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ALL AUTOMORPHISMS OF ALL CALKIN ALGEBRAS

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Abstract. The Proper Forcing Axiom implies all automorphisms of every Calkin algebra associated with an infinite-dimensional complex Hilbert space and the ideal of compact operators are inner. As a means of the proof we introduce notions of metric ω_1-trees and coherent families of Polish spaces and develop their theory parallel to the classical theory of trees of height ω_1 and coherent families indexed by a σ-directed ordering.

Fix an infinite-dimensional complex Hilbert space H. Let $\mathcal{B}(H)$ be its algebra of bounded linear operators, $\mathcal{K}(H)$ its ideal of compact operators and $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ the Calkin algebra. Answering a question first asked by Brown–Douglas-Fillmore, in [10] and [5] it was proved that the existence of outer automorphisms of the Calkin algebra associated with a separable H is independent from ZFC. In the present paper we consider the existence of outer automorphisms of the Calkin algebra associated with an arbitrary complex, infinite-dimensional Hilbert space.

PFA stands for the Proper Forcing Axiom, MA for Martin's Axiom and TA stands for Todorcevic's Axiom (see e.g., [12] or [8] for PFA and TA and [7, Chapter II] for MA). It is well-known that both MA and TA are consequences of PFA.

Theorem 1. *MA and TA together imply all automorphisms of the Calkin algebra associated with Hilbert space with basis of cardinality $\aleph_1$ are inner.*

Theorem 2. *PFA implies all automorphisms of every Calkin algebra are inner.*

The only use of TA in the present paper is implicit via the following result from [5].

Theorem 3. *TA implies all automorphisms of the Calkin algebra on a separable, infinite-dimensional Hilbert space are inner.* □

All of these results are part of the program of finding set-theoretic rigidity results for algebraic quotient structures. This program can be traced back to Shelah's seminal construction of a model of ZFC in which all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial ([11]). At present we have a non-unified collection of results and it is unclear how far-reaching this phenomenon is (see [2, §3.2], [3], [4] and the last section of [5]).

The rudimentary idea of the proofs of Theorem 1 and Theorem 2 is taken from the analogous Velickovic's results on automorphisms of the Boolean algebra $\mathcal{P}(\kappa)/\operatorname{Fin}$ in [13, §4]. A sketch of Velickovic's argument is in order for the reader's benefit.

If Φ is an automorphism of $\mathcal{P}(\omega_1)/\operatorname{Fin}$ then there is a closed unbounded set $C \subseteq \omega_1$ such that for every $\alpha \in C$ the restriction of Φ to $\mathcal{P}(\alpha)/\operatorname{Fin}$ is an automorphism of $\mathcal{P}(\alpha)/\operatorname{Fin}$. Since MA and TA imply that all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial ([13, Theorem 2.1]), for each $\alpha \in C$ we can fix a map $h_\alpha\colon \alpha \to \alpha$ such that the map $\mathcal{P}(\alpha) \ni A \mapsto h_\alpha[A] \in \mathcal{P}(\alpha)$ is a representation of the restriction of Φ to $\mathcal{P}(\alpha)/\operatorname{Fin}$.

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For $\alpha < \beta < \gamma$ with β and γ in C we have that $h_\beta \restriction \alpha$ and $h_\gamma \restriction \alpha$ agree modulo finite. Therefore

$$T = \{h_\beta \restriction \alpha : \alpha < \beta, \beta \in C\},$$

considered as a tree with respect to the extension ordering, has countable levels. Automorphism Φ is trivial if and only if T has a cofinal branch. For every $f\colon \omega_1 \to 2$ the tree

$$T[f] = \{f \circ t : t \in T\}$$

has a cofinal branch, determined by Y such that $[Y]_{\mathrm{Fin}} = \Phi([X]_{\mathrm{Fin}})$, where $f = \chi_X$. On the other hand, if $\dot{f}$ is added by forcing with finite conditions $\mathbb{P}$ (i.e., if $\dot{f}$ codes a set of $\aleph_1$ side-by-side Cohen reals over V) then $\mathbb{P}$ forces that $T[\dot{f}]$ has no cofinal branches. Applying MA to the poset for adding $\dot{f}$ followed by the ccc poset for specializing $T[\dot{f}]$ one obtains a contradiction.

Velickovic's proof of triviality of automorphisms of $\mathcal{P}(\kappa)/\operatorname{Fin}$ for $\kappa \geq \aleph_2$ uses a PFA-reflection argument, in which the above proof is preceded by a Levy collapse of κ to $\aleph_1$.

While the structure of our proof of Theorem 1 loosely resembles the above sketch, a number of nontrivial additions and modifications were required. For example, it is not clear whether for every automorphism Φ of $\mathcal{C}(\ell_2(\aleph_1))$ the set C of countable ordinals α such that the restriction of Φ to $\mathcal{C}(\ell_2(\alpha))$ is an automorphism of the latter algebra is closed and unbounded. This follows from MA+TA by Theorem 1, but I don't know whether this fact is true in ZFC. This problem is dealt with in §3.1. An another inconvenience was caused by the fact that the natural 'quantized' analogue of the poset for adding $\aleph_1$ Cohen reals is not ccc (Lemma 4.1), as well as the expected non-commutativity complications.

Also, the appropriate analogues of Velickovic's trees T and $T[f]$ are continuous rather than discrete. Therefore the proof of Theorem 1 required introduction and analysis of 'metric ω_1-trees,' analogous to the classical theory of ω_1-trees. This was done in §1. This section is independent of the rest of the paper and it is 'purely set-theoretic' in the sense that C*-algebras are not being mentioned in it.

The structure of the paper. Metric ω_1-trees and metric coherent families are introduced and treated using MA and PFA, respectively, in §1. The short §2 contains a few simple and well-known general facts about inner automorphisms of C*-algebras. In §3 we define analogues of trees T and $T[f]$ from Velickovic's proof, and in §4 we analyze $T[\tau]$ for an appropriately defined generic operator τ. Proof of Theorem 2 and brief concluding remarks can be found in §5 and §6, respectively.

Our notation and terminology are standard and excellent references for the background on C*-algebras and set theory are [1] and [7], respectively. Introductions to applications of combinatorial set theory to C*-algebras can be found in [14] and [6].

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1. Polish ω_1-trees

In this section we introduce a continuous version of Aronszajn trees. A note on terminology is in order. In operator algebras 'contraction' commonly refers to a map that is *distance-non-increasing*. In some other areas of mathematics such maps are referred to as *1-Lipshitz* and 'contraction' refers to a *distance-decreasing* map. The latter type of a map is referred to as a *strict contraction* by operator algebraists. In what follows I use the operator-algebraic terminology, hence a *contraction* f is assumed to satisfy $d(x,y) \geq d(f(x), f(y))$. Other than this concession, the theory of operator algebras does not make appearance in the present section.

A *metric ω_1-tree* is a family $T = (X_\alpha, d_\alpha, \pi_{\beta\alpha}$, for $\alpha \leq \beta < \omega_1)$, such that

(1) X_α is a complete metric space with compatible metric d_α,
(2) $\pi_{\beta\alpha}\colon X_\beta \to X_\alpha$ is a contractive surjection,
(3) projections $\pi_{\beta\alpha}$ are commuting and $\pi_{\alpha\alpha} = \mathrm{id}_{X_\alpha}$ for all α.

If all spaces X_α are separable we say T is a *Polish ω_1-tree.* If in addition the inverse limit $\varprojlim_\alpha X_\alpha$ is empty then we say that T is a *Polish Aronszajn tree.* Otherwise, the elements of the inverse limit $\varprojlim_\alpha X_\alpha$ are considered to be *branches* through T. In our terminology all branches and all ε-branches are assumed to be cofinal.

