In the following lemma we identify  $\hat{\mathbb{A}}$  and  $Y$  by identifying  $i \in Y$  with  $z(i) \in \hat{\mathbb{A}}$ .

**Lemma 1.7.** *Assume  $Y, \mathbb{A}$ , and  $\hat{\mathbb{A}}$  are as above.*

- (1) *the dual,  $\hat{\mathbb{A}}$ , of  $\hat{\mathbb{A}}$  is equal to  $\mathbb{A}$ .*
- (2)  *$\mathbb{A}$  is dense in  $2^Y$  if and only if  $\hat{\mathbb{A}}$  is independent.*
- (3)  *$\hat{\mathbb{A}}$  is dense in  $2^{\mathbb{A}}$  if and only if  $\mathbb{A}$  is independent.*

*Proof.* The first assertion is obvious and the third follows immediately from the first two.

Assume  $\mathbb{A}$  is not dense in  $2^Y$  and fix a nonempty basic open set  $U \subseteq 2^Y$  disjoint from  $\mathbb{A}$ . For some finite and disjoint  $F \subseteq Y$  and  $G \subseteq Y$  we have that  $U = \{x \in 2^Y : x \cap F = \emptyset \text{ and } G \subseteq x\}$ . The Boolean combination  $\bigcap F \setminus \bigcup G = \emptyset$  witnesses that  $\hat{\mathbb{A}}$  (identified with  $Y$ ) is not independent. Now assume  $\mathbb{A}$  is dense in  $2^Y$ . This implies that its intersection with every nonempty basic open set is nonempty (and moreover infinite if  $Y$  is infinite), and by the above argument  $\hat{\mathbb{A}}$  is independent.  $\square$

We include a proof of the following classical result ([9, (A6) on p. 288]) for reader's convenience.

**Lemma 1.8** (Fichtenholz–Kantorovich). *There exists an independent family of subsets of  $\mathbb{N}$  of cardinality continuum.*

*Proof.* Let  $2^{<\mathbb{N}}$  denote the set of all finite sequences of 0s and 1s. For  $f \in 2^{\mathbb{N}}$  by  $f \upharpoonright m$  we denote its initial segment of length  $m$  and for  $s \in 2^{<\mathbb{N}}$  by  $|s|$  we denote its length. For  $f \in 2^{\mathbb{N}}$  let

$$X_f = \{T \subseteq 2^{<\mathbb{N}} : (\exists m)|s| = m \text{ for all } s \in T \text{ and } f \upharpoonright m \in T\}.$$

Assume  $m < n$  and  $f_1, \dots, f_m, f_{m+1}, \dots, f_n$  are distinct elements of  $2^{\mathbb{N}}$ . Fix  $k$  large enough so that  $f_i \upharpoonright k \neq f_j \upharpoonright k$  for all  $i \neq j$ . Then

$$s = \{f_i \upharpoonright k : i \leq m\}$$

belongs to  $\bigcup_{i=1}^m X_{f_i} \setminus \bigcup_{j=m+1}^n X_{f_j}$ . Therefore the family  $\{X_f : f \in 2^{\mathbb{N}}\}$  is independent.  $\square$

*Proof of Theorem 1.* Let  $\mathbb{A}$  be an independent family of subsets of  $\mathbb{N}$  of size continuum as in Lemma 1.8. The remark after Definition 1.5 implies that if  $x \in \mathbb{A}$  is replaced with  $x'$  such that the symmetric difference  $x\Delta x'$  is finite, then  $(\mathbb{A} \cup \{x'\}) \setminus \{x\}$  is still independent. By making finite changes to countably many of the members of  $\mathbb{A}$  we can therefore assure  $\mathbb{A}$  both is dense and independent in  $2^{\mathbb{N}}$ . By Lemma 1.4 the C\*-algebra  $B = B(G(\mathbb{N}, \mathbb{A}))$  has an irreducible representation on a separable Hilbert space. Since the graphs  $G(Y, \mathbb{A})$  and  $G(\mathbb{A}, \hat{\mathbb{A}})$  are isomorphic,  $B$  is isomorphic to  $B(G(\mathbb{A}, \hat{\mathbb{A}}))$ . Since  $|\mathbb{A}| = 2^{\aleph_0}$ , Lemma 1.4 implies that  $B$  has an irreducible representation on a nonseparable Hilbert space.  $\square$

