TRIVIAL AUTOMORPHISMS

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Abstract. We prove that the statement 'For all Borel ideals $\mathcal{I}$ and $\mathcal{J}$ on ω, every isomorphism between Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ has a continuous representation' is relatively consistent with ZFC. In a model of this statement we have that for a number of Borel ideals $\mathcal{I}$ on ω every isomorphism between $\mathcal{P}(\omega)/\mathcal{I}$ and any other quotient $\mathcal{P}(\omega)/\mathcal{J}$ over a Borel ideal is trivial.

We can also assure that in this model the dominating number, $\mathfrak{d}$, is equal to $\aleph_1$ and that $2^{\aleph_1}$ is arbitrarily large. In this model Calkin algebra has outer automorphisms while all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial.

1. INTRODUCTION

We start with a fairly general setting. Assume X/I and Y/J are quotient structures (such as groups, Boolean algebras, C*-algebras,...) with π_I and π_J denoting the respective quotient maps. Also assume Φ is an isomorphism between X/I and Y/J. A *representation* of Φ is a map $F\colon X \to Y$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\Phi_*"] \arrow[d, "\pi_I"'] & Y \arrow[d, "\pi_J"'] \\
X/I \arrow[r, "\Phi"'] & Y/J
\end{tikzcd}$$

commutes. Since representation is not required to have any algebraic properties its existence follows from the Axiom of Choice and is therefore inconsequential to the relation of X/I, X/J and Φ.

We shall say that Φ is *trivial* if it has a representation that is itself a homomorphism between X and Y. Requiring a representation to be an isomorphism itself would be too strong since in many situations of interest there exists an isomorphism which has a representation that is a homomorphism but does not have one which is an isomorphism.

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In a number of cases of interest X and Y are structures of cardinality of the continuum and quotients X/I and Y/J are countably saturated in the model-theoretic sense (see e.g., [4]). In this situation Continuum Hypothesis, CH, makes it possible to use a diagonalization to construct nontrivial automorphisms of X/I and, if the quotients are elementarily equivalent, an isomorphism between X/I and Y/J. For example, CH implies that Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$ has nontrivial automorphisms ([25]) and Calkin algebra has outer automorphisms ([23] or [12, §1]). This is by no means automatic and for example the quotient group S_∞/G (where G is the subgroup consisting of finitely supported permutations) has the group of outer automorphisms isomorphic to $\mathbb{Z}$ and all of its automorphisms are trivial ([1]). Also, some quotient Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ for Borel ideals $\mathcal{I}$ are not countably saturated and it is unclear whether nontrivial automorphisms exist (see [9]). One problem is that these quotients are not necessarily countably saturated. A construction of isomorphism between quotients over two different density ideals that are not countably saturated in classical sense in [17] should be revisited using the logic of metric structures developed in [3]. As observed in [17], these two quotients have the natural structure of complete metric spaces and when considered as models of the logic of metric structures two algebras are countably saturated. This fact can be extracted from the proof in [17] or from its generalization given in [9].

We shall consider the opposite situation, but only after noting that by Woodin's Σ^2_1 absoluteness theorem ([35], [20]) Continuum Hypothesis provides the optimal context for finding nontrivial isomorphisms whenever X and Y have Polish space structure with Borel-measurable operations and I and J are Borel ideals (see [7, §2.1]).

The line of research to which the present paper belongs was started by the second author's proof that the assertion 'all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial' is relatively consistent with ZFC ([30]). A weak form of this conclusion was extended to some other Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ in [15] and [14]. This line of research took a new turn when it was realized that forcing axioms imply all isomorphisms between quotients over Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$, for certain Borel ideals $\mathcal{I}$, are trivial ([27], [34], [16], [6], [7], [10]). The first author conjectured in [11] that the Proper Forcing Axiom, PFA, implies all isomorphisms between any two quotient algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, for a Borel ideal $\mathcal{I}$, are trivial. This conjecture naturally splits in following two rigidity conjectures.

(RC1) PFA implies every isomorphism has a continuous representation, and
(RC2) Every isomorphism with a continuous representation is trivial.

Noting that in our situation Shoenfield's Absoluteness Theorem implies that (RC2) cannot be changed by forcing and that no progress on it has been made in the last ten years, we shall concentrate on (RC1).

In the present paper we construct a forcing extension in which all isomorphisms between Borel quotients have continuous representations. This does not confirm (RC1) but it does give some positive evidence towards it.

The assumption of the existence of a measurable cardinal in the following result is used only to assure sufficient forcing-absoluteness[1] and it is very likely unnecessary.

Theorem 1. *Assume there exists a measurable cardinal. Then there is a forcing extension in which all of the following are true.*

1. *Every automorphism of a quotient Boolean algebra* $\mathcal{P}(\omega)/\mathcal{I}$ *over a Borel ideal* $\mathcal{I}$ *has a continuous representation.*
2. *Every isomorphism between quotient Boolean algebras* $\mathcal{P}(\omega)/\mathcal{I}$ *and* $\mathcal{P}(\omega)/\mathcal{J}$ *over Borel ideals has a continuous representation.*
3. *Every homomorphism between quotient Boolean algebras* $\mathcal{P}(\omega)/\mathcal{I}$ *over Borel ideals has a locally continuous representation.*
4. *All of the above and in addition the dominating number,* $\mathfrak{d}$*, is equal to* $\aleph_1$*.*
5. *All of the above, and in addition we can have either* $2^{\aleph_0} = 2^{\aleph_1}$ *or* $2^{\aleph_0} < 2^{\aleph_1}$*.*

Thus the full rigidity conjecture, ‘it is relatively consistent with ZFC that all isomorphisms between quotients over Borel ideals are trivial’ reduces to (RC2) above.

Corollary 2. *It is relatively consistent with ZFC that all automorphisms of* $\mathcal{P}(\omega)/\operatorname{Fin}$ *are trivial while the Calkin algebra has outer automorphisms. In addition, the corona of every separable, stable C*-algebra has outer automorphisms.*

Proof. By the above, the triviality of all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ together with $\mathfrak{d} = \aleph_1$ and Luzin's weak Continuum Hypothesis, is relatively consistent with ZFC. By [7, §1], the two latter assumptions imply the existence of an outer automorphism of the Calkin algebra. An analogous result for coronas of some other C*-algebras, including separable stable algebras, is proved in [5]. □

If α is an indecomposable countable ordinal, the *ordinal ideal* $\mathcal{I}_\alpha$ is the ideal on α consisting of all subsets of α of strictly smaller order type. If α is multiplicatively indecomposable, then the *Weiss ideal* $\mathcal{W}_\alpha$ is the ideal of all subsets of α that don't include a closed copy of α in the ordinal topology. See [7] for more on these ideals and the definition of nonpatholigical analytic p-ideals.

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[1] More precisely, we need to know that in all forcing extensions by a small proper forcing all $\mathbf{\Sigma}^1_2$ sets have the property of Baire, $\mathbf{\Pi}^1_2$-unformization and that all $\mathbf{\Pi}^1_2$ sets have the Property of Baire. By Martin–Solovay ([22]) it suffices to assume that $H(\mathfrak{c}^+)^\#$ exists

Corollary 3. *It is relatively consistent with ZFC that every isomorphism between $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ is trivial whenever $\mathcal{I}$ is Borel and $\mathcal{J}$ is in any of the following classes of ideals is trivial:*

1. *Nonpathological analytic p-ideals,*
2. *Ordinal ideals,*
3. *Weiss ideals,*

In particular, quotient over an ideal of this sort and any other Borel ideal can be isomorphic if and only if the ideals are isomorphic.

Proof. If an isomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ has a continuous representation and $\mathcal{J}$ is in one of the above classes, then Φ is trivial. This was proved in [7], [19] and [18]. $\square$

In the presence of sufficient large cardinals and forcing absoluteness, the forcing notion used in the proof Theorem 1 gives a stronger consistency result. Universally Baire sets of reals were defined in [13] and well-studied since. A reader not familiar with the theory of universally Baire sets may safely skip all references to them.

Theorem 4. *Assume there are class many Woodin cardinals. Then all conclusions of Theorem 1 hold simultaneously for arbitrary universally Baire ideals in place of Borel ideals.*

The proof of Theorem 4 will be sketched in §6.1.

1.1. Definitions.

We frequently simplify the notation and for $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ and $a \subseteq \omega$ write $\Phi \restriction a$ instead of the correct $\Phi \restriction \mathcal{P}(a)/(a \cap \mathcal{I})$.

We say that a homomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ is $\mathbf{\Delta}^1_2$ if the set

$$\{(a,b) : \Phi([a]_{\mathcal{I}}) = [b]_{\mathcal{J}}\}$$

includes a $\mathbf{\Delta}^1_2$ set $\mathcal{X}$ such that for every a there exists b for which $(a,b) \in \mathcal{X}$. We similarly define when Φ is Borel, $\mathbf{\Pi}^1_2$, or in any other pointclass.