When each d_α is a discrete metric then the above definitions reduce to the usual definitions of ω_1-trees and Aronszajn trees (see e.g., [7]). Similarly, ε-branches, ε-antichains and ε-special trees as defined below are branches, antichains, and special trees, respectively, when $0 < \varepsilon < 1$.

Spaces X_α are assumed to be disjoint and we shall identify T with the union $\bigcup_\alpha X_\alpha$ of its levels when convenient and the projections are clear from the context. On T we have a map $\mathrm{Lev}\colon T \to \omega_1$ defined by $\mathrm{Lev}(x) = \alpha$ if and only if $x \in X_\alpha$.

It will be convenient to write π_α for the map $\bigcup_{\beta\geq\alpha} \pi_{\beta,\alpha}$ from T into T_α. Define a map ρ on T^2 as follows. For x, y in T let $\alpha = \min(\mathrm{Lev}(x), \mathrm{Lev}(y))$ and let

$$\rho(x,y) = d_\alpha(\pi_\alpha(x), \pi_\alpha(y)).$$

Note that ρ is not a metric or even a quasi-metric. The triangle inequality is violated by any triple such that $x \neq z$ but $y = \pi_\alpha(x) = \pi_\alpha(z)$.

For $\varepsilon > 0$ a subset A of T is an *ε-antichain* of T if $\rho(x,y) > \varepsilon$ for all distinct x and y in A. We say that T is *ε-special* if there are ε-antichains A_n, for $n \in \mathbb{N}$, such that $X_\alpha \cap \bigcup_n A_n$ is dense in X_α, for all $\alpha < \omega_1$.

For $\varepsilon > 0$ a subset A of T is an *ε-branch* if $A = \{x_\alpha : \alpha < \omega_1\}$, $\mathrm{Lev}(x_\alpha) = \alpha$ for all α, and $\rho(x_\alpha, x_\beta) \leq \varepsilon$ for all α, β. A *subtree* of T is a subset $S \subseteq T$ that is closed under projection maps and intersects every level X_α.

Lemma 1.1. *The following are equivalent for every metric ω_1-tree T and $\varepsilon > 0$.*

(1) *T has an ε-branch,*
(2) *There is $B \subseteq T$ that intersects cofinally many levels such that $\rho(x,y) \leq \varepsilon$ for all x, y in B,*
(3) *T has a subtree of diameter $\leq \varepsilon$.*

Proof. For $B \subseteq T$ let its *downwards closure* $S(B)$ be the subset of T such that its intersection with X_α is the metric closure of $\{\pi_\alpha(x) : x \in B, \alpha \leq \mathrm{Lev}(x)\}$. Since each π_α us ρ-nonincreasing, the 'ρ-diameter' of $S(B)$ is equal to the 'ρ-diameter' of B. This shows that (2) implies (3), and the other implications do not require a proof. □

Lemma 1.2. *Assume T is a metric ω_1-tree such that each of its subtrees has an ε-branch for every $\varepsilon > 0$. Then T has a branch.*

Proof. Choose B_n, for $n \in \mathbb{N}$, so that B_n is a $1/n$-branch and $B_{n+1} \subseteq S(B_n)$. Then for every α we have that $B_n \cap X_\alpha$, for $n \in \mathbb{N}$, is a decreasing sequence of subsets of X_α with diameters converging to 0. If x_α is the unique point in $\bigcap_n (B_n \cap X_\alpha)$ then the fact that the projections are commuting contractions easily implies that x_α, for $\alpha < \omega_1$, is a branch of T. □

There is a Polish Aronszajn tree with an ε-branch for all $\varepsilon > 0$ but no branches. To see this, fix any special Aronszajn tree T. Let X_α be the disjoint union of countably many copies of the α-th level of T and define d_α so that the the n-th copy has diameter $1/n$ and the distance between two distinct copies is 1. With the natural projection maps, the n-th copy of T includes a $1/n$-branch but T has no branches.

In the following lemma and elsewhere no attempt was made to find optimal numerical estimates.

Lemma 1.3. *If T is an ε-special metric ω_1-tree then it has no $\varepsilon/2$-branches.*

Proof. Let A_n, for $n \in \mathbb{N}$, be ε-antichains with dense union in each level. Assume x_α, for $\alpha < \omega_1$, is an ε-branch. Let n be such that $d_\alpha(x_\alpha, z_\alpha) < \varepsilon/4$ for some $z_\alpha \in A_n \cap X_\alpha$ for uncountably many α. Since projections are contractions, for such $\alpha < \beta$ we have $\rho(z_\alpha, z_\beta) < \varepsilon$, a contradiction. □

The proof of the following lemma is a straightforward modification of the well-known analogous fact for ω_1-trees.

Lemma 1.4 (MA)**.** *Assume T is a Polish ω_1-tree with no ε-branches. Then T is $\varepsilon/2$-special.*

Proof. For each α fix a countable dense subset Z_α of X_α. Let $\mathbb{P}_0$ be the poset of finite $\varepsilon/2$-antichains included in $\bigcup_\alpha Z_\alpha$ ordered with $\mathbf{p} \geq \mathbf{q}$ if $\mathbf{p} \subseteq \mathbf{q}$.

We shall prove $\mathbb{P}_0$ is ccc. Fix $\mathbf{p}_\alpha$, $\alpha < \omega_1$ in $\mathbb{P}_0$. Since each Z_α is countable, by a Δ-system argument we can find $\bar{\alpha}$, an uncountable $J \subseteq \omega_1$, and (writing $Z = \bigcup_{\beta \leq \bar{\alpha}} Z_\beta$) $\bar{\mathbf{p}} \subseteq Z$ and $\bar{\mathbf{q}} \subseteq Z$ so that the following hold for all $\alpha \in J$. First, $\mathbf{p}_\alpha = \bar{\mathbf{p}} \cup \mathbf{q}_\alpha$. Second, $\pi_{\bar{\alpha}}$ maps $\mathbf{q}_\alpha$ injectively onto $\bar{\mathbf{q}}$. Third, $\gamma(\alpha) = \min\{\mathrm{Lev}(x) : x \in \mathbf{q}_\alpha\}$ converges to ω_1.

It suffices to find $\alpha < \beta$ in J such that $\mathbf{q}_\alpha \cup \mathbf{q}_\beta$ is an $\varepsilon/2$-antichain. Let $n = |\bar{\mathbf{q}}|$ and fix an enumeration $\mathbf{q}_\alpha = \{z_\alpha(i) : i < n\}$ for all $\alpha \in J$. Let $\mathcal{U}$ be a uniform ultrafilter on J. Assuming α and β as above cannot be found, there are $i < j < n$ such that the set $J_1 = \{\alpha \in J : \{\beta : \rho(z_\alpha(i), z_\beta(j)) < \varepsilon/2\} \in \mathcal{U}\}$ belongs to $\mathcal{U}$. But then $\rho(z_\alpha(i), z_\gamma(i)) < \varepsilon$ for all $\alpha < \gamma$ in J_1, and therefore $\{z_\alpha(i) : \alpha \in J_1\}$ defines an ε-branch of T.

This proof that $\mathbb{P}_0$ is ccc shows that it is powefully ccc, i.e., the finitely supported product $\mathbb{P}_0^{<\omega}$ of countably many copies of $\mathbb{P}$ is ccc. Apply MA to the ccc poset $\mathbb{P} = \mathbb{P}_0^{<\omega}$ and $\aleph_1$ many dense sets assuring that $\mathbb{P}$ ads countably many ε-antichains A_n whose union is equal to $\bigcup_\alpha Z_\alpha$. □

1.1. Coherent families of Polish spaces. The material of this subsection plays a role only in the proof of Theorem 2 and the reader may safely skip it in the first reading.