The assumptions of Lemma 1.6 can be weakened. Instead of requiring  $\mathbb{A}$  to be independent, we may require that for every  $x \in \mathbb{A}$  and every finite  $F \subseteq \mathbb{A} \setminus \{x\}$  the set  $x \setminus \bigcup F$  is nonempty. Instead of requiring  $\mathbb{A}$  to be dense, we can require that for every finite  $s \subseteq Y$  and every  $j \in s$  there is  $x \in \mathbb{A}$  such that  $x \cap s = \{j\}$ . The proof of Lemma 1.7 shows that  $\mathbb{A}$  satisfies these two conditions if and only if  $\hat{\mathbb{A}}$  satisfies these two conditions. Therefore instead of an independent family, in the proof of Theorem 1 we could have used an *almost disjoint* family, i.e., a family  $\mathbb{A}$  of infinite subsets of  $Y$  such that  $x \cap y$  is finite for all distinct  $x$  and  $y$  in  $\mathbb{A}$ . Uncountable almost disjoint families in  $2^{\mathbb{N}}$  are well-studied set-theoretic objects.

## 2. MORE ON ALGEBRAS AND GRAPHS

Note that if  $|V| = n$  then  $B(G)$  is a  $2^n$ -dimensional vector space over  $\mathbb{C}$  since it is spanned by  $v_s = \prod_{x \in s} v_x$  for  $s \subseteq V$  ( $v_s$  are defined using a fixed linear order on  $V$  for definiteness). On the collection of all graphs define the equivalence relation  $\sim$  by  $G_1 \sim G_2$  if  $B(G_1)$  and  $B(G_2)$  are isomorphic.

For a graph  $G = (V, E)$ , a finite subset  $s$  of  $V$  and  $x \in s$  define the graph  $G - x + s$  as follows. Its vertex set is  $V' = V \setminus \{x\} \cup \{s\}$ , hence  $s$  is considered as a vertex in the new graph. The adjacency relation for vertices in  $V \setminus \{x\}$  is unchanged, and we let  $s$  be adjacent to  $u \in V \setminus \{x\}$  if and only if  $|\{w \in s : \{w, u\} \in E\}|$  is an odd number.

**Lemma 2.1.** *For  $G, x$  and  $\mathbf{s}$  as above the algebras  $B(G)$  and  $B(G - x + \mathbf{s})$  are isomorphic.*

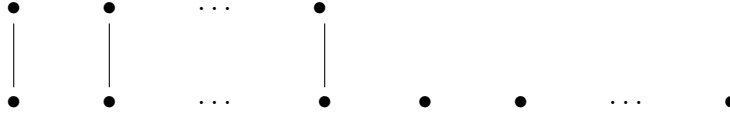
*Proof.* In  $B(G)$  consider the product  $u_{\mathbf{s}} = \prod_{i \in \mathbf{s}} u_i$  (for definiteness, we are assuming that  $V$  is well-ordered and the unitaries in the product are taken in this order). Then  $u_{\mathbf{s}}$  is a unitary and one of  $u_{\mathbf{s}}$  and  $iu_{\mathbf{s}}$  is self-adjoint, depending on whether the number of edges between the vertices in  $\mathbf{s}$  is even or odd. Let  $w_{\mathbf{s}}$  denote this self-adjoint unitary. Then the unitaries  $\{u_x: x \in V \setminus \{x\}\} \cup \{w_{\mathbf{s}}\}$  clearly satisfy the relations corresponding to  $G - x + \mathbf{s}$ .

Since  $x \in \mathbf{s}$ , in  $B(G - x + \mathbf{s})$  we can similarly define a unitary  $w_x$  such that the unitaries  $\{u_x: x \in V \setminus \{x\}\} \cup \{w_x\}$  satisfy the relations corresponding to  $G$ .

We have shown that every algebra generated by unitaries satisfying relations corresponding to  $G$  is also generated by unitaries satisfying relations corresponding to  $G - x + \mathbf{s}$ , and vice versa. Since this correspondence is given in a canonical way, we conclude that the universal algebras are isomorphic.  $\square$

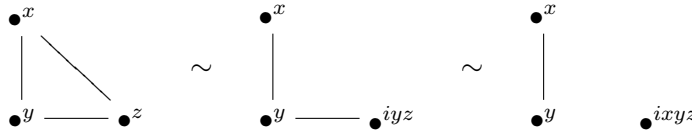
**Lemma 2.2.** *For every graph  $G$ , if  $|G| = n$  then there is  $k \leq n/2$  such that with  $l = n - 2k$  we have that  $B(G)$  is isomorphic to  $M_{2^k}(\mathbb{C}) \otimes \mathbb{C}^{2^l}$ .*

*Proof.* We need to show that every graph  $G$  with  $n$  vertices is equivalent to a graph of the form



where there are  $k$  pairs of vertices on the left hand side and  $l = n - 2k$  vertices on the right hand side. We shall refer to this graph as ‘the canonical graph representing  $M_{2^k}(\mathbb{C}) \otimes \mathbb{C}^{2^l}$ .’