For a homomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ consider the ideals

$$\mathrm{Triv}^0_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ is trivial}\},$$

$$\mathrm{Triv}^1_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ has a continuous representation}\},$$

and

$$\mathrm{Triv}^2_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ is } \mathbf{\Delta}^1_2\}.$$

We say that Φ is *locally trivial* if Triv^0_Φ is nonmeager, that it is *locally topologically trivial* if Triv^1_Φ is nonmeager and hat it is *locally* $\mathbf{\Delta}^1_2$ if Triv^2_Φ is nonmeager.

[7, Theorem 3.3.5] implies that a fairly weak consequence of PFA implies every homomorphism between quotients over Borel p-ideals is locally continuous (and a bit more). See [7] for additional definitions.

Following [32] we denote the theory obtained from ZFC by removing the power set axiom and adding '$\beth_\omega$ exists' by ZFC*.

2. Creatures

We now describe the forcing notion whose iteration will be used in the proof of Theorem 1. Instead of the creature forcing defined here one can use the more classical 'groupwise Silver forcing' (cf. §2.2). For background on creatures see [24]. Fix a partition $I = (I_n : n \in \omega)$ of ω into consecutive finite intervals. Also fix another fast partition $J = (J_n : n \in \omega)$ into consecutive finite intervals. For $s \subseteq \omega$ write

$$I_s = \bigcup_{j \in s} I_j \text{ and } I_{<n} = \bigcup_{j<n} I_j.$$

Let $\mathbf{x}$ denote the pair (I, J), called 'relevant parameter.' Define $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ as follows (in terms of [24], this will be a 'creating pair').

Let $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ if

$$\mathfrak{c} = (n_{\mathfrak{c}}, u_{\mathfrak{c}}, \eta_{\mathfrak{c}}, \mathcal{F}_{\mathfrak{c}}, m_{\mathfrak{c}}, k_{\mathfrak{c}})$$

(we omit the subscript $\mathfrak{c}$ whenever it is clear from the context) provided the following conditions hold

(1) $u \subseteq J_n$,
(2) $\eta : I_u \to \{0, 1\}$,
(3) $\mathcal{F} \subseteq \{0, 1\}^{I_{J_n}}$ and each $\mu \in \mathcal{F}$ extends η,
(4) $k \leq |J_n| - |u|$,
(5) if $v \subseteq J_n \setminus u$, $|v| \leq k$, and $\nu \colon I_v \to \{0, 1\}$ then some $\mu \in \mathcal{F}$ extends $\eta \cup \nu$,
(6) $m < 3^{-|I_{<n}|} \log_2 k$

For $\mathfrak{c}$ and $\mathfrak{d}$ in $\mathrm{CR}_{\mathbf{x}}$ let $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ if the following conditions hold

(7) $n_{\mathfrak{d}} = n_{\mathfrak{c}}$,
(8) $\eta_{\mathfrak{c}} \subseteq \eta_{\mathfrak{d}}$,
(9) $k_{\mathfrak{c}} \geq k_{\mathfrak{d}}$,
(10) $\mathcal{F}_{\mathfrak{c}} \supseteq \mathcal{F}_{\mathfrak{d}}$,
(11) $m_{\mathfrak{c}} \leq m_{\mathfrak{d}}$.

For $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ we define the following

(12) $\mathrm{nor}_0(\mathfrak{c}) = \lfloor 3^{-|I_{<n}|} \log_2 k \rfloor$
(13) $\mathrm{nor}(\mathfrak{c}) = \mathrm{nor}_0(\mathfrak{c}) - m$,
(14) $\mathrm{pos}(\mathfrak{c}) = \mathcal{F}$.

Therefore $\mathfrak{c}$ is a finite 'forcing notion' that 'adds' a function from I_{J_n} into $\{0, 1\}$. Its 'working part' (or, the already decided part of the 'generic' function) is $\eta_{\mathfrak{c}}$ and $\mathcal{F}_{\mathfrak{c}}$ is the set of 'possibilities' for the generic function (thus the redundant notation (14) included here for the purpose of compatibility with [24]). The 'norm' $\mathrm{nor}(\mathfrak{c})$ provides a lower bound on the amount of freedom allowed by $\mathfrak{c}$ in determining the generic function.

For a relevant parameter $\mathbf{x}$ we now define the creature forcing $\mathbb{Q} = \mathbb{Q}_{\mathbf{x}}$. Let $\mathbf{H}(n) = 2^k$, where $k = I_{J_n}$. This is the number of 'generics' for $\mathfrak{c} \in \Sigma_{\mathbf{x}}$ with $n_{\mathfrak{c}} = n$. Also let

$$\phi_{\mathbf{H}}(j) = |\prod_{i<j} \mathbf{H}(i)|.$$

Fix a function $f\colon \omega\times\omega\to\omega$ which satisfies the following conditions for all k and l in ω:

(15) $f(k,l)\leq f(k,l+1)$,
(16) $f(k,l)<f(k+1,l)$,
(17) $\phi_{\mathbf{H}}(l)(f(k,l)+\phi_{\mathbf{H}}(l)+2)<f(k+1,l)$.

We say such f is $\mathbf{H}$-*fast*

We now let $\mathbb{Q}_{\mathbf{x}}$ be $\mathbb{Q}_f(\mathrm{CR}_{\mathbf{x}},\Sigma_{\mathbf{x}})$, as in [24, Definition 1.1.10 (f)]. This means that a typical condition in $\mathbb{Q}$ is a triple $p=(f_p,i(p),\bar{\mathfrak{c}}(p))$ such that (we drop subscript p when convenient)

(18) $f\colon I_{<i(p)}\to\{0,1\}$ for some $i(p)\in\omega$,
(19) $\bar{\mathfrak{c}}(p)=\langle\mathfrak{c}(p,j):j\geq i(p)\rangle$
(20) Each $\mathfrak{c}(j)$ is in $\mathrm{CR}_{\mathbf{x}}$ and satisfies $n_{\mathfrak{c}(j)}=j$,
(21) $(\forall k)(\forall^\infty i)(\mathrm{nor}(p(j))>f(k,n(p,j))$.

We let $q\leq p$ (where q is a condition stronger than p) if the following conditions are satisfied

(22) $f_p\subseteq f_q$,
(23) $\mathfrak{c}(q,j)\in\Sigma_{\mathbf{x}}(\mathfrak{c}(p,j))$ for $j\geq i(q)$,
(24) $f_q\restriction I_j\in\mathrm{pos}(\mathfrak{c}(p,j))$ for $j\in[i(p),i(q))$.

The idea is that $\mathbb{Q}_{\mathbf{x}}$ adds a function $\dot{f}$ from ω into $\{0,1\}$. A condition $p=(f_p,i(p),\bar{\mathfrak{c}}(p))$ decides that $\dot{f}$ extends f_p as well as $f_{\mathfrak{c}(p,j)}$ for all $j\geq i(p)$. Also, $\mathrm{pos}(\mathfrak{c}(p,j))$ is the set of possibilities for the restriction of $\dot{f}$ to I_{J_j}. The 'norms on possibilities' condition (21) affects the 'rate' at which decisions are being made.

Experts may want to take note that with our creating pair $(\mathrm{CR}_{\mathbf{x}},\Sigma_{\mathbf{x}})$ there is no difference between $\mathbb{Q}_f$ and $\mathbb{Q}^*_f$ (cf. [24, Definition 1.2.6]) since the intervals J_n form a partition of ω. This should be noted since the results from [24] quoted below apply to $\mathbb{Q}^*_f$ and not $\mathbb{Q}_f$ in general.

2.1. **Properties of $\mathbb{Q}_{\mathbf{x}}$.**

We shall need several results from [24] where the class of forcings to which $\mathbb{Q}_{\mathbf{x}}$ belongs was introduced and studied.

Lemma 2.1. *The forcing notion $\mathbb{Q}_{\mathbf{x}}$ is nonempty and nonatomic.*

Given $h\colon\omega\to\omega$ (typically increasing), we say that the creating pair $(\mathrm{CR}_{\mathbf{x}},\Sigma_{\mathbf{x}})$ is *h-big* ([24, Definition 2.2.1]) if for each $\mathfrak{c}\in\mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c})>1$ and $\chi\colon\ \mathrm{pos}(\mathfrak{c})\to h(n(\mathfrak{c}))$ there is $\mathfrak{d}\in\Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d})\geq\mathrm{nor}(\mathfrak{c})-1$ and $\chi\restriction\mathrm{pos}(\mathfrak{d})$ is constant. We need only h-bigness in the case when $h(n)=2$ for all n.

Lemma 2.2. *If $h(n)=3^{|I_{<n}|}$ then the pair $(\mathrm{CR}_{\mathbf{x}},\Sigma_{\mathbf{x}})$ is h-big.*

Proof. Fix $\mathfrak{c}=(n,u,\eta,\mathcal{F},m,k)\in\mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c})=\lfloor 2^{-|I_{<n}|}\log_2 k\rfloor-m>0$ and a partition $\mathcal{F}=\bigcup_{j<r}\mathcal{F}$, with $r=3^{|I_{<n}|}$. We need to find $\mathfrak{d}\in\Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d})\geq\mathrm{nor}(\mathfrak{c})-1$ and $\mathcal{F}_{\mathfrak{d}}\subseteq\mathcal{F}_j$ for some j.

Since $\mathrm{nor}(\mathfrak{c})=\lfloor r\log_2 k\rfloor-m>0$, we have that $\log_2 k\geq r$ and therefore $k'=\lceil rk\rceil>0$.