A system $\mathbb{F} = (X_\lambda, d_\lambda, \pi_{\lambda'\lambda} : \lambda < \lambda' \text{ in } \Lambda)$ is a *coherent family of Polish spaces* if

(1) Λ is upwards σ-directed set and a lower semi-lattice,
(2) X_λ is a Polish space with compatible metric d_λ,
(3) $\pi_{\lambda'\lambda}\colon X_{\lambda'} \to X_\lambda$ is a contractive surjection,
(4) projections $\pi_{\lambda'\lambda}$ are commuting and $\pi_{\lambda\lambda} = \mathrm{id}_{X_\lambda}$ for all λ.

The family is *trivial* if $\varprojlim_\lambda X_\lambda \neq \emptyset$. Hence if $\Lambda = \omega_1$ with its natural ordering then $\mathbb{F}$ is a Polish ω_1-tree.

Spaces X_λ are assumed to be disjoint and we shall identify $\mathbb{F}$ with the union $\bigcup_\lambda X_\lambda$ of its levels when convenient and when the choice of projections is clear from the context. On $\mathbb{F}$ we have a map $\mathrm{Lev}\colon \mathbb{F} \to \Lambda$ defined by $\mathrm{Lev}(x) = \lambda$ if and only if $x \in X_\lambda$. It will be convenient to write π_λ for the map $\bigcup_{\lambda' \geq \lambda} \pi_{\lambda'\lambda}$.

Define a map ρ on $\mathbb{F}^2$ as follows. For x, y in $\mathbb{F}$ let $\lambda = \mathrm{Lev}(x) \wedge \mathrm{Lev}(y)$ and let

$$\rho(x, y) = d_\lambda(\pi_\lambda(x), \pi_\lambda(y)).$$

For $\varepsilon > 0$ a subset A of T is an *ε-antichain* of $\mathbb{T}$ if $\rho(x, y) > \varepsilon$ for all distinct x and y in A. A set $\{x_\lambda : \lambda \in \Lambda\}$ is an *ε-branch* of $\mathbb{F}$ if $x_\lambda \in X_\lambda$ for all λ and $\rho(x_\lambda, x_{\lambda'}) \leq \varepsilon$ for all λ and λ'.

If $Y_\lambda \subseteq X_\lambda$ is a nonempty Polish subspace for all λ and the family Y_λ, for $\lambda \in \Lambda$, is closed under the projection maps then (with d'_λ denoting the restriction of d_λ to Y_λ) we say that $\mathbb{F}' = (Y_\lambda, d'_\lambda, \pi_{\lambda'\lambda}, \text{ for } \lambda < \lambda' \text{ in } \Lambda)$ is a *cofinal subfamily of* $\mathbb{F}$.

Proof of the following is analogous to the proof of Lemma 1.2.

Lemma 1.5. *Assume $\mathbb{F}$ is a coherent family of Polish spaces such that each of its cofinal subfamilies has an ε-branch for every $\varepsilon > 0$. Then $\mathbb{F}$ is trivial.* □

Assume $\mathbb{F}$ is a coherent family of Polish spaces. If $f\colon \omega_1 \to \mathbb{F}$ is a strictly increasing map then we say the Polish ω_1-tree $(X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < \omega_1)$ is a *Polish subtree* of $\mathbb{F}$.

Lemma 1.6 (PFA). *Assume $\mathbb{F} = (X_\lambda, d_\lambda, \pi_{\lambda'\lambda} : \lambda < \lambda' \text{ in } \Lambda)$ is a coherent family of Polish spaces with no ε-branches. Then $\mathbb{F}$ has an $\varepsilon/6$-special Polish subtree.*

Proof. Let $\mathbb{P}$ denote the σ-closed collapse of $|\Lambda|$ to $\aleph_1$. Then $\mathbb{P}$ forces that there is a strictly increasing, cofinal map $f\colon \omega_1 \to \Lambda$. We first prove that $\mathbb{P}$ forces the Polish ω_1-tree $T_f = (X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < \omega_1)$ has no $\varepsilon/3$-branches.

Assume otherwise and let $\dot{B}$ be a name for an $\varepsilon/3$-branch of T_f. Let $\theta = (2^{|\Lambda|})^+$ and let M be a countable elementary submodel of H_θ containing $\mathbb{F}$, $\mathbb{P}$, and a name $\dot{f}$ for f. Let $\mathcal{D}_n$, for $n \in \mathbb{N}$, enumerate all dense open subsets of $\mathbb{P}$ that belong to M. Pick conditions $\mathbf{p}_s$, x_s and y_s for $s \in 2^{<\mathbb{N}}$, satisfying the following for all s.

(1) $\mathbf{p}_s \geq \mathbf{p}_t$ if t extends s,
(2) $\mathbf{p}_s \in M \cap \mathcal{D}_n$, where $n = |s|$,
(3) $\mathbf{p}_s \Vdash \check{x}_s \in \dot{B}$,
(4) $x_s \in M$, and
(5) $\rho(x_{s0}, x_{s1}) \geq \varepsilon$.

These objects are chosen by recursion. If $\mathbf{p}_s$ has been chosen, then the set $\{x \in \mathbb{F} : (\exists \mathbf{q} \leq \mathbf{p}_s)\mathbf{q} \Vdash x \in \dot{B}\}$ is not an ε-branch and therefore we can choose x_{s0} and x_{s1} in this set such that $\rho(x_{s0}, x_{s1}) \geq \varepsilon$. Let $\mathbf{p}_{s0}$ and $\mathbf{p}_{s1}$ be (necessarily incompatible) extensions of $\mathbf{p}_s$ forcing that x_{s0} and x_{s1}, respectively, belong to $\dot{B}$. Since all the relevant parameters are in M, $\mathbf{p}_{s0}$, $\mathbf{p}_{s1}$, x_{s0} and x_{s1} can also be chosen to belong to M.

Since Λ is σ-directed, let $\lambda(M) \in \Lambda$ be an upper bound for $M \cap \Lambda$. For each $g \in 2^{\mathbb{N}}$ let $\mathbf{p}_g$ be $(M, \mathbb{P})$-generic condition extending all $\mathbf{p}_{g\restriction n}$ and deciding $x_g \in X_{\lambda(M)}$ in $\dot{B}$. For $g \neq g'$ let s be the longest common initial segment of g and g'. We may assume g extends $s0$ and g' extends $s1$. Let $\alpha = \min(\mathrm{Lev}(x_{s0}), \mathrm{Lev}(x_{s1}))$ and let y_0, y_1, x_0, x_1 be the projections of $x_g, x_{g'}, x_{s0}$ and x_{s1}, respectively, to X_α. Then

$$d_\alpha(y_0, y_1) \geq d_\alpha(x_0, x_1) - d_\alpha(y_0, x_0) - d_\alpha(y_1, x_1) \geq \varepsilon/3,$$

and therefore $d_{\lambda(M)}(x_g, x_{g'}) \geq \varepsilon/3$. This contradicts the assumed separability of $X_{\lambda(M)}$.

Since $\mathbb{P}$ forces that $\mathbb{F}$ has no $\varepsilon/3$-branches, by Lemma 1.4 we have a $\mathbb{P}$-name for a ccc poset that $\varepsilon/6$-specializes T_f. By applying PFA to the iteration and an appropriate collection of dense sets we obtain the desired conclusion. □

Coherent families of discrete Polish spaces and their uniformization using PFA have been used in different contexts. See e.g., [12] and [8].

2. Inner automorphisms

In this short section we state and prove some well-known results about inner automorphisms of C*-algebras. Recall that for a partial isometry v in algebra A by $\mathrm{Ad}\, v$ we denote the conjugation map $\mathrm{Ad}\, v(a) = vav^*$.

Lemma 2.1. *Assume that unitaries v and w in a C*-algebra A are such that* $\mathrm{Ad}\, v$ *and* $\mathrm{Ad}\, w$ *agree on A. Then $vw^* \in \mathcal{Z}(A)$.*

Proof. We have $vav^* = waw^*$ and therefore $w^*va = aw^*v$ for all $a \in A$. □

In the following $\dot{a}$ denotes the image of $a \in \mathcal{B}(H)$ in the Calkin algebra under the quotient map, not a forcing name.