The proof is by induction on  $n$ . If  $n = 1$  or  $n = 2$  then the assertion is vacuous. We shall first prove the case  $n = 3$  both as a warmup and because it will be used in the inductive step. We shall prove that each graph  $G$  on three vertices is isomorphic either to the null graph or to the graph with a single edge. By using Lemma 2.1 we have the following.

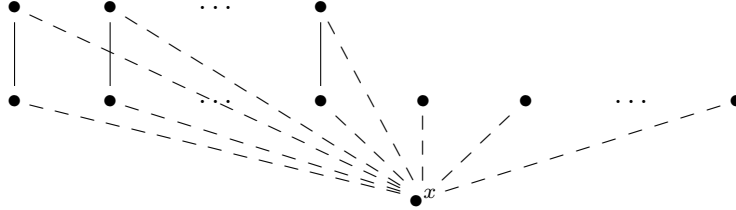


Since  $G_1, G_2$  and  $G_3$ , together with the null graph, are all graphs with three vertices, this concludes the proof of the case  $n = 3$ .

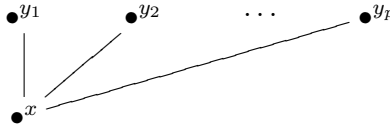
Assume the assertion is true for  $n$  and fix  $G$  such that  $|V| = n + 1$ . Applying the inductive hypothesis, we may assume that the induced graph



of  $G$  to the first  $n$  vertices is the canonical graph representing  $M_{2^k}(\mathbb{C}) \otimes \mathbb{C}^{2^{2l}}$  for some  $k$  and  $l$ . Then  $G$  is of the form



By the case  $n = 3$  treated above, each of the triangles on the left hand side of the graph can be turned into a graph with exactly one edge, by multiplying  $x$  with some of the other generators and (if necessary)  $i$ . It therefore remains to check that every graph of the form

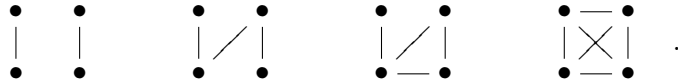


is equivalent to a graph with exactly one edge. This is obtained by replacing  $y_j$ , for  $j \geq 2$ , with  $y_1 y_j$  and using Lemma 2.1.  $\square$

Lemma 2.2 implies there are exactly  $1 + \lfloor n/2 \rfloor$  nonisomorphic algebras of the form  $B(G)$ , where  $G$  is a graph with  $n$  vertices. For example, in the case  $n = 4$  the algebras are  $\mathbb{C}^{16}$  (corresponding to the null graph),  $M_2(\mathbb{C}) \otimes \mathbb{C}^4$ , corresponding to any of the graphs



and  $M_4(\mathbb{C})$ , corresponding to any of the graphs



I don't know whether there is a simpler description of the relation  $\sim$ , even on the finite graphs, then the one given by Lemma 2.3 below.

For a graph  $G = (V, E)$  let  $G^{<\infty}$  be the graph whose vertices are all finite nonempty subsets of  $V$  and two such vertices  $\mathbf{s}$  and  $\mathbf{t}$  are adjacent if and only if the cardinality of the set

$$\{(i, j) : i \in \mathbf{s}, j \in \mathbf{t}, \text{ and } \{i, j\} \in E\}$$

is an odd number.

**Lemma 2.3.** *Assume  $G$  and  $K$  are graphs.*

- (1) *If graphs  $G^{<\infty}$  and  $K^{<\infty}$  are isomorphic then algebras  $B(G)$  and  $B(K)$  are isomorphic.*
- (2) *Assume  $G$  and  $K$  are finite. Then the following are equivalent*
  - (a)  *$G^{<\infty}$  and  $K^{<\infty}$  are isomorphic,*

- (b)  $G$  can be obtained from  $K$  by a finite number of applications of Lemma 2.1,
- (c)  $B(G)$  and  $B(K)$  are isomorphic.