We shall find $\mathfrak{d}$ of the form $(n, v, \zeta, \mathcal{F}_j, m, k')$ for appropriate v, ζ and $j < r$. Note that $\mathrm{nor}(\mathfrak{d}) = \lfloor r \log_2 k' \rfloor - m = \mathrm{nor}(\mathfrak{c}) - 1$. We shall try to find u_j and $\eta_j \colon u_j \to \{0, 1\}$ for $j < r$ as follows. If $\mathfrak{d}_0 = (n, u, \eta, \mathcal{F}_0, m, k') \in \Sigma_{\mathbf{x}}(\mathfrak{c})$, we let $\mathfrak{d} = \mathfrak{d}_0$ and stop. Otherwise, there are $v_0 \subseteq J_n \setminus u$ and $\zeta_0 \colon v_0 \to \{0, 1\}$ such that $\eta \cup \zeta_0$ has no extension in $\mathcal{F}_0$. Let $u_1 = u \cup v_0$ and $\eta_1 = \eta \cup \zeta_0$. If $\mathfrak{d}_1 = (n, u_1, \eta_1, \mathcal{F}_1, m, k') \in \Sigma_{\mathbf{x}}(\mathfrak{c})$, we let $\mathfrak{d} = \mathfrak{d}_1$ and stop. Otherwise, there are $v_1 \subseteq J_n \setminus u_1$ and $\zeta_1 \colon v_1 \to \{0, 1\}$ such that $\eta_1 \cup \zeta_1$ has no extension in $\mathcal{F}_1$. Let $u_2 = u_1 \cup v_1$ and $\eta_2 = \eta \cup \zeta_1$. Proceeding in this way, for $j < r$ we construct v_j, u_j, ζ_j and η_j such that η_j has no extension in $\mathcal{F}_j$ or we find $\mathfrak{d}_j$ witnessing r-bigness of $\mathfrak{c}$. If u_j and η_j are constructed for $j < r - 1$, then $v = \bigcup_{j<r} v_j$ has cardinality $rk' = k$ and $\nu = \bigcup_{j<r} \zeta_j$ has no extension in $\mathcal{F}$. But this contradicts the assumption (4) on $\mathfrak{c}$. Therefore one of $\mathfrak{d}_j$ is as required. □

A creating pair $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ has the *halving property* ([24, Definition 2.2.7]) if for each $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c}) > 0$ there is $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ (usually denoted $\mathrm{half}(\mathfrak{c})$) such that

1. $\mathrm{nor}(\mathfrak{d}) \geq \frac{1}{2} \mathrm{nor}(\mathfrak{c})$,
2. If in addition $\mathrm{nor}(\mathfrak{c}) \geq 2$ then for each $\mathfrak{d}_1 \in \Sigma_{\mathbf{x}}(\mathfrak{d})$ such that $\mathrm{nor}(\mathfrak{d}_1) > 0$ there is $\mathfrak{c}_1 \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{c}_1) \geq \frac{1}{2} \mathrm{nor}(\mathfrak{c})$ and $\mathrm{pos}(\mathfrak{c}_1) \subseteq \mathrm{pos}(\mathfrak{d}_1)$.

Lemma 2.3. *The pair* $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ *has the halving property.*

Proof. $\mathfrak{c} = (n, u, \eta, \mathcal{F}, m, k) \in \mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c}) = 2^{-|I_{<n}|} - m > 0$. Write $r = 3^{-|I_{<n}|}$. Since $m < rk$ by (6) we have that $m_{\mathfrak{d}} = \frac{1}{2}(rk + m)$ satisfies $m' < rk$ and therefore $\mathfrak{d} = (n, u, \eta, \mathcal{F}, m_{\mathfrak{d}}, k)$ is in $\Sigma_{\mathbf{x}}(\mathfrak{c})$.

Now let us assume $\mathrm{nor}(\mathfrak{c}) \geq 2$ since otherwise there is nothing left to do. Assume $\mathfrak{d}_1 = (n, u_1, \eta_1, \mathcal{F}_1, k_1, m_1) \in \Sigma_{\mathbf{x}}(\mathfrak{d})$ is such that $\mathrm{nor}(\mathfrak{e}) > 0$. Note that $\mathrm{nor}(\mathfrak{d}_1) = r \log_2(k_1) - m_1$, $m_1 \geq m_{\mathfrak{d}}$ and $k_1 \leq k_{\mathfrak{d}} = k$.

Let $\mathfrak{c}_1 = (n, u_1, \eta_1, \mathcal{F}_1, k_1, m)$. Then

$$\mathrm{nor}(\mathfrak{c}_1) = \lfloor r \log_2 k_1 \rfloor - m = \mathrm{nor}(\mathfrak{d}_1) - m + m_1 \geq \frac{1}{2} \mathrm{nor}(\mathfrak{c}_1)$$

as required. □

Recall that a forcing notion $\mathbb{P}$ is *ω^ω-bounding* if for every name $\dot{f}$ for an element of ω^ω and every $p \in \mathbb{P}$ there are $q \leq p$ and $g \in \omega^\omega$ such that $q \Vdash \dot{f}(n) \leq \check{g}(n)$ for all n.

Proposition 2.4. *Forcing notion* $\mathbb{Q}_{\mathbf{x}}$ *is proper,* ω^ω*-bounding, and both the ordering and the incomparability relation on* $\mathbb{Q}_{\mathbf{x}}$ *are Borel.*

Proof. In addition to bigness and halving properties of $\mathbb{Q}_{\mathbf{x}}$ proved in two lemmas above, we note that this forcing is finitary (i.e,, each $\mathrm{CR}_{\mathbf{x}}$ is finite) and simple (i.e., $\Sigma_{\mathbf{x}}(S)$ is not defined for $S \subseteq \mathrm{CR}_{\mathbf{x}}$ that contains more than one element). By [24, Corollary 2.2.12 and Corollary 3.1.2], or rather by [24, Theorem 2.2.11], it is proper and ω^ω-bounding.

It is clear that $\leq_{\mathbb{Q}_{\mathbf{x}}}$ is Borel. We check the remaining fact, that the relation $\perp_{\mathbb{Q}_{\mathbf{x}}}$ is Borel. Function $g\colon (\mathbb{Q}_{\mathbf{x}})^2 \to \omega^\omega$ defined by

$$g(p,q)(n) = \max\{\mathrm{nor}(\mathfrak{d}) : \mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c}(p,n)) \cap \Sigma_{\mathbf{x}}(\mathfrak{c}(q,n))\}$$

(with $\max\emptyset = 0$) is continuous. Since p and q are compatible if and only if $g(p,q)$ satisfies the largeness requirement (21) from §2, the incompatibility relation is Borel. $\square$

2.2. **Groupwise Silver forcing.** We sketch a forcing notion that can be used in our proof in place of $\mathbb{Q}_{\mathbf{x}}$ defined above. The 'relevant parameter' is $\bar{I} = (I_n : n \in \omega)$, a partition of ω into finite intervals. Forcing $\mathbb{S}_{\bar{I}}$ consists of partial functions f from a subset of ω into $\{0,1\}$ such that the domain of f is disjoint from infinitely many of the I_n. Every condition f can be identified with the compact subset p_f of $\mathcal{P}(\omega)$ consisting of all functions extending f. Special cases of $\mathbb{S}_{\bar{I}}$ are Silver forcing (the case when $I_n = \{n\}$ for all n) and 'infinitely equal,' or EE, forcing (the case when $|I_n| = n$ for all n, see [2, §7.4.C]).

Since this forcing will not be used in our proof, we only sketch the properties of $\mathbb{S}_{\bar{I}}$ leaving straightforward modifications of standard proofs to $\mathbb{S}_{\bar{I}}$ to the reader. This is a real forcing (Definition 3.1) and a fusion argument shows that it is proper, ω^ω-bounding and has continuous reading of names. Also, the proof that this forcing is ω^ω-bounding and proper are analogous to proofs of the corresponding facts for EE, [2, Lemma 7.4.14] and [2, Lemma 7.4.12], respectively).

3. Continuous reading of names in the iteration

A crucial property of the forcing iteration used in our proof is that it has the continuous reading of names. This is an immediate consequence of well-known results, and it can be proved in at least three different ways: as a very special case of the results in [32], by using a multi-ideal version of [38, Theorem 3.10.19 and Theorem 6.3.16] or by using results from [37]. One problem with each of these is that the proof published proof is only sketched, it needs to be slightly modified, or is difficult to find. We shall therefore outline a proof that combines results from [37] with an iteration theorem from [26].

One last remark before we present the proof. The preservation theorem quoted here requires forcing iterands to be 'definable' in some way so that they can be re-interpreted in a forcing extension. There is a number of different ways to formalize properties that we need: suslin proper, nep (non-elementary proper), creature forcings, real forcings, forcings of the form P_I for a σ-ideal of Borel sets on $\mathbb{R}$,.... We choose to essentially follow [37] since this approach seems to require the least amount of additional work on our side towards presenting a complete proof.