Lemma 2.2. *If v and w in $\mathcal{B}(H)$ are such that $\dot{v}$ and $\dot{w}$ are unitaries in $\mathcal{C}(H)$ and $(\mathrm{Ad}\, v)a - (\mathrm{Ad}\, w)a$ is compact for all $a \in \mathcal{B}(H)$, then there is $z \in \mathbb{T}$ such that $v - zw$ is compact.*

Proof. We first check (a well-known fact) that $\mathcal{Z}(\mathcal{C}(H)) = \mathbb{C}$. Since it is a C*-algebra, it suffices to see that the only self-adjoint elements of $\mathcal{Z}(\mathcal{C}(H))$ are scalar multiples of the identity. Assume $\dot{a}$ is self-adjoint and its essential spectrum is not a singleton, say it contains some $\lambda_1 < \lambda_2$. Fix $\varepsilon < |\lambda_1 - \lambda_2|/3$. In $\mathcal{B}(H)$ fix infinite-dimensional projections p and q such that $\|pap - \lambda_1 p\| < \varepsilon$ and $\|qaq - \lambda_2 q\| < \varepsilon$. A noncompact partial isometry v such that $vv^* \leq p$ and $v^*v \leq q$ clearly does not commute with a modulo the compacts.

By Lemma 2.1 applied to $\dot{v}$ and $\dot{w}$ and the above there is a scalar z such that $z\dot{v} = \dot{w}$, as required. □

Lemma 2.3. *Assume H is an infinite-dimensional Hilbert space and Φ and Ψ are automorphisms of $\mathcal{C}(H)$ that agree on the corner $\dot{p}\mathcal{C}(H)\dot{p}$ for every projection $p \in \mathcal{B}(H)$ with separable range. Then $\Phi = \Psi$.*

Proof. We may assume H is nonseparable. Assume the contrary and let $a \in \mathcal{B}(H)$ be such that $\dot{b} = \Phi(\dot{a}) - \Psi(\dot{a}) \neq 0$. Let r be a projection with separable range such that rbr is not compact and let p be such that $\Phi(\dot{p}) = \dot{r}$. By our assumption, $\Psi(\dot{p}) = \dot{r}$. Also $\dot{r}\Psi(\dot{a})\dot{r} = \Psi(\dot{p}\dot{a}\dot{p}) = \Phi(\dot{p}\dot{a}\dot{p}) = \dot{r}\Phi(\dot{a})\dot{r}$, contradicting the choice of a. □

3. Part I of the proof of Theorem 1: Trees T and $T[a]$

Let H denote $\ell_2(\aleph_1)$. Throughout this section we assume Φ is an automorphism of $\mathcal{C}(H)$ and $\Phi_*\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is its *representation*, i.e., any map such that the diagram

$$\begin{array}{ccc} \mathcal{B}(H) & \xrightarrow{\Phi_*} & \mathcal{B}(H) \\ \downarrow & & \downarrow \\ \mathcal{C}(H) & \xrightarrow{\Phi} & \mathcal{C}(H) \end{array}$$

commutes. Since every projection in $\mathcal{C}(H)$ lifts to a projection in $\mathcal{B}(H)$ ([14, Lemma 3.2]) we may assume Φ_* maps projections to projections.

Lemma 3.1. *If p is a projection in $\mathcal{B}(H)$ with separable range, then $\Phi_*(p)$ is a projection with separable range and $\Phi(\dot{p}\mathcal{C}(H)\dot{p}) = \Phi(\dot{p})\mathcal{C}(H)\Phi(\dot{p})$.*

Proof. Since a nonzero projection in $\mathcal{C}(H)$ generates the minimal nontrivial ideal of $\mathcal{C}(H)$ if and only if it is of the form $\dot{q}$ for some q with a separable range, the first claim follows. For the second part note that $A = \dot{p}\mathcal{C}(H)\dot{p}$ is a hereditary subalgebra (i.e., if $0 \leq a \leq b$ for $a \in \mathcal{C}(H)$ and $b \in A$, then $a \in A$) and therefore Φ maps it to a hereditary subalgebra. □

3.1. Localization. A straightfoward recursive construction produces an increasing family of projections with separable range p_α, $\alpha < \omega_1$ in $\mathcal{B}(H)$ such that

(1) $\bigvee_{\alpha<\omega_1} p_\alpha = 1$ and for a limit δ we have $p_\delta = \bigvee_{\alpha<\delta} p_\alpha$,
(2) p_0 and each $p_{\alpha+1} - p_\alpha$ are noncompact,
(3) for some projection r_α such that $\dot{r}_\alpha = \Phi(\dot{p}_\alpha)$ we have $p_\alpha \leq r_{\alpha+1}$ and $r_\alpha \leq p_{\alpha+1}$.

For convenience we write $p_{-1} = 0$. For each α fix a basis of the range $p_{\alpha+1} - p_\alpha$ and enumerate it as e_β, for $\alpha \cdot \omega \leq \beta < (\alpha+1)\cdot\omega$. We therefore have a basis $(e_\alpha)_{\alpha<\omega_1}$ for H such that

(4) p_α is the closed linear span of $\{e_\beta : \beta < \alpha\cdot\omega\}$.

For every $\alpha < \omega_1$ Lemma 3.1 implies that the restriction of Φ to $\dot{p}_\alpha\mathcal{C}(H)\dot{p}_\alpha$ is an isomorphism between Calkin algebras associated with separable Hilbert spaces, $p_\alpha[H]$ and $r_\alpha[H]$. Therefore by Theorem 3 we can fix a partial isometry v_α such that

(5) $v_\alpha v_\alpha^* \leq r_\alpha$, $v_\alpha^* v_\alpha \leq p_\alpha$, and $\operatorname{Ad} v_\alpha$ is a representation of Φ on $\dot{p}_\alpha\mathcal{C}(H)\dot{p}_\alpha$.

For each $\alpha > 1$ by Lemma 2.2 we can find $z_\alpha \in \mathbb{T}$ such that $v_0 - z_\alpha v_\alpha p_0$ is compact. Replace v_α with $z_\alpha v_\alpha$ and note that $\operatorname{Ad} v_\alpha$ still satisfies (5). Let us prove that in addition (with $a =^{\mathcal{K}} b$ standing for '$a - b$ is compact')

(6) $v_\alpha =^{\mathcal{K}} v_\beta p_\alpha$ whenever $\alpha < \beta$.

By Lemma 2.2, there is $z \in \mathbb{T}$ such that $v_\alpha - z v_\beta p_\alpha$ is compact. Since p_0 is non-compact and since $v_\alpha p_0 =^{\mathcal{K}} v_0 =^{\mathcal{K}} v_\beta p_0$, we must have $z = 1$.

For $a \in \mathcal{B}(H)$ define the *support* of a as

$$\operatorname{supp}(a) = \{\alpha < \omega_1 : \|a e_\alpha\| > 0 \text{ or } \|a^* e_\alpha\| > 0\}.$$

All compact operators are countably supported and the set of finitely supported operators is a dense subset of $\mathcal{K}(H)$. An easy analogue of the Δ-system lemma (e.g., [7, Theorem II.1.5]) is worth stating explicitly (here $H = \ell^2(\aleph_1)$ and p_α are as in (4)).

Lemma 3.2. *Assume a_α, $\alpha < \omega_1$, belong to $\mathcal{K}(H)$. Then for every $\varepsilon > 0$ there is a stationary $X \subseteq \omega_1$, a finitely supported projection r, and an operator a such that $rar = a$ and*

(a) $\|p_\alpha(r a_\alpha r - a_\alpha)p_\alpha\| < \varepsilon$ *for all* $\alpha \in X$,
(b) $\|p_\alpha(a - a_\alpha)p_\alpha\| < \varepsilon$ *for all* $\alpha \in X$, *and*
(c) $\|p_\alpha a_\alpha p_\alpha - p_\beta a_\beta p_\beta\| < 2\varepsilon$ *for all* $\alpha < \beta$ *in* X.