*Proof.* (1) Consider the algebra  $B(G)$ . The linear span of unitaries of the form  $w_s$  (as defined in the proof of Lemma 2.1) is dense in  $A(G)$ . If  $G^{<\infty}$  is isomorphic to  $K^{<\infty}$  then  $A(G)$  and  $A(K)$  have isomorphic—and therefore isometric—dense  $*$ -algebras and are, therefore, isomorphic.

It is clear that (2b) implies (2a) and (2a) implies (2c) by part (1).

For the remaining implication we need to assume  $G$  and  $K$  are finite. Assume (2c). Lemma 2.2 implies that by a finite number of applications of Lemma 2.1 graph  $G$  can be turned into the canonical graph representing  $M_{2^k}(\mathbb{C}) \otimes \mathbb{C}^{2^l}$  for some  $k$  and  $l$ . Similarly, by a finite number of applications of Lemma 2.1 graph  $K$  can be turned into the canonical graph representing  $M_{2^{k'}}(\mathbb{C}) \otimes \mathbb{C}^{2^{l'}}$  for some  $k'$  and  $l'$ . If these algebras are isomorphic then  $k = k'$  and  $l = l'$ , and (2b) follows by transitivity.  $\square$

Although the equivalence of (3) and (4) of the following lemma is a version of Lemma 1.6 in a wider context, the latter is not an immediate consequence of the former.

**Lemma 2.4.** *For an infinite graph  $G = (V, E)$  the following are equivalent.*

- (1) *The family of finite induced subgraphs  $G_0$  of  $G$  such that  $B(G_0)$  is isomorphic to a full matrix algebra is cofinal in all finite induced subgraphs of  $G$ .*
- (2)  *$B(G)$  is AM.*
- (3)  *$B(G)$  is simple and has a unique trace.*
- (4) *For all finite nonempty  $s \subseteq V$  there is  $v \in V$  such that  $|\{u \in s : u \text{ is adjacent to } v\}|$  is odd.*

*Proof.* The implication from (1) to (2) is immediate and (2) implies (3) was proved in [3]. Assume (4) fails for some  $s$ . Then the unitary  $w_s$  (as defined in the proof of Lemma 2.1) commutes with every generating unitary  $u_x$  of  $B(G)$  and therefore belongs to its center, hence (3) fails.

Now assume (4). We first prove that it is equivalent to

- (5) *For all finite nonempty  $s \subseteq V$  there is a finite  $t \subseteq V$  such that  $|\{(u, v) \in s \times t : u \text{ is adjacent to } v\}|$  is odd.*

Clearly (4) implies (5). The reverse implication holds because if a sum of integers is odd then at least one of them has to be odd.

In order to prove (1) fix a finite induced subgraph  $G_0$  of  $G$ . By Lemma 2.2  $G_0$  is  $\sim$ -equivalent to the canonical graph representing  $M_{2^k}(\mathbb{C}) \otimes \mathbb{C}^{2^l}$  for some  $k$  and  $l$ . By Lemma 2.3 (2), we can assume  $G_0$  is equal to the latter graph (note that the condition (5) is invariant under this change).

If  $l = 0$  there is nothing to prove. Otherwise let  $x$  be one of the  $l$  unmatched vertices in  $G_0$ . By (4) pick  $y$  in  $G$  adjacent to  $x$ . Note that  $y$  is not

a vertex of  $G_0$ . The construction of Lemma 2.2 shows that the induced subgraph of  $G$  on  $V(G_0) \cup \{y\}$  is equivalent to the canonical graph representing  $M_{2^{k+1}}(\mathbb{C}) \otimes \mathbb{C}^{2^{l-1}}$ .

Repeating this construction  $l - 1$  more times we find an induced subgraph  $G_1$  of  $G$  including  $G_0$  such that  $B(G)$  is isomorphic to  $M_{2^{k+l}}(\mathbb{C})$  and (1) follows.  $\square$

Lemma 2.4 implies  $B(G)$  is isomorphic to the CAR algebra  $M_{2^\infty}$  for the generic countable graph  $G$ . This is not surprising since the generic countable graph is isomorphic to the Rado graph, also known as the random graph. Lemma 2.4 also implies that the algebras of the form  $B(G)$  do not give new examples of separable  $C^*$ -algebras.