Definition 3.1. Following a special case of [37, Definition 2.1.5] we say that a forcing $\mathbb{P}$ is *real* if its conditions are nonempty compact subsets of $\mathcal{P}(\omega)$, $\mathbb{P}$

is a Borel subset of $K(\mathcal{P}(\omega))$ (the compact metric space of compact subsets of $\mathcal{P}(\omega)$), the ordering on $\mathbb{P}$ is given by reverse inclusion (so that a smaller set is a stronger condition), and every basic clopen $U \subseteq \mathcal{P}(\omega)$ belongs to $\mathbb{P}$.

The definition of real forcings in [37, Definition 2.1.5] is more general, in particular conditions of $\mathbb{P}$ are not necessarily compact sets. It is clear that for a real forcing $\mathbb{P}$ a generic filter $G \subseteq \mathbb{P}$ is determined by a real $\dot{g}$ which is the only element of $\bigcap G$.

A real forcing $\mathbb{P}$ has *continuous reading of names* if for every $\mathbb{P}$-name $\dot{x}$ for a subset of ω there is $p \in \mathbb{P}$ and a continuous function $f \colon p \to \mathcal{P}(\omega)$ such that p forces $\dot{x} = f(\dot{g})$, where g is the canonical name for the generic element of $\mathcal{P}(\omega)$.

In [37, Lemma 2.1.6] a characterization of such forcings was given, but we shall short-circuit it and only note that by [37, Lemma 2.2.3], a proper real forcing is ω^ω-bounding if and only if it has continuous reading of names.

Lemma 3.2. *The following forcing notions are real forcings and have continuous reading of names.*

1. *Poset for adding a random real*
2. *Poset $\mathbb{Q}_{\mathbf{x}}$ defined in §2.*
3. *Groupwise Silver forcing defined in §2.2.*

Proof. The fact that the forcings are naturally identified with forcings in which conditions are compact subsets of $\mathcal{P}(\omega)$ is immediate, and definitions of all three forcings are clearly Borel. Since all three forcings are ω^ω-bounding, continuous reading of names follows by Zapletal's lemma quoted above.

This can also be proved directly by a fusion argument in cases (2) and (3) and by Luzin's theorem in case (1). □

Definition 3.3. A forcing notion $\mathbb{P}$ is *locally real* if for every $\mathbb{P}$-name $\dot{x}$ for a real there is a real forcing $\mathbb{Q} \lessdot \mathbb{P}$ such that $\dot{x}$ is a $\mathbb{Q}$-name and $\mathbb{Q}$ is isomorphic to a Borel forcing.

The following is a minor modification of results from [37, §3], where unfortunately only the case when a single forcing is being iterated was considered. Although the proof remains unchanged, we shall give some details for the benefit of the reader. We have a cardinal κ (equal to $\aleph_2$ in our application) and a map $\mathbb{Q}$ from κ into real forcings. The following applies to a countable support forcing iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \mathfrak{c}^+, \eta < \mathfrak{c}^+)$ such that $\dot{\mathbb{Q}}_\eta$ is equal to $\mathbb{Q}(\eta)$ as interpreted in the intermediate extension.

Lemma 3.4. *Assume $\mathbb{P}$ is a countable support iteration of ground-model proper real posets (as above). Then $\mathbb{P}$ is proper and locally real. If moreover each iterand $\dot{\mathbb{Q}}_\eta$ is forced to have continuous reading of names then $\mathbb{P}$ has continuous reading of names.*

Proof. We first note that by Zapletal's lemma all iterands are ω^ω-bounding and by [26] the iteration is proper and ω^ω-bounding.

The iteration can be defined in terms of I-perfect sets of reals, as defined in [37, Definition 3.1.4]. In our context, some $B \subseteq \mathcal{P}(\omega)^\kappa$ is $\mathbb{Q}$-*perfect* if

(1) for every $\beta < \kappa$ and very $\bar{s} \in B \restriction \beta$ the set $\{x \in \mathcal{P}(\omega) : \bar{s}^\frown x \in B \restriction (\beta+1)\}$ belongs to $\mathbb{Q}(\beta)$, and
(2) for every increasing sequence (β_n) of ordinals $< \kappa$ with $\beta = \sup_n \beta_n$ and every sequence $\bar{s} \in \mathcal{P}(\omega)^\beta$ we have that $\bar{s} \restriction \beta_n \in B \restriction \beta_n$ for all n implies $\bar{s} \in B$.

The forcing notion whose conditions are $\mathbb{Q}$-perfect sets of reals with respect to reverse inclusion (corresponding to P_κ from [37, p. 49]) is forcing-equivalent to the iteration $\mathbb{P}$. This follows from definitions and absoluteness of Borel real forcings (note that we are circumventing mention of P_I and corresponding R_κ, and that a bit of the work in [37] is to show that P_κ and R_κ are forcing-equivalent).

Note that every condition $p \in \mathbb{P}_\kappa$ has a countable support $X \subseteq \kappa$, so that we can write $\mathcal{P}(\omega)^\kappa = \mathcal{P}(\omega)^X \times \mathcal{P}(\omega)^{\kappa\setminus X}$ and there is p_1 such that $p = p_1 \times \mathcal{P}(\omega)^{\kappa\setminus X}$. We can therefore identify a Borel function $f\colon p_1 \to \mathcal{P}(\omega)$ with its composition with the projection from $\mathcal{P}(\omega)^\kappa$ onto $\mathcal{P}(\omega)^X$.

Finally, by (the proof of) [37, Corollary 3.1.10] for every $\mathbb{P}_\kappa$-name $\dot{x}$ for a real and $p \in \mathbb{P}_\kappa$ there is $q \leq p$ and a Borel function $f\colon q \to \mathcal{P}(\omega)$ such that $q \Vdash \dot{x} = f(\dot{g} \restriction X)$. Since $\mathbb{P}$ is ω^ω-bounding, by Zapletal's lemma we can further shrink q to a condition r such that the restriction of f to r is continuous. □

4. Counting quantifiers

A $\mathbb{P}$-name $\dot{x}$ for a real is *localized* by $p \in \mathbb{P}$ if there are countably many countable antichains $\mathcal{A}_n$, for $n \in \omega$, each maximal below p, such that $\mathcal{A}_n$ decides $\check{n} \in \dot{x}$. Hence if $\mathbb{P}$ is proper then every $\mathbb{P}$-name for a real is localized by some condition in $\mathbb{P}$.

With additional absoluteness assumptions the main result of this paper can be extended to a class of ideals larger than Borel. We shall need the fact that, assuming the existence of class many Woodin cardinals, all projective sets of reals are universally Baire and, more generally, that every set projective in a universally Baire set is universally Baire. Proofs of these results use Woodin's stationary tower forcing and they can be found in [20].

Assumptions of the following lemma is well-known to hold for a wider class of nep forcings (see [32]).

Lemma 4.1. *Assume $\mathbb{P}$ is proper and locally real with continuous reading of names, $\dot{x}$ is a $\mathbb{P}$-name for a real localized by p, $A \subseteq \mathbb{R}$ is in pointclass Γ and $F\colon \mathbb{R}^2 \to \mathbb{R}$ is a Borel function. Then the set*

$$\{a : p \Vdash F(\check{a}, \dot{x}) \in A\}$$

is $\mathbf{\Delta}^1_2(\Gamma)$.

In particular, it is $\mathbf{\Delta}^1_2$ *if* A *is Borel, projective if* A *is projective, and universally Baire if* A *is universally Baire and there exist class many Woodin cardinals.*

Proof. We may assume $\mathbb{P}$ is Borel proper, $p \in \mathbb{P}$ (therefore p is a compact subset of $\mathcal{P}(\omega)$ and $H\colon p \to \mathcal{P}(\omega)$ is a continuous function such that $p \Vdash H(\dot g) = \dot x$, where $\dot g$ is the canonical name for a $\mathbb{P}$-generic (see §4).

For $a \subseteq \omega$ we now have that $p \Vdash F(\check a, \dot x) \in A$ if there exists a countable well-founded model M of ZFC* containing everything relevant such that for every M-generic $G \subseteq M \cap \mathbb{P}$ with $p \in G$ we have that $\dot y[G] = a$. This is a $\mathbf{\Sigma}^1_2$ statement with A as a parameter.

Alternatively, $p \Vdash F(\check a, \dot x) \in A$ if for every countable well-founded model M of ZFC* and every M-generic $G \subseteq M \cap \mathbb{P}$ with $p \in G$ we have that $\dot y[G] = a$. This is a $\mathbf{\Pi}^1_2$-statement with A as a parameter. □

We record an immediate consequence of Lemma 4.1 whose version for Silver forcing plays a key role in the last section of [29].