Proof. For a_α find a finitely supported b_α with complex rational coefficients with support in p_α such that $\|p_\alpha(a_\alpha - b_\alpha)p_\alpha\| < \varepsilon/2$. By the Pressing Down Lemma ([7, Theorem II.6.15]) we can find a stationary set X_0 such that all b_α with $\alpha \in X_0$ have the same support, S. Let r be the projection to $\operatorname{span}\{e_i : i \in S\}$. By a counting argument we can refine X_0 further and find a. The third inequality is an immediate consequence of the second. □

3.2. The tree T. For $\alpha < \omega_1$ let (with r_α and p_α as in (3) of §3.1)

$$X_\alpha = \{r_{\alpha+1} w p_\alpha : w \in \mathcal{B}(H), w =^{\mathcal{K}} v_\alpha\}.$$

Note the 'extra room' provided by defining X_α in this way instead of the apparently more natural $\{r_\alpha w p_\alpha : w \in \mathcal{B}(H), w =^{\mathcal{K}} v_\alpha\}$. Let us prove a few properties of X_α.

(7) X_α is a norm-separable complete metric space.
(8) If $\alpha < \beta$ then the map $\pi_{\beta\alpha} \colon X_\beta \to X_\alpha$ defined by

$$\pi_{\beta\alpha}(w) = r_{\alpha+1} w p_\alpha$$

is a surjection and a contraction.

Only the latter property requires a proof. It is clear that the range of $\pi_{\beta\alpha}$ is included in X_α and that the map is contraction. For $u \in X_\alpha$ let $w = v_\beta + u - r_{\alpha+1} v_\beta p_\alpha$. Then $w - v_\beta$ is compact since $u \in X_\alpha$ and clearly $r_{\alpha+1} w p_\alpha = r_{\alpha+1} u p_\alpha = u$.

Consider the Polish ω_1-tree T with levels X_α and connecting maps $\pi_{\alpha\beta}$.

Lemma 3.3. *The following are equivalent.*

(9) Φ *is inner.*
(10) *There is a $v \in \mathcal{B}(H)$ such that $\dot{v}$ is a unitary in $\mathcal{C}(H)$ and for all $\alpha < \omega_1$ we have $r_{\alpha+1} v p_\alpha \in X_\alpha$.*
(11) *T has a branch.*

Proof. Clearly (10) and (11) are equivalent, hence it suffices to prove (9) implies (10) and that (10) implies (11). Assume Φ is inner and v implements it. Then by Lemma 2.2 for every $\alpha < \omega_1$ there is $z_\alpha \in \mathbb{T}$ such that $z_\alpha v p_\alpha - v_\alpha$ is compact. Since

$v_\alpha p_0 - v_0$ is compact for each α and p_0 is noncompact, we have $z_\alpha = z_0$ for all α. Therefore $z_0 v$ defines a branch of T.

Now assume (11) and fix a v that defines a branch of T. Then the automorphism of $\mathcal{C}(H)$ with representation $\operatorname{Ad} v$ agrees with Φ on the ideal of all operators with separable range. By Lemma 2.3, this automorphism agrees with Φ on all of $\mathcal{C}(H)$, hence (9) follows. □

A minor modification of the proof that (10) implies (11) above gives an another equivalent reformulation of Φ being inner. Although we shall not need it, it deserves mention:

(12) Every subtree of T has a branch.

We proceed with the analysis of T and the corresponding 'local trees' $T[a]$.

For $b \in \mathcal{B}(H)$ and $\alpha < \omega_1$ let

$$Z[b]_\alpha = \{p_\alpha w b w^* p_\alpha : w \in X_{\alpha+1}\}.$$

Then for every $c \in Z[b]_\alpha$ we have $p_\alpha \Phi_*(b) p_\alpha =^{\mathcal{K}} c$ because

$$\begin{aligned} p_\alpha w b w^* p_\alpha &=^{\mathcal{K}} p_\alpha v_{\alpha+1} p_{\alpha+1} b p_{\alpha+1} v_{\alpha+1}^* p_\alpha \\ &=^{\mathcal{K}} p_\alpha \Phi_*(p_{\alpha+1} b p_{\alpha+1}) p_\alpha \\ &=^{\mathcal{K}} p_\alpha r_{\alpha+1} \Phi_*(b) r_{\alpha+1} p_\alpha \\ &=^{\mathcal{K}} p_\alpha \Phi_*(b) p_\alpha. \end{aligned}$$

Also, for $\alpha < \beta$ the map $\varpi^b_{\beta\alpha}$ (denoted $\varpi_{\beta\alpha}$ when b is clear from the context) from $Z[b]_\beta$ to $Z[b]_\alpha$ defined by

$$\varpi_{\beta\alpha}(c) = p_\alpha c p_\alpha$$

is clearly a contractive surjection.

For $a \in \mathcal{B}(H)$ let $T[a]$ denote the Polish ω_1-tree with levels $Z[a]_\alpha$ and commuting projections $\varpi_{\beta\alpha}$. By 'subtree' we always mean a downwards closed subtree of height ω_1.

Lemma 3.4. *For every $a \in \mathcal{B}(H)$ every subtree S of $T[a]$ has a branch.*

Proof. Let $b = \Phi_*(a)$. For every $\alpha < \omega_1$ fix $w_\alpha \in X_{\alpha+1}$ such that

$$b_\alpha = p_\alpha w_\alpha a w_\alpha^* p_\alpha$$

belongs to $S \cap Z[a]_\alpha$. Let $u_\alpha = p_\alpha w_\alpha$.

Fix $\varepsilon > 0$. Recall that the fixed basis e_α, for $\alpha < \omega_1$, of H spans all p_α (see (4)). Apply 'Δ-system' Lemma 3.2 to operators $p_\alpha(b - b_\alpha)p_\alpha$ to find uncountable $J \subseteq \omega_1$ and finitely supported c and c_α, $\alpha \in J$, with disjoint supports, so that

$$\|(b - b_\alpha) - (c + c_\alpha)\| < \varepsilon \text{ and } \|p_\alpha(b - b_\alpha)p_\alpha - c\| < \varepsilon.$$

By going to a further subset of J we may assume that for $\alpha < \beta$ in J the support of c_α is included in $\beta \cdot \omega$ (or more naturally stated, that $p_\beta c_\alpha p_\beta = c_\alpha$). For each $\alpha \in J$ let α^+ be the minimal element of J above α and let $b'_\alpha = p_\alpha(b_{\alpha^+})p_\alpha$. For α in J we have $\|b'_\alpha - (p_\alpha b p_\alpha - c)\| < \varepsilon$, and therefore $\|b'_\alpha - p_\alpha b'_\beta p_\alpha\| < 2\varepsilon$ for $\alpha < \beta$ in J. Hence b'_α, for $\alpha \in J$, defines a 2ε-branch in $T[a]$. Since S has a 2ε-branch for an arbitrarily small ε it has a branch by Lemma 1.2. □

4. Proof of Theorem 1, part II: A generic operator

In this section we apply Martin's Axiom. First, we add a generic operator τ to $\mathcal{B}(H)$ by a poset with finite conditions which forces that $T[\tau]$ has a branch. Second, we use the properties of τ to argue that T has a branch.

4.1. Adding $\aleph_1$ Cohen reals. For a Hilbert space K with a fixed basis e_j, $j \in J$, let $\mathbb{P}(K)$ be the forcing defined as follows. A condition in $\mathbb{P}(K)$ is a pair (F, M) where F is a finite subset of J and M is an $F \times F$ matrix with entries in the complex rationals, $\mathbb{Q} + i\mathbb{Q}$, such that the operator norm of M satisfies $\|M\| < 1$. We order $\mathbb{P}(K)$ by extension, setting $(F', M') \leq (F, M)$ if $F' \supseteq F$ and $M' \restriction F \times F \equiv M$.