**Corollary 2.5.** *If  $G$  is a countably infinite graph then  $B(G)$  is isomorphic to an algebra of the form  $M_{2^m}(\mathbb{C}) \otimes \mathbb{C}^{2^n}$  for  $m$  and  $n$  in  $\mathbb{N} \cup \infty$ , where  $\mathbb{C}^{2^\infty}$  is defined to be  $C(2^\mathbb{N})$ , the algebra of continuous functions on the Cantor space.*  $\square$

The algebras of the form  $B(G)$  associated with uncountable graphs have other interesting properties. For example, it is not difficult to show that under the assumptions of Lemma 1.6 and using the notation from its proof the masas generated by  $\{u_i : i \in Y\}$  and  $\{v_x : x \in \mathbb{A}\}$  have the extension property. (Recall that a *masa* in a  $C^*$ -algebra is its maximal abelian  $C^*$ -subalgebra and that it has the *extension property* if each of its pure states has the unique extension to a state of the algebra.) This assertion is an immediate consequence of the following lemma.

**Lemma 2.6.** *Assume  $A$  is a  $C^*$ -algebra and  $\phi$  is a state on  $A$ . Also assume  $u$  is a self-adjoint unitary in  $A$  and  $b \in A$  is such that  $ub = -bu$ . If  $\phi(u) = 1$  or  $\phi(u) = -1$  then  $\phi(b) = 0$ .*

*Proof.* Assume for a moment that  $\phi(u) = 1$ . The projection  $p = (1 + u)/2$  satisfies  $\phi(p) = 1$  and  $pbu = -pub = -p(2p - 1)b = -pb$  hence  $pb(u + 1) = 0$  and  $pbp = 0$ . But the Cauchy–Schwartz inequality for  $\phi$  easily implies  $\phi(b) = \phi(pbp) = 0$  (see e.g., [4, Lemma 3.5]). The case when  $\phi(u) = -1$  is analogous.  $\square$

### 3. CONCLUDING REMARKS

It would be interesting to investigate algebras  $B(G)$  associated with uncountable graphs with strong partition properties (see [10]).

For every  $n$  the algebra  $M_n(\mathbb{C})$  is the universal algebra generated by unitaries  $u$  and  $v$  such that  $u^n = v^n = 1$  and  $uv = \gamma vu$ , where  $\gamma$  is a primitive  $n$ -th root of unity. Using this observation one can generalize algebras  $B(G)$  by associating an AF algebra to a digraph with labelled edges. At present I am not aware of any applications of these algebras.

A positive answer to the Kishimoto–Ozawa–Sakai question would have rather interesting consequences. In [1] it was proved that if Jensen’s diamond

principle holds on  $\aleph_1$  then there is a counterexample to Naimark's problem. If the results of [8] and [5] extended to nonseparable nuclear  $C^*$ -algebras then the argument from [1] would show that Jensen's diamond principle on any cardinal implies the existence of a counterexample to Naimark's problem. It is not known whether positive answer to Naimark's problem is consistent with the standard axioms of set theory.

**Question 3.1.** *Assume  $A$  is simple  $C^*$ -algebra and  $\phi$  and  $\psi$  are its pure states. Is there a  $C^*$ -algebra  $B$  that has  $A$  as a subalgebra and such that*

- (1) *both  $\phi$  and  $\psi$  have unique state extensions,  $\tilde{\phi}$  and  $\tilde{\psi}$ , to  $B$ ,*
- (2) *these extensions are equivalent, i.e., there is an automorphism  $\alpha$  of  $B$  such that  $\tilde{\phi} = \tilde{\psi} \circ \alpha$ .*

If the answer to Question 3.1 is positive, or if, for example, for every simple nuclear algebra  $A$  one can find a simple nuclear algebra  $B$  satisfying its requirements, then the argument from [1] shows that Jensen's diamond principle on any cardinal implies the existence of a counterexample to Naimark's problem.

The following was suggested by Todor Tsankov. One can clearly ask a number of questions along these lines.

**Question 3.2.** *Consider the algebra  $B(G)$  constructed in the proof of Theorem 1. Are all of its irreducible representations on a separable Hilbert space equivalent?*

Consider  $\Gamma = G^{<\infty}$  (see the paragraph before Lemma 2.3) with respect to the symmetric difference  $\Delta$  as a discrete abelian group. The map  $b: \Gamma^2 \rightarrow \{-1, 1\}$  defined by

$$b(\mathbf{s}, \mathbf{t}) = (-1)^{|\{(x,y) \in \mathbf{s} \times \mathbf{t}: x \text{ is adjacent to } y\}|}$$

satisfies relations  $b(\mathbf{s}, \mathbf{t}) = 1/b(\mathbf{t}, \mathbf{s})$  and  $b(\mathbf{s}, \mathbf{t}_1)b(\mathbf{s}, \mathbf{t}_2) = b(\mathbf{s}, \mathbf{t}_1 \mathbf{t}_2)$ .