Lemma 4.2. *Assume forcing* $\mathbb{P} * \dot{\mathbb{Q}}$ *is locally real with continuous reading of names and* $\dot y$ *is a* $\mathbb{P}$*-name for a ground-model subset of* ω*. Then for every* $p \in \mathbb{P}$ *the set* $\{(q,a) \in (\mathbb{P} * \dot{\mathbb{Q}}) \times \mathcal{P}(\omega) : q \leq p, q \Vdash \dot y = \check a\}$ *is* $\mathbf{\Delta}^1_2$. □

Assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is a forcing iteration in some model M of a large enough fragment of ZFC and $G_\kappa \subseteq \mathbb{P}_\kappa$ is an M-generic filter. Then $G \restriction \xi$ denotes $G \cap \mathbb{P}_\xi$. If $\dot A$ is a $\mathbb{P}_\kappa$-name for a set of reals we can consider it as a collection of nice names for reals. Furthermore, if $\mathbb{P}_\kappa$ is proper then we can identify $\dot A$ with a collection of pairs $(p, \dot x)$ where $p \in \mathbb{P}_\kappa$ and $\dot x$ is a name that involves only countable antichains below p. The intention is that p forces $\dot x$ is in A. With this convention we let $\dot A \restriction \xi$ denote the subcollection of $\dot A$ consisting only of those pairs $(p, \dot x)$ such that $p \in \mathbb{P}_\xi$ and $\dot x$ is a $\mathbb{P}_\xi$ name,

The following proposition will be used repeatedly in proof of the main theorem. It ought to be well-known but it does not seem to appear explicitly in the literature.

Proposition 4.3. *Assume* $\kappa > \mathfrak{c}$ *is a regular cardinal and*

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$$

is an iteration of proper forcings of cardinality $< \kappa$*. Assume* $\dot A$ *is a* $\mathbb{P}_\kappa$ *name for a set of reals. Then the set of ordinals* $\xi < \kappa$ *such that*

$$(H(\aleph_1), \mathrm{int}_{G\restriction\xi}(\dot A \restriction \xi))^{V[G\restriction\xi]} \prec_2 (H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot A))^{V[G]}$$

includes a club relative to $\{\xi < \kappa : \mathrm{cf}(\xi) \geq \omega_1\}$.

Proof. For every $\mathbf{\Sigma}^1_2$ statement $\phi(Z)$ (here Z is a second-order variable) and name $\dot A$ for a set of reals let

$$\mathbf{S}_{\phi,\dot A} = \{\xi < \kappa : \mathrm{cf}(\xi) \geq \omega_1 \text{ and } (H(\aleph_1), \mathrm{int}_{G\restriction\xi}(\dot A \restriction \xi))^{V[G\restriction\xi]} \models \varphi(\dot A \restriction \xi)\}.$$

We shall show that the following are equivalent for every $\phi(Z)$ and every name $\dot A$ for a set of reals:

(i) $(H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot{A}))^{V[G]} \models \phi(A)$.
(ii) there is a club $\mathbf{C}$ such that $\mathbf{S}_{\phi,\dot{A}} \supseteq \{\xi \in \mathbf{C} : \mathrm{cf}(\xi) \geq \omega_1\}$,
(iii) $\mathbf{S}_{\phi,\dot{A}}$ is stationary,

Fix $\varphi(Z)$ and $\dot{A}$. By going to an intermediate extension we may assume that all parameters are in V. Let $\varphi_0(x, y, Z)$ be a $\mathbf{\Delta}_0$ formula such that $\varphi(Z)$ is equal to $(\exists x)(\forall y)\varphi_0(x, y, Z)$.

We first prove that (i) implies (ii). Assume that $(H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot{A}))^{V[G]} \models \varphi(\mathrm{int}_{\mathrm{G}}(\dot{A}))$, and let $\dot{a}$ be a name for a real such that $(H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot{A}))^{V[G]} \models (\forall y)\varphi_0(\mathrm{int}_{\mathrm{G}}(\dot{a}), y, \mathrm{int}_{\mathrm{G}}(\dot{A}))$. Since φ_0 is Δ_0, the formula on the right hand side is Π_1 and therefore downward absolute and satisfied in each $V[G \restriction \xi]$ large enough to absorb $\mathrm{int}_{\mathrm{G}}(\dot{a})$.

Since (ii) obviously implies (iii), it remains to prove (iii) implies (i). For each $\xi \in \mathbf{S}$ fix a $\mathbb{P}_\xi$-name $\dot{a}_\xi$ for a real such that

$$(H(\aleph_1), \mathrm{int}_{G\restriction\xi}(\dot{A} \restriction \xi))^{V[G\restriction\xi]} \models (\forall y)\varphi_0(\dot{a}_\xi, y, \dot{A} \restriction \xi)\}.$$

Since $\mathbb{P}_\kappa$ is proper and $\mathrm{cf}(\xi)$ is uncountable, there is a condition $p_\xi \in G \restriction \xi$ which extends $p \restriction \xi$ and decides an ordinal $\alpha_\xi < \xi$ such that $\dot{a}_\xi$ was added by $\mathbb{P}_{\alpha_\xi}$ and $p_\xi \in \mathbb{P}_{\alpha(\xi)}$. Again since $\mathrm{cf}(\xi)$ is uncountable, we may also assume $p_\xi \in G \restriction \alpha_\xi$, possibly by increasing α_ξ. By the pressing down lemma there are $\alpha < \kappa$, $q \in G \restriction \alpha$, and a stationary $\mathbf{S}_1 \subseteq \mathbf{S}$ such that $\alpha_\xi = \alpha$ and $p_\xi = q$ for all $\xi \in \mathbf{S}_1$. Since $|\mathbb{P}_\alpha| < \kappa$ we have that $|\mathbb{R}^{V[G\restriction\alpha]}| < \kappa$ and we can find a stationary $\mathbf{S}_2 \subseteq \mathbf{S}_1$ and $\mathbb{P}_\alpha$-name $\dot{a}$ for a real such that $(\alpha, q, \dot{a}) = (\alpha_\xi, p_\xi, \dot{a}_\xi)$ for all $\xi \in \mathbf{S}_2$. We therefore have that for cofinally many $\xi < \kappa$

$$q \Vdash_\xi (\forall y)\varphi_0(\dot{a}, y, \dot{A} \restriction \xi).$$

Since $\varphi_0(\dot{a}, y, \dot{A} \restriction \xi)$ is Δ_0, for every fixed y it is absolute between $V[G \restriction \xi]$ and $V[G]$. Since the structure $(H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot{A}))^{V[G]}$ is a direcf limit of structures such that q forces $(\forall y)\varphi_0(\dot{a}, y, \dot{A} \restriction \xi)$ holds in each one of them, we conclude that $q \Vdash_\kappa (\forall y)\varphi(\dot{a}, y, \dot{A})$ and therefore $q \Vdash \varphi(\dot{A})$. Since $q \in G$ this implies $(H(\aleph_1), \mathrm{int}_{\mathrm{G}}(\dot{A}))^{V[G]} \models \varphi_0(\mathrm{int}_{\mathrm{G}}(\dot{A}))$.

This concludes the proof that (i), (ii) and (iii) are equivalent. Since there are only countably many relevant statements $\phi(\dot{Z})$, the intersection of all $\mathbf{S}_{\phi,\dot{A}}$ (note that $\dot{A}$ is fixed) is a club as required. □

By a well-known result of Talagrand (for a proof see [2] or [7, Theorem 3.10.1]) for each meager ideal $\mathcal{I}$ that includes Fin there is a partition $\bar{I} = (I_n : n \in \omega)$ of ω into finite intervals such that for every infinite $c \subseteq \omega$ the set $\bar{I}_c = \bigsqcup_{n \in C} I_n$ is positive. In other words, the ideal $\mathcal{I}$ is meagre if and only if for some partition $\bar{I}$ of ω into finite intervals $\mathcal{I}$ is included in the hereditary F_σ set

$$\mathcal{H}(\bar{I}) = \{a \subseteq \omega : (\forall^\infty n) I_n \not\subseteq a\}.$$

We say that $\bar{I}$ *witnesses* $\mathcal{I}$ is meagre. If $\mathcal{I}$ is an ideal that has the property of Baire and includes Fin, then it is necessarily meager.

The following is a well-known consequence of the above.

Lemma 4.4. *Assume $\mathcal{I}$ is a Borel ideal and $\mathcal{K}$ is a nonmeager ideal. Then for every $c \in \mathcal{I}^+$ there is $d \in \mathcal{K}$ such that $c \cap d \in \mathcal{I}^+$.*

Proof. Since the ideal $\mathcal{I} \cap \mathcal{P}(c)$ is a proper Borel ideal on c, it is meager and we can find a partition of c into intervals $c = \bigsqcup_n I_n$ such that $\bigsqcup_{n \in y} I_n \notin \mathcal{I}$ for every infinite $y \subseteq \omega$. Let $\omega = \bigsqcup_n J_n$ be a partition such that $J_n \cap c = I_n$ for all n. Since $\mathcal{K}$ is nonmeager, there is an infinite y such that $d = \bigcup_{n \in y} J_n$ belongs to $\mathcal{K}$. Then $d \cap c = \bigsqcup_{n \in y} I_n$ is not in $\mathcal{I}$ and therefore d is as required. □

In the following corollary we assume $\kappa > \mathfrak{c}$ is a regular cardinal and

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$$

is an iteration of proper forcings of cardinality $< \kappa$, $G_\kappa \subseteq \mathbb{P}_\kappa$ is a V-generic filter and $G \restriction \xi$ is its intersection with $\mathbb{P}_\xi$.