Lemma 4.1. *Poset $\mathbb{P}(K)$ is ccc if and only if K is separable.*

Proof. if K is separable then $\mathbb{P}(K)$ is countable, so we only need to show the other direction. This direction will not be used in our proof, but we nevertheless include it since it shows why Lemma 4.2 below does not use $\mathbb{P}(H)$.

We may assume $0 \in J$. For each $j \in J \setminus \{0\}$ define a condition $\mathbf{a}_j = (F^j, M^j)$ by $F^j = \{0, j\}$ and the $(0, j)$ entry of M^j is equal to $1/\sqrt{2}$, while the other three entries are 0. Then the norm of any matrix including M_j and M_k is at least 1, hence a_j, for $j \in J$, is an uncountable antichain. □

4.2. Adding a generic operator τ. By (2) in §3.1 the projection

$$s_\alpha = p_{\alpha+1} - p_\alpha$$

has an infinite-dimensional and separable range. Let

$$\mathcal{D} = \{a \in \mathcal{B}(H) : a = \textstyle\sum_{\alpha<\omega_1} s_\alpha a s_\alpha\}$$

where the sum is taken in the strong operator topology. This subalgebra of $\mathcal{B}(H)$ is an analogue of algebras $\mathcal{D}[\vec{E}]$ that played a prominent part in the proof of Theorem 3 in [5]. Although much of the theory of $\mathcal{D}[\vec{E}]$ has analogues in the nonseparable case, we shall not develop this theory since the role of $\mathcal{D}$ in the proof of Theorem 1 is different.

For each $\alpha < \omega_1$ let $H_\alpha = s_\alpha H$, with the basis $\{e_\xi : \alpha \cdot \omega \leq \xi < (\alpha + 1) \cdot \omega\}$ and let $\mathbb{P}_\alpha$ be $\mathbb{P}(H_\alpha)$. The finitely supported product $\mathbb{P}$ of $\mathbb{P}_\alpha$, for $\alpha < \omega_1$ is ccc. Actually, being a finitely supported product of countable posets, it is forcing-equivalent to the poset for adding $\aleph_1$ Cohen reals.

If $\dot{G} \subseteq \mathbb{P}$ is a generic filter, then it defines a sesquilinear form whose norm is, by genericity, equal to 1. This in turn defines an operator on H in the unit ball of $\mathcal{B}(H)$ ([9, Lemma 3.2.2]) This operator belongs to the von Neumann algebra $\mathcal{D}$ (see §4.2) and we let τ denote its $\mathbb{P}$-name.

Lemma 4.2. *Poset $\mathbb{P}$ forces that every subtree of $T[\tau]$ has a branch.*

Proof. If not, then by Lemma 3.4 we fix a condition $p \in \mathbb{P}$ deciding $\varepsilon > 0$ such that some subtree $T'[\tau]$ of $T[\tau]$ has no ε-branch and consider $\mathbb{P} * \dot{\mathbb{S}}$ (below p) where $\dot{\mathbb{S}}$ is a ccc poset for $\varepsilon/2$-specializing $T'[\tau]$. By applying MA we can find $a \in \mathcal{B}(H)$ and an $\varepsilon/2$-special subtree of $T[a]$. By Lemma 1.3 this subtree has no branches, and this contradicts Lemma 3.4. □

Fix $\varepsilon > 0$. By Lemma 4.2, if S is a subtree of T then for $\alpha < \omega_1$ we can fix w_α and a condition $\mathbf{a}_\alpha$ in $\mathbb{P}$ that forces $\mathrm{Ad}(p_\alpha w_\alpha)\tau$ belongs to a cofinal ε-branch of $T[\tau]$. Here $w_\alpha \in S \cap X_{\alpha+1}$ and w_α is in the ground model. Identify $\mathbf{a}_\alpha$ with a finitely supported operator in $\mathcal{B}(H)$ and note that it belongs to the algebra $\mathcal{D}$ as defined in §4.2. Apply Lemma 3.2 to $\{\mathrm{Ad}(p_\alpha w_\alpha)\mathbf{a}_\alpha\}$ to find a finitely supported $\mathbf{b}$ such that

(13) $\|\mathbf{b} - \mathrm{Ad}(p_\alpha w_\alpha)\mathbf{a}_\alpha\| < \varepsilon$

for all α in a stationary set J_0. Since the coefficients of $\mathbf{a}_\alpha$ are complex rationals, by the Δ-system lemma and a counting argument there are a stationary set $J_1 \subseteq J_0$, a finitely-supported projection q, and $\mathbf{a}$ such that

(14) $q\mathbf{a}q = \mathbf{a}$ and $p_\alpha \mathbf{a}_\alpha p_\alpha = \mathbf{a}$

for all $\alpha \in J_1$. Note that $\mathbf{a}_\alpha = \mathbf{a} + (I - p_\alpha)\mathbf{a}_\alpha(I - p_\alpha)$ for all $\alpha \in J_1$. Find $\bar\alpha$ such that $p_{\bar\alpha} q = q$. Applying Lemma 3.2 to $(w_\beta - v_{\bar\alpha})p_{\bar\alpha}$ find a stationary $J \subseteq J_1$ such that

(15) $\|(w_\beta - w_\gamma)p_{\bar\alpha}\| < \varepsilon$

for all $\beta < \gamma$ in J. Let q_α denote the support of $\mathbf{a}_\alpha$. For $\beta \in J$ let $u_\beta = w_\beta p_\beta$. Then for $\alpha + 1 \le \beta$ we have $p_\alpha u_\beta =^{\mathcal{K}} p_\alpha w_\beta$.

Lemma 4.3. *The set $\{r_{\alpha+2} u_\beta p_{\alpha+1} : \alpha + \omega < \beta, \beta \in J\}$ is a 5ε-branch of T.*

Preparations for the proof of Lemma 4.3 take up the remainder of this section, with the main points being Claim 4.6 and Lemma 4.7.

Claim 4.4. *If $a \in \mathcal{D}$, $\alpha < \beta$ are in J, $q_\alpha a q_\alpha = \mathbf{a}_\alpha$, and $q_\beta a q_\beta = \mathbf{a}_\beta$, then*

$$\|\mathrm{Ad}(p_\alpha w_\alpha)a - \mathrm{Ad}(p_\alpha w_\beta)a\| \le \varepsilon.$$

Proof. Otherwise, there is $\delta > 0$ and a finitely supported projection $s \ge q_\alpha \vee q_\beta$ such that for every $c \in \mathcal{D}$ satisfying $scs = sas$ we have $\|\mathrm{Ad}(p_\alpha w_\alpha)c - \mathrm{Ad}(p_\alpha w_\beta)c\| > \varepsilon + \delta$. Making a small change to coefficients of sas one obtains a condition in $\mathbb{P}$ forcing that $\|\mathrm{Ad}(p_\alpha w_\alpha)\tau - \mathrm{Ad}(p_\alpha w_\beta)\tau\| > \varepsilon$, a contradiction. □

Claim 4.5. *Assume a and b are in $\mathcal{D}$, $qaq = qbq = 0$, $p_\alpha a p_{\alpha+\omega} = p_\alpha b p_{\alpha+\omega}$, and $\alpha + \omega < \beta$ for $\beta \in J$. Then*

$$\|\mathrm{Ad}(p_\alpha w_\beta)(a + \mathbf{a}_\beta) - \mathrm{Ad}(p_\alpha w_\beta)(b + \mathbf{a}_\beta)\| \le 2\varepsilon.$$

Proof. Assume otherwise and let

$$\delta = \|\mathrm{Ad}(p_\alpha w_\beta)(a + \mathbf{a}_\beta) - \mathrm{Ad}(p_\alpha w_\beta)(b + \mathbf{a}_\beta)\| - 2\varepsilon.$$