In [12], J. Slawny associates a universal  $C^*$ -algebra to a group  $\Gamma$  and a function  $b: \Gamma^2 \rightarrow \mathbb{T}$  as above. For a Boolean group  $\Gamma$ , the CCR algebra associated to the pair  $\Gamma, b$  is always isomorphic to a group of the form  $B(G)$ . (Note that Slawny considered only second countable groups, while in the present paper we consider uncountable discrete groups.) Consider  $\Gamma$  as a vector space over  $\mathbb{F}_2$  and fix its basis  $V$ . Proclaim two elements  $x, y$  of  $V$  to be adjacent if and only if  $b(x, y) = -1$ . Then Lemma 2.3 shows that  $B(G)$  is isomorphic to Slawny's algebra. Among other things, Slawny proved that this algebra is simple if and only if the cocycle  $c$  is nontrivial. This is related to the implication from (4) to (3) of Lemma 2.4.

All algebras of the form  $B(G)$  are clearly AF, and as proved in Lemma 2.4 every simple algebra of the form  $B(G)$  is a direct limit of full matrix algebras  $M_{2^n}(\mathbb{C})$  for  $n \in \mathbb{N}$ . While every separable algebra of this form is UHF, i.e., isomorphic to a tensor product of full matrix algebras, in [3] the author and T. Katsura proved that this may fail for nonseparable  $C^*$ -algebras.

For example, if  $\mathcal{Z}$  is the Jiang–Su algebra ([6]) then the algebra (below  $\aleph_1$  denotes the least uncountable cardinal)

$$A_{\aleph_0 \aleph_1} := M_{2^\infty} \otimes \bigotimes_{\aleph_1} \mathcal{Z}$$

is a direct limit of full matrix algebras but not UHF ([3, Proposition 3.2 and Theorem 1.3]). The algebra constructed in the proof of Theorem 1 gives another example of an AM C\*-algebra that is not UHF. This is because a nonseparable UHF algebra cannot have a representation on a separable Hilbert space ([3, Proposition 7.6]).

**Conjecture 3.3.** *The algebra  $A_{\aleph_0 \aleph_1}$  is not isomorphic to  $B(G)$  for any graph  $G$ .*

Clearly, not every AM algebra is isomorphic to an algebra of the form  $B(G)$ —take, for example,  $M_3(\mathbb{C})$ . (This is an immediate consequence of Elliott’s classification of AF algebras, see e.g., [11].) However, there is no K-theoretic obstruction to having a graph  $G$  such that  $A_{\aleph_0 \aleph_1}$  is isomorphic to  $B(G)$ , since the K-theory of all these algebras coincides with the K-theory of the CAR algebra. Thus a confirmation of Conjecture 3.3 would essentially confirm that the AM algebras that are also CCR algebras form a nontrivial intermediate class between AM algebras and UHF algebras.

Recall that for two graphs  $G$  and  $K$  we write  $G \sim K$  if C\*-algebras  $B(G)$  and  $B(K)$  are isomorphic (see Lemma 2.3). The proof of Lemma 2.2 gives an algorithm that associates a natural number  $k = k(G)$  to every finite graph  $G$  such that  $G \sim K$  if and only if  $|V(G)| = |V(K)|$  and  $k(G) = k(K)$ .

**Question 3.4.** *What is the computational complexity of the relation  $G \sim K$  for finite graphs  $G$  and  $K$ ?*

Shortly after seeing a preliminary version of this paper, A. Kishimoto sketched a proof of Theorem 1 using crossed products ([7]).

A word on precursors of the class of algebras considered here is in order. A variant of algebras of the form  $B(G(Y, \mathbb{A}))$  with uncountable  $Y$  was used in [3] to answer a question of Jacques Dixmier. After my presentation of [3] at the COSy in Toronto in May 2008, Bruce Blackadar suggested what is essentially a variant of  $B(\mathbb{N}, \mathbb{A})$  with an uncountable  $\mathbb{A}$ . This example was reproduced in [3, §7] to give a partial answer to a question of Masamichi Takesaki. In all algebras used in [3] the family  $\mathbb{A}$  includes all singletons of  $Y$ .

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