Still assuming $\kappa > \mathfrak{c}$ is a regular cardinal and

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$$

is an iteration of proper forcings of cardinality $< \kappa$, we say that formula $\phi(x, Y)$ (with parameters $x \in \mathbb{R}$ and $Y \subseteq \mathbb{R}$) *reflects* if for every name $\dot{a}$ for a real and every name $\dot{B}$ for a set of reals the following are equivalent.

(1) $V[G] \models \phi(\dot{a}, \dot{B})$, and
(2) There is a club $\mathbf{C} \subseteq \kappa$ such that for all $\xi \in \mathbf{C}$ with $\mathrm{cf}(\xi) \geq \omega_1$ we have $V[G \restriction \xi] \models \phi(\dot{a}, \dot{B} \restriction \xi)$.

Corollary 4.5. *With $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ as above, assume $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ are $\mathbb{P}_\kappa$-names for Borel ideals on ω and $\dot{\Phi}$ is a $\mathbb{P}_\kappa$-name for an isomorphism between their quotients.*

(1) *for every name $\dot{a}$ for a real the statement $\dot{a} \in \mathrm{Triv}^1_{\dot{\Phi}}$ reflects.*
(2) *For $0 \leq j \leq 2$ the statement "$\mathrm{Triv}^j_{\dot{\Phi}}$ is meagre" reflects.*
(3) *For every $\mathbb{P}_\kappa$-name $\dot{I}$ for a partition of ω into finite sets the statement $\dot{\mathcal{I}} \subseteq \mathcal{H}(\dot{I})$ reflects.*

Proof. Since the pertinent statements are either Σ^1_2 or Π^1_2, each of the assertions is a consequence of Proposition 4.3. □

5. Trivializing automorphisms

Given a partition $I = (I_n : n \in \omega)$ of ω into finite intervals we say that a forcing notion $\mathbb{P}$ *captures* I if there is a $\mathbb{P}$-name $\dot{r}$ for a subset of ω such that for every $p \in \mathbb{P}$ there is an infinite $c \subseteq \omega$ with the following property:

(1) For every $d \subseteq \bigcup_{n \in c} I_n$ there is $q_d \leq p$ such that q_d forces

$$\dot{r} \cap \bigcup_{n \in c} \check{I}_n = \check{d}.$$

By $[a]_{\mathcal{I}}$ we denote the equivalence class of set a modulo the ideal $\mathcal{I}$. When the ideal is clear from the context we may write $[a]$ instead of $[a]_{\mathcal{I}}$.

The assumption of the following lemma follows from the assumption that there exists a measurable cardinal by [22].

Lemma 5.1. *Assume that all $\mathbf{\Sigma}^1_2$ sets of reals have the property of Baire. If a homomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ is $\mathbf{\Delta}^1_2$ then it has a continuous representation.*

Proof. By the Novikov–Kondo–Addison uniformization theorem, Φ has a $\mathbf{\Sigma}^1_2$ representation. Since this map is Baire-measurable, by a well-known fact (e.g., [7, Lemma 1.3.2]) Φ has a continuous representation. □

Let us say that a forcing $\mathbb{Q}$ *covers* some $a \subseteq \omega$ if there is a $\mathbb{Q}$-name $\dot{x}$ for a subset of ω such that for every $b \subseteq a$ there exists $p_b \in \mathbb{Q}$ such that

$$p_b \Vdash \dot{x} \cap \check{a} = \check{b}.$$

Proof of the following resembles a proof present in the last section of [29] (but cf. the first paragraph of §7).

Lemma 5.2. *Assume Φ is an isomorphism between $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ and both $\mathcal{I}$ and $\mathcal{J}$ are Borel. Moreover assume $\mathbb{P}$ is a locally real with continuous reading of names forcing notion and $a \subseteq \omega$ is such that*

1. *$\mathbb{P}$ forces that Φ extends to an isomorphism, and*
2. *$\mathbb{P}$ covers a.*

Then $\Phi \restriction a$ is $\mathbf{\Delta}^1_2$.

Proof. Let $\dot{z}$ be a $\mathbb{P}$-name for the image of $\dot{x}$ under the extension of Φ and fix $d \subseteq \omega$ such that $\Phi([a]_{\mathcal{I}}) = [d]_{\mathcal{J}}$. Consider the set

$$\mathcal{Z} = \{(b,c) : (\exists q \in \mathbb{P}) q \Vdash \dot{x} \cap \check{a} =^{\mathcal{I}} \check{b} \text{ and } \dot{z} \cap \check{d} =^{\mathcal{J}} \check{c}\}$$

By Lemma 4.1 this set is $\mathbf{\Delta}^1_2$. By our assumptions, for every $b \subseteq a$ there is c such that $(b,c) \in \mathcal{Z}$ and moreover $\Phi([b]_{\mathcal{I}}) = [c]_{\mathcal{J}}$ for all $(b,c) \in \mathcal{Z}$. □

We identify $\mathcal{P}(\omega)$ with 2^ω and $(\mathbb{Z}/2\mathbb{Z})^\omega$ and equip it with the corresponding Haar measure. The following lemma will be instrumental in the proof of one of our key lemmas, Lemma 5.6.

Lemma 5.3. *Assume $\mathcal{J}$ is a Borel ideal and f and h are continuous functions such that each one of them is a representation of a homomorphism from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)/\mathcal{J}$*

$$\Delta_{f,g,\mathcal{J}} = \{c \subseteq \omega : f(c) =^{\mathcal{J}} g(c)\}$$

is null. Then $\Delta_{f,g,\mathcal{J}} = \emptyset$.

Proof. By the inner regularity of Haar measure we can find a compact set K disjoint from $\Delta_{f,g,\mathcal{J}}$ of measure $> 1/2$. Fix any $c \subseteq \omega$. The sets K and $K\underline{\Delta}c = \{b\Delta c : b \in K\}$ both have measure $> 1/2$ and therefore we can find $b \in K$ such that $b\Delta c \in K$. But then

$$f(c) =^{\mathcal{J}} f(c\Delta b)\Delta f(b) =^{\mathcal{J}} g(c\Delta b)\Delta g(b) =^{\mathcal{J}} g(c)$$

completing the proof. $\square$

In the following $\mathcal{R}$ denotes the forcing for adding a random real and $\dot{x}$ is the canonical $\mathcal{R}$-name for the random real.

Corollary 5.4. *Assume $\mathcal{J}$ is a Borel ideal and f and h are continuous functions such that each is a representation of a homomorphism from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)/\mathcal{J}$. Furthermore assume $\mathcal{R}$ forces $f(\dot{x}) =^{\mathcal{J}} g(\dot{x})$. Then $f(c) =^{\mathcal{J}} g(c)$ for all $c \subseteq \omega$.*

Proof. It will suffice to show that the assumptions of Lemma 5.3 are satisfied. This is a standard fact but we include the details. Since the set $\Delta_{f,g,\mathcal{J}}$ is Borel, if it is not null then there exists a compact set $K \subseteq \Delta_{f,g,\mathcal{J}}$ of positive measure. If M is a countable transitive model of a large enough fragment of ZFC containing codes for K, f, g, and $\mathcal{J}$ and $x \in K$ is a random real over M, then $M[x] \models f(x) =^{\mathcal{J}} g(x)$ by the assumption on f and g. However, this is a Δ^1_1 statement and is therefore true in V. But $x \in \Delta_{f,g,\mathcal{J}}$ and therefore $f(x) \neq^{\mathcal{J}} g(x)$, a contradiction. $\square$

5.1. **An iteration.** Throughout this subsection we assume

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \mathfrak{c}^+, \eta < \mathfrak{c}^+)$$

is a forcing iteration such that each of its end-segments is forced to be locally real with continuous reading of names. We shall write $p \Vdash_\xi \phi$ instead of $p \Vdash_{\mathbb{P}_\xi} \phi$.

If $\mathcal{X}$ is a predicate for a subset of $\mathbb{R}$ and a is a real, a statement ϕ is $\mathbf{\Sigma}^1_2(a, \mathcal{X})$ if it is of the form $(\exists x)(\forall y)\phi_0(x, y, a, \mathcal{X})$ where ϕ_0 is a $\mathbf{\Delta}_0$ statement. For example, the statement $a \in \mathrm{Triv}^1_\Phi$ is $\mathbf{\Sigma}^1_2(a, \Phi)$ since it asserts that there exists a continuous function $f \colon \mathcal{P}(a) \to \mathcal{P}(\omega)$ such that for every $x \subseteq a$ we have $(a, f(a)) \in \Phi$ (where Φ is identified with its graph).

Lemma 5.5. *Assume that for every name $\dot{I}$ for a partition of ω into finite intervals the set $\{\xi < \mathfrak{c}^+ \Vdash_\xi \dot{\mathbb{Q}}_\xi$ captures $\dot{I}$ and $\mathrm{cf}(\xi)$ is uncountable$\}$ is stationary. Then every isomorphism between quotients over Borel ideals is locally $\mathbf{\Delta}^1_2$ (see §1.1).*

Proof. Fix a name $\dot{\Phi}$ for an isomorphism between quotients over Borel ideals. By moving to an intermediate forcing extension containing relevant Borel codes, we may assume these ideals $\mathcal{I}$ and $\mathcal{J}$ are in the ground model. Let $G \subseteq \mathbb{P}_{\mathfrak{c}^+}$ be a generic filter.