For $n < \omega$ write $s_n = p_{\alpha+\omega} - p_{\alpha+n}$. By continuity fix $n < \omega$ such that for all $c \in s_n\mathcal{D}$ ($= s_n\mathcal{D}s_n$ since s_n in the commutant of $\mathcal{D}$) with $\|c\| \le 1$ we have

$$\|\mathrm{Ad}(p_\alpha w_\beta)(a + \mathbf{a}_\beta) - \mathrm{Ad}(p_\alpha w_\beta)((1 - s_n)(a + \mathbf{a}_\beta) + c)\| < \delta/2$$

and

$$\|\mathrm{Ad}(p_\alpha w_\beta)(b + \mathbf{a}_\beta) - \mathrm{Ad}(p_\alpha w_\beta)((1 - s_n)(b + \mathbf{a}_\beta) + c)\| < \delta/2.$$

Let $c = \mathbf{a}_{\alpha+n} - \mathbf{a}$. Then Claim 4.4 applied to $(1 - s_n)(a + \mathbf{a}_\beta) + c$ and to $(1 - s_n)(b + \mathbf{a}_\beta) + c$ implies

$$\|\mathrm{Ad}(p_\alpha w_\beta)((1 - s_n)(a + \mathbf{a}_\beta) + c) - \mathrm{Ad}(p_\alpha w_{\alpha+n})((1 - s_n)(a + \mathbf{a}_\beta) + c)\| \le \varepsilon$$
$$\|\mathrm{Ad}(p_\alpha w_\beta)((1 - s_n)(b + \mathbf{a}_\beta) + c) - \mathrm{Ad}(p_\alpha w_{\alpha+n})((1 - s_n)(b + \mathbf{a}_\beta) + c)\| \le \varepsilon$$

leading to $2\varepsilon + \delta < 2\varepsilon + \delta$. □

Claim 4.6. *For $\alpha+\omega<\beta<\gamma$ such that β and γ are in J we have*

$$\Delta = \|\operatorname{Ad}(p_\alpha u_\beta)a - \operatorname{Ad}(p_\alpha u_\gamma)a\| \leq 5\varepsilon$$

for all $a\in\mathcal{D}$ with $\|a\|\leq 1$ and $(1-p_\beta)a=0$.

Proof. Fix $a\in\mathcal{D}$ with $\|a\|\leq 1$. We have that $\mathbf{c}=\mathbf{a}_\beta+(1-p_\gamma)\mathbf{a}_\gamma$ is a condition in $\mathbb{P}$ with support $q'=q_\beta\vee q_\gamma$ extending both $\mathbf{a}_\beta$ and $\mathbf{a}_\gamma$. Let

$$a' = a - q'aq' + \mathbf{c}.$$

With $\bar\alpha$ as in (15) we have $p_{\bar\alpha}a = ap_{\bar\alpha}$ since $a\in\mathcal{D}$. Therefore

$$\begin{aligned}\operatorname{Ad}(p_\alpha u_\beta)a - \operatorname{Ad}(p_\alpha u_\beta)a' &= \operatorname{Ad}(p_\alpha u_\beta p_{\bar\alpha})(a-a') + \operatorname{Ad}(p_\alpha u_\beta(p_\beta - p_{\bar\alpha}))(a-a')\\ &= \operatorname{Ad}(p_\alpha u_\beta p_{\bar\alpha})(a-a').\end{aligned}$$

By this and an analogous computation for γ we have

$$\begin{aligned}\operatorname{Ad}(p_\alpha u_\beta)a - \operatorname{Ad}(p_\alpha u_\gamma)a =& \operatorname{Ad}(p_\alpha u_\beta p_{\bar\alpha})(a-a') - \operatorname{Ad}(p_\alpha u_\gamma p_{\bar\alpha})(a-a')\\ &+ \operatorname{Ad}(p_\alpha u_\beta)a' - \operatorname{Ad}(p_\alpha u_\gamma)a'\end{aligned}$$

Using (15) and $p_\beta\mathbf{a}_\beta = p_\gamma\mathbf{a}_\gamma = \mathbf{a}$ we conclude that each of the first two summands has norm $\leq\varepsilon$, hence Δ is within 2ε of $\|\operatorname{Ad}(p_\alpha u_\beta)a' - \operatorname{Ad}(p_\alpha u_\gamma)a'\|$. Since $a'\in\mathcal{D}$ we have $(1-p_\beta)a' = (1-p_\beta)\mathbf{a}_\beta$ and the following.

$$\operatorname{Ad}(p_\alpha u_\beta)a' = \operatorname{Ad}(p_\alpha w_\beta)a' - \operatorname{Ad}(w_\beta(1-p_\beta))\mathbf{a}_\beta.$$

By this and an analogous computation for γ we have

$$\begin{aligned}\operatorname{Ad}(p_\alpha u_\beta)a' - \operatorname{Ad}(p_\alpha u_\gamma)a' =& \operatorname{Ad}(p_\alpha w_\beta)a' - \operatorname{Ad}(p_\alpha w_\gamma)a'\\ &+ \operatorname{Ad}(w_\beta(1-p_\beta))\mathbf{a}_\beta - \operatorname{Ad}(w_\gamma(1-p_\gamma))\mathbf{a}_\gamma.\end{aligned}$$

By Claim 4.4 the first difference has norm $\leq\varepsilon$ and by (13) the second difference has norm $\leq 2\varepsilon$. The conclusion follows. □

4.3. Metrics on $X_{\alpha+1}$. We are now within one page worth of definitions and computations from completing the proof. In order to complement Claim 4.6 in the proof of Lemma 4.3, we digress a little bit. For $\alpha<\omega_1$ define the following metrics on $X_{\alpha+1}$ (only d_4 and d_2 will be needed in our proof).

$$\begin{aligned}d_{1,\alpha}(u,w) &= \|u-w\|\\ d_{2,\alpha}(u,w) &= \sup_{a\in\mathcal{D},\|a\|=1}\|\operatorname{Ad}ua - \operatorname{Ad}wa\|\\ d_{3,\alpha}(u,w) &= \sup_{a\in\mathcal{B}(H),\|a\|=1}\|\operatorname{Ad}ua - \operatorname{Ad}wa\|\\ d_{4,\alpha}(u,w) &= \|p_\alpha(u-w)\|\end{aligned}$$

We shall drop the subscript α whenever it is clear from the context.

Lemma 4.7. *For all α, on $X_{\alpha+1}$ we have $d_4\leq d_2\leq d_3\leq 2d_1$.*

Proof. The inequality $d_2\leq d_3$ is trivial, and $d_3\leq 2d_1$ follows from the following computation.

$$\begin{aligned}\|\operatorname{Ad}ua - \operatorname{Ad}wa\| &\leq \|uau^* - uaw^*\| + \|uaw^* - waw^*\|\\ &\leq \|ua\|\cdot\|u^*-w^*\| + \|u-w\|\cdot\|ua\|\end{aligned}$$

It remains to prove $d_4 \leq d_2$.

Let $v, w \in X_{\alpha+1}$ be given, and put $d = \|p_\alpha(v-w)\|$. Fix $\delta > 0$ and a unit vector ξ such that $\|(v^* - w^*)p_\alpha \xi\| > d - \delta$. Clearly we may assume $p_\alpha \xi = \xi$. Let ζ be a unit vector colinear with $v^*\xi - w^*\xi$ and let ι be a unit vector orthogonal to ζ such that $v^*\xi$ and $w^*\xi$ belong to the linear span of ζ and ι. Fix scalars x, y, x', y' such that

$$v^*\xi = x\zeta + y\iota$$
$$w^*\xi = x'\zeta + y'\iota$$

Since $v^*\xi - w^*\xi$ is colinear with ζ, we have $y = y'$. Therefore $\|v^*\xi - w^*\xi\| = |x - x'|$.