Assume $\mathrm{Triv}^2_{\mathrm{int}_{\mathrm{G}}(\dot{\Phi})}$ is meager in $V[G]$ with a witnessing partition $\mathrm{int}_{\mathrm{G}}(\dot{I})$ (cf. discussion before Corollary 4.5). By Corollary 4.5 the set of $\xi < \mathfrak{c}^+$ of uncountable cofinality such that $\mathrm{int}_{G \restriction \xi}(\dot{I})$ witnesses $\mathrm{Triv}^2_{\mathrm{int}_{G\restriction\xi}(\dot{\Phi}\restriction\xi)}$ is meager in $V[G \restriction \xi]$ includes a relative club.

Since by Zapletal's lemma (cf. §3) the forcing is ω^ω-bounding, we may assume $\mathrm{int}_{\mathrm{G}}(\dot{I})$ is a ground-model partition, $\bar{I} = (I_n : n \in \omega)$ By our assumption, there is a stationary set $\mathbf{S}$ of ordinals of uncountable cofinality such that for all $\xi \in \mathbf{S}$ we have

(1) $\Vdash_\xi$"$\dot{\mathbb{Q}}_\xi$ adds a real $\dot{x}$ that captures $\bar{I}$".

Fix $\xi \in \mathbf{S}$ for a moment. For every infinite $c \subseteq \omega$ there is an infinite $d \subseteq c$ such that $a = I_d$ and for every $b \subseteq a$ there is $p_b \in \dot{\mathbb{Q}}_\xi$ such that $p_b \Vdash \dot{x} \cap \check{a} = \check{b}$. Let $\dot{y}$ be a $\mathbb{P}_{[\xi,\mathfrak{c}^+)}$-name for $\Phi(\dot{x})$. Since this forcing notion is locally real with continuous reading of names, there is a proper real forcing $\mathbb{Q} \lessdot \mathbb{P}_{[\xi,\mathfrak{c}^+)}$ such that both $\dot{x}$ and $\dot{y}$ are $\mathbb{Q}$-names.

Therefore a and $\dot{\mathbb{Q}}_\xi$ satisfy the assumptions of Lemma 5.1 and in $V[G \restriction \xi]$ the restriction of $\mathrm{int}_{G\restriction\xi}(\dot{\Phi} \restriction \xi)$ to $\mathcal{P}(I_d)/\mathcal{I}$ is $\mathbf{\Delta}^1_2$, contradicting our assumption. Since assuming $\mathrm{Triv}^2_{\mathrm{int}_G(\dot{\Phi})}$ was trivial lead to a contradiction, this concludes the proof. $\square$

A topologically trivial isomorphism Φ *extends to an isomorphism that respects triviality of* Φ (in a forcing extension) if the continuous representation of Φ is forced to be a continuous representation of an isomorphism. This is not a consequence of the assumption that Φ extends to an isomorphism in the forcing extension. By a result of Steprāns, there is a σ-linked forcing notion such that a trivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ extends to a trivial automorphism, but the triviality is not implemented by the same function ([33]). Steprāns used this to show that there is a forcing iteration $\mathbb{P}_\kappa$ that forces Martin's Axiom and the existence of a nontrivial automorphism Φ of $\mathcal{P}(\omega)/\operatorname{Fin}$ that is trivial in $V[G \restriction \xi]$ for cofinally many ξ.

The following key lemma shows that in our forcing extension local topological triviality is always witnessed by a $\mathbf{\Pi}^1_2$ set.

Lemma 5.6. *Assume isomorphism Φ between quotients over Borel ideals is locally topologically trivial. Assume $\mathbb{P}$ is Cohen or random forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a proper locally real with continuous reading of names forcing notion and that $\mathbb{P}*\dot{\mathbb{Q}}$ forces that Φ extends to an automorphism that respects its local triviality. Then the set $\{(c,d) : \Phi_*(c) =^{\mathcal{J}} d\}$ is $\mathbf{\Pi}^1_2$.*

Proof. We shall provide the proof only for the random real. The Cohen real case has a similar proof and it will not be used in the proof.

We have $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$. Let $\dot{x}$ be a $\mathbb{P}$-name for a real (Cohen or random) and let $\dot{y}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for the image of $\dot{x}$ by the extension of Φ. Fix a condition p that localizes $\dot{y}$ and note that $\dot{x}$ is already localized since $\mathbb{P}$ is ccc.

Consider the set $\mathcal{Z}$ of all (a, b, f) such that

(1) a and b are subsets of ω.
(2) $f\colon \mathcal{P}(a) \to \mathcal{P}(b)$ is a continuous map,
(3) f is a representation of a homomorphism from $\mathcal{P}(a)/\mathcal{I}$ into $\mathcal{P}(b)/\mathcal{J}$,
(4) $f(c) \in \mathcal{J}$ if and only if $c \in \mathcal{I}$,
(5) p forces that $f(\dot{x} \cap \check{a}) =^{\mathcal{J}} \dot{y} \cap \check{b}$.

Conditions (1) and (2) state that $\mathcal{Z}$ is a subset of the compact metric space $\mathcal{P}(\omega)^2 \times C(\mathcal{P}(\omega), \mathcal{P}(\omega))$, where $C(X, Y)$ denotes the compact metric space of continuous functions between compact metric spaces X and Y. Since (3) states that

$$(\forall x \subseteq a)(\forall y \subseteq a) f(x \cup y) =^{\mathcal{J}} f(x) \cap f(y)$$

$$(\forall x \subseteq a) f(a) \setminus f(x) =^{\mathcal{J}} f(a \setminus x)$$

this is a $\mathbf{\Pi}^1_1$ condition. Similarly (4) is $\mathbf{\Pi}^1_1$. Lemma 4.1 implies that the remaining condition, (5), is $\mathbf{\Delta}^1_2$ (recall that $\mathcal{J}$ was assumed to be Borel). Therefore the set $\mathcal{Z}$ is $\mathbf{\Delta}^1_2$. The set

$$\mathcal{K} = \{a : (a, b, f) \in \mathcal{Z} \text{ for some } (b, f)\}$$

is easily seen to be an ideal that includes Triv^1_Φ. Since Φ is locally topologically trivial it is nonmeager.

We shall now prove a few facts about the elements of $\mathcal{Z}$.

An $(a, b, f) \in \mathcal{Z}$ can be re-interpreted in the forcing extension, and in particular we identify function f with the corresponding continuous function. Properties (1)–(4) are $\mathbf{\Pi}^1_1$ and therefore still hold in the extension. In particular f is forced to be a representation of an isomorphism.

For $a \in \mathcal{K}$ let f_a denote some function such that $(a, b, f_a) \in \mathcal{Z}$ for some b and g. For every $a \in \mathrm{Triv}^1_\Phi$ let $h_a \colon \mathcal{P}(a) \to \mathcal{P}(\omega)$ be a continuous representation of $\Phi \restriction a$. Let Φ_* denote a representation of the extension of Φ in the forcing extension.

(6) If $a \in \mathrm{Triv}^1_\Phi$ then $a \in \mathcal{K}$ and $h_a(c) =^{\mathcal{J}} \Phi_*(c) =^{\mathcal{J}} f_a(c)$ for all $c \subseteq a$.

Let us prove (6). That $a \in \mathcal{K}$ is immediate from the definition of $\mathcal{Z}$ and $\mathcal{K}$, and $h_a(c) =^{\mathcal{J}} \Phi_*(c)$ is immediate from $a \in \mathrm{Triv}^1_\Phi$ and the definition of h_a. The fact that $h_a(c) =^{\mathcal{J}} f_a(c)$ for all $c \subseteq a$ follows from Corollary 5.4.

(7) Assume $d \subseteq a$ are such that $d \in \mathrm{Triv}^1_\Phi$ and $a \in \mathcal{K}$. Then $h_d(c \cap d) =^{\mathcal{J}} f_a(c) \cap \Phi_*(d)$ for all $c \subseteq a$.

By a standard absoluteness argument (see the proof of Corollary 5.4) the set

$$\{c \subseteq a : h_d(c \cap d) =^{\mathcal{J}} f_a(c) \cap \Phi_*(d)\}$$

has full measure. If we fix a compact set K of measure $> 1/2$ included in this set, then the argument from the proof of Lemma 5.3 implies (7).

(8) $\mathcal{K} = \mathrm{Triv}^1_\Phi$.

Since one inclusion was proved in (6), it remains to show that $a \in \mathcal{K}$ implies $a \in \mathrm{Triv}^1_\Phi$. Assume the contrary and choose c such that $f_a(c) \neq^{\mathcal{J}} \Phi_*(c)$ for some $c \subseteq \omega$. Since $c \in \mathcal{I}$ implies $f_a(c) \in \mathcal{J}$ and $\Phi_*(c) \in \mathcal{J}$, we have $c \notin \mathcal{I}$.