Find representations $\zeta = \sum_{\gamma<\alpha} x_\gamma \zeta_\gamma$ and $\iota = \sum_{\gamma<\alpha} y_\gamma \iota_\gamma$ so that ζ_γ and ι_γ belong to the range of $s_\gamma = p_{\gamma+1} - p_\gamma$ for all γ. Since the range of s_γ is infinite-dimensional and since $v - w$ is compact, we can find a unit vector ν_γ in this range orthogonal to both ζ_γ and ι_γ and such that $\|v\nu_\gamma\| = 1$ but $\|v\nu_\gamma - w\nu_\gamma\| < \delta/d$. Let

$$\nu = \sum_{\gamma<\alpha} x_\gamma \nu_\gamma$$

Then ζ, ι, and ν are mutually orthogonal unit vectors and the rank two operator $a \in \mathcal{B}(H)$ defined by $a(\nu) = \zeta$ and $a(\zeta) = \nu$ has norm equal to one. Moreover, $a \in \mathcal{D}$, since for each γ the operator $as_\gamma = s_\gamma a$ is just the rank-two operator which transposes the orthogonal unit vectors ν_γ and ζ_γ. Note that $((\operatorname{Ad} v)a)\xi = vav^*\xi = va(x\zeta + y\iota) = xw\nu$ and $((\operatorname{Ad} w)a)\xi = waw^*\xi = wa(x'\zeta + y\iota) = x'w\nu$. Hence,

$$\|((\operatorname{Ad} v)a - (\operatorname{Ad} w)a)\xi\| = \|(x - x')w\nu\| = |x - x'| > d - \delta.$$

Since $\delta > 0$ was arbitrary, we conclude that $d_2(v, w) \geq d$. □

Proof of Lemma 4.3. In order to show $\{r_{\alpha+2}u_\beta p_{\alpha+1} : \alpha + \omega < \beta, \beta \in J\}$ is a 5ε-branch, it suffices to show that $\|p_{\alpha+3}(u_\beta - u_\gamma)p_{\alpha+2}\| \leq 5\varepsilon$ whenever $\alpha + \omega < \beta < \gamma$ for β, γ in J. But the inequality $d_{4,\alpha+1} \leq d_{2,\alpha+1}$ from Lemma 4.7 implies

$$\|p_{\alpha+3}(u_\beta - u_\gamma)p_{\alpha+2}\| \leq \sup_{a\in\mathcal{D}} \|\operatorname{Ad}(p_{\alpha+3}u_\beta p_{\alpha+2})a - \operatorname{Ad}(p_{\alpha+3}u_\gamma p_{\alpha+2})a\|$$

and the right hand side is $\leq 5\varepsilon$ by Claim 4.6 □

Since ε was arbitrary, Lemma 4.3 and Lemma 1.2 imply that T has a cofinal branch. By Lemma 3.3, Φ is inner.

5. The proof of Theorem 2

The proof of Theorem 2 is reasonably similar to the proof of the analogous result from [13, §4]. All we need is the analysis of coherent families of Polish spaces from §1.1 and a fragment of PFA. Fix $\kappa \geq \aleph_2$, write $H = \ell^2(\kappa)$ and let Φ be an automorphism of the Calkin algebra $\mathcal{C}(H)$. Fix a basis $\{e_\alpha : \alpha < \kappa\}$ of H and denote the projection to $\overline{\operatorname{span}\{e_\alpha : \alpha \in \lambda\}}$ by p_λ.

Recall that $\mathcal{P}_{\omega_1}(\kappa)$ denotes the family of all countable subsets of κ. This set is σ-directed under the inclusion and it is a lower semilattice. For every countable subset $\lambda \subseteq \kappa$ fix projection r_λ with separable range such that $\Phi(\dot{p}_\lambda) = \dot{r}_\lambda$. For $\lambda \leq \lambda'$ in Λ we have $\dot{r}_\lambda \leq \dot{r}_{\lambda'}$ but not necessarily $r_\lambda \leq r_{\lambda'}$. By [5] we can fix a partial isometry v_λ such that $\operatorname{Ad} v_\lambda$ implements the restriction of Φ to $\dot{p}_\lambda \mathcal{C}(H) \dot{p}_\lambda$. For $\lambda \in \mathcal{P}_{\omega_1}(\kappa)$ let

$$X_\lambda = \{r_\lambda w p_\lambda : w \in \mathcal{B}(H), w =^{\mathcal{K}} v_\lambda\}.$$

Let us prove a few properties of X_λ.

(16) X_λ is a norm-separable complete metric space.
(17) If $\lambda \subseteq \lambda'$ then the map $\pi_{\lambda'\lambda}\colon X_{\lambda'} \to X_\lambda$ defined by

$$\pi_{\lambda'\lambda}(w) = r_\lambda w p_\lambda$$

is a contraction.

The proof is analogous to the proof of (8) in §3.2.

Consider the coherent family of Polish spaces

$$\mathbb{F} = (X_\lambda, \pi_{\lambda'\lambda}, \pi_{\lambda'\lambda}, \text{ for } \lambda \in \mathcal{P}_{\omega_1}(\kappa)).$$

The omitted proof of the following uses Lemma 2.3 and is analogous to the proof of Lemma 3.3.

Lemma 5.1. *The following are equivalent.*

(18) Φ *is inner.*
(19) *There is* $v \in \mathcal{B}(H)$ *such that* $\dot{v}$ *is a unitary in* $\mathcal{C}(H)$ *and for all* $\lambda \in \mathcal{P}_{\omega_1}(\kappa)$ *we have* $r_\lambda v p_\lambda \in X_\lambda$.
(20) *The coherent family of Polish spaces* $\mathbb{F}$ *is trivial.* □

If Φ is not inner, then by Lemma 5.1 and Lemma 1.5 there is an $\varepsilon > 0$ and a cofinal subfamiy $\mathbb{F}'$ of $\mathbb{F}$ with no ε-branches. By PFA and Lemma 1.6, there is a strictly increasing map $f\colon \omega_1 \to \mathbb{F}$ such that the Polish ω_1-tree $(X_{f(\alpha)}, d_{f(\alpha)}, \pi_{f(\beta)f(\alpha)}, \alpha \leq \beta < \omega_1)$ is $\varepsilon/6$-special. Then $Z = \bigcup f[\omega_1]$ is an $\aleph_1$-sized subset of κ. Let $\mathcal{C}(Z)$ denote the Calkin algebra associated with $\mathcal{B}(\ell^2(Z))$. By modifying the proof of Lemma 1.6 and meeting some additional dense sets, we can assure that the restriction Φ_Z of Φ to $\mathcal{C}(Z)$ is an automorphism of $\mathcal{C}(Z)$.

Theorem 1 implies Φ_Z is inner and Lemma 3.3 implies Φ_Z is outer. This contradiction concludes the proof of Theorem 2.

6. Concluding remarks

The existence of a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ clearly implies the existence of a nontrivial automorphism of $\mathcal{P}(\kappa)/\operatorname{Fin}$ for every infinite κ. Velickovic announced that it is possible to construct a nontrivial automorphism of $\mathcal{P}(\aleph_2)/\operatorname{Fin}$ by other means (see [13, p. 13]) but the proof of this result is unfortunately not available. The situation with automorphisms of Calkin algebras is even less clear. There are no obvious implications between the existence of outer automorphisms of the Calkin algebra associated with Hilbert spaces of different densities. I don't even know whether it is relatively consistent with ZFC that the Calkin algebra associated with some nonseparable Hilbert space has an outer automorphism?

While $\mathcal{B}(H)$ has the unique nontrivial two-sided closed ideal if H is separable, in the nonseparable case there are as many such ideals as there are infinite cardinals less or equal than the character density of H. Therefore there are several 'Calkin algebras' associated with a large Hilbert space H. The existence of outer automorphisms of these algebras will be investigated in a forthcoming joint paper with Ernest Schimmerling and Paul McKenney.

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