We shall first show a well-known fact about isomorphisms of Boolean algebras, that we may choose $c \notin \mathcal{I}$ so that $f(c) \cap \Phi_*(c) \in \mathcal{J}$. For c such that $\Phi_*(c) \Delta f_a(c) \notin \mathcal{J}$ at last one of $f(c) \setminus \Phi_*(c)$ and $\Phi_*(c) \setminus f(c)$ is not in $\mathcal{J}$. Since Φ is an isomorphism we can choose c_1 and c_2 so that

$$\Phi_*(c_1) =^{\mathcal{J}} f(c) \setminus \Phi_*(c)$$

$$\Phi_*(c_2) =^{\mathcal{J}} \Phi_*(c) \setminus f(c)).$$

Then at least one of c_1 and c_2 is not in $\mathcal{I}$, denote it c_j. We have $c_j \subseteq^{\mathcal{I}} c$, $f(c_j) \subseteq^{\mathcal{J}} f(c)$ and $\Phi_*(c_j) \subseteq^{\mathcal{J}} \Phi_*(c)$. Therefore $\Phi_*(c_j) \cap f(c_j) \in \mathcal{J}$ and can replace c with c_j.

Since Triv^1_Φ is nonmeager, by Lemma 4.4, there is $d \in \mathrm{Triv}^1_\Phi$ such that $d \cap c \notin \mathcal{I}$. Then we have (using (6) and (7))

$$\Phi_*(d \cap c) =^{\mathcal{J}} h_d(d \cap c) =^{\mathcal{J}} f_a(c) \cap \Phi_*(d).$$

However, the left-hand side is included in $\Phi_*(c)$ modulo $\mathcal{J}$ and the right-hand side is included in $f_a(c)$ modulo $\mathcal{J}$. But this implies that $\Phi_*(d \cap c) \in \mathcal{J}$, contradicting the assumption that $\ker(\Phi) = \mathcal{I}$.

(9) We have

$$\{(c,d) : \Phi_*(c) =^{\mathcal{J}} d\} = \{(c,d) : (\forall (a,b,f) \in \mathcal{Z}) f(c \cap a) =^{\mathcal{J}} b \cap d\}$$

Take (c,d) such that $\Phi_*(c) =^{\mathcal{J}} d$. Then for every $(a,b,f) \in \mathcal{Z}$ we have $\Phi_*(c \cap a) =^{\mathcal{J}} f(c \cap a)$ by (6) and (7), and therefore (c,d) belongs to the right-hand side set.

Now take (c,d) such that $\Phi_*(c) \Delta d$ is not in $\mathcal{J}$.

Assume for a moment that $e = \Phi_*(c) \setminus d \notin \mathcal{J}$. Since Φ is an isomorphism, we can find a such that $\Phi_*(a) =^{\mathcal{J}} e$. We have that a is $\mathcal{I}$ positive. Since $\mathcal{K}$ is nonmeager, by Lemma 4.4 we can find $a' \subseteq a$ such that $a' \in \mathcal{K} \setminus \mathcal{I}$. Then $f_{a'}(c \cap a')$ is $\mathcal{J}$-positive, included (modulo $\mathcal{J}$) in e, and disjoint (modulo $\mathcal{J}$) from d. Therefore $(a', f_{a'})$ witness that (c,d) does not belong to the right-hand side of (9).

We must therefore have $e = d \setminus \Phi_*(c) \notin \mathcal{J}$ (there is no harm in denoting this set by e, since the existence of the set denoted by e earlier lead us to a contradiction). Applying the above argument we can find $a' \in \mathcal{K}$ such that $c \cap a'$ is $\mathcal{I}$-positive, but its image under $f_{a'}$ is included (modulo $\mathcal{J}$) in d and disjoint (modulo $\mathcal{J}$) from $\Phi_*(c)$, which is again a contradiction.

By (9) we have the required $\mathbf{\Pi}^1_2$ definition of Φ. $\square$

6. Proof of Theorem 1

By §2 for every partition I of ω into finite intervals there is a forcing notion of the form $\mathbb{Q}_{\mathbf{x}}$ that adds a real which captures I. We can use either $\mathbb{Q}_{\mathbf{x}}$ or $\mathbb{S}_{\bar{I}}$ (see §2 and §2.2, respectively). Each of these forcings is proper, real, has continuous reading of names and is ω^ω-bounding. Starting from a model of CH consider a countable support iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \omega_2, \eta < \omega_2)$ of forcings of the form $\mathbb{Q}_{\mathbf{x}}$ and random reals such that for every $\dot{I}$ the set $\{\xi : \mathrm{cf}(\xi) = \omega_1 \text{ and } \dot{\mathbb{Q}}_\xi \text{ is } \mathbb{Q}_{\mathbf{x}}\}$ is stationary and also $\{\xi : \mathrm{cf}(\xi) = \omega_1$ and $\dot{\mathbb{Q}}_\xi$ is the poset for adding a random real$\}$ is stationary. Now fix names $\dot{\mathcal{I}}$ an $\dot{\mathcal{J}}$ for Borel ideals and a name $\dot{\Phi}$ for an automorphism between Borel quotients $\mathcal{P}(\omega)/\dot{\mathcal{I}}$ and $\mathcal{P}(\omega)/\dot{\mathcal{J}}$. By Lemma 5.5, $\dot{\Phi}$ is forced to be locally $\mathbf{\Delta}^1_2$ and by Corollary 4.5 there is a stationary set $\mathbf{S}$ of ξ such that $\mathrm{cf}(\xi) = \omega_1$ such that $\dot{\Phi} \restriction \xi$ is a $\mathbb{P}_\xi$ name for a a locally $\mathbf{\Delta}^1_2$-isomorphism, and $\dot{\mathbb{Q}}_\xi$ is the standard poset for adding a random real.

By our assumption that all $\mathbf{\Sigma}^1_2$ sets have the property of Baire and Lemma 5.1, $\dot\Phi$ is forced to be locally topologically trivial. By Lemma 5.6, if $\xi \in \mathbf{S}$ then $\dot\Phi$ is $\mathbf{\Pi}^1_2$ in $V[G \restriction \xi]$. Therefore $\dot\Phi$ is $\mathbf{\Pi}^1_2$ in $V[G]$.

Since our assumption that there exists a measurable cardinal implies that we have $\mathbf{\Pi}^1_2$-uniformization of this graph, $f\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and all $\mathbf{\Pi}^1_2$ sets have the Property of Baire, Φ has a Baire-measurable representation. By a well-known fact (e.g., [7, Lemma 1.3.2]) Φ has a continuous representation.

Since forcing is a countable support iteration of proper ω^ω-bounding forcings it is proper and ω^ω-bounding (by [31, §VI.2.8(D)]) and therefore $\mathfrak{d} = \aleph_1$ in the extension.

In order to add $2^{\aleph_0} < 2^{\aleph_1}$ to the conclusions, start from a model of CH and add Cohen subsets of $\aleph_1$ to increase $2^{\aleph_1}$ at least to $\aleph_3$. Now force with the iteration defined above. The above argument was not sensitive to the value of $2^{\aleph_1}$ therefore all isomorphisms still have continuous representations. Also, the iteration does not collapse $2^{\aleph_1}$ because it has $\aleph_2$-cc.

6.1. **Universally Baire ideals.** Not much remains to be said about a proof of Theorem 4. Assume there exist class many Woodin cardinals, consider the very same forcing iteration as in the proof of Theorem 1 and fix names for universally Baire ideals $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ as well as for an isomorphism $\dot\Phi$ between their quotients. Lemmas from §4 show that the graph of $\dot\Phi$ is forced to be projective in $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ and therefore universally Baire itself (see [20]). It can therefore be uniformized on a dense G_δ set by a continuous function, and therefore $\dot\Phi$ is forced to have a continuous representation.

7. Concluding remarks

As pointed out earlier, some of the ideas used here were present in the last section of [29]. However, in the latter only automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ were considered and, more importantly, model constructed there does have nontrivial automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$. This is because Silver forcing preserves p-points (e.g., [2]) and therefore in this model there exists a p-point of character $\aleph_1$. By [28] this implies the existence of a nontrivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Questions of whether isomorphisms with continuous representations are necessarily trivial and what can be said about triviality of homomorphisms as compared to isomorphisms (see [7, Question 3.14.2]) are as interesting as ever, but since we have no new information on these questions we shall move on. Problem 7.1 reiterates one of the conjectures from [11], and a positive answer to (1) below may require an extension of results about freezing gaps in Borel quotients from [10].

Problem 7.1.
1. Prove that PFA implies that all isomorphisms between quotients over Borel ideals have continuous representations.
2. Prove that all isomorphisms between quotients over Borel ideals have continuous representations in standard $\mathbb{P}_{\max}$ extension ([36], [21]).

We end with two fairly ambitious questions. A positive answer to the following would be naturally conditioned on a large cardinal assumption.

Question 7.2. *Is there a metatheorem, parallel to Woodin's Σ^2_1 absoluteness theorem (*[20], [35]*) and the Π_2-maximality of $\mathbb{P}_{\max}$ extension (*[36], [21]*), that provides a positive answer to Problem 7.1 (1) or (2) automatically from Theorem 1?*

Let us now move on from Boolean algebras and briefly return to the general situation described in the introduction. Attempts to generalize these rigidity results to other categories were made, with limited success, in [8]. For example, quotient group $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}/\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ clearly has nontrivial automorphisms in ZFC. One should also mention the case of semilattices, when isomorphisms are locally trivial but not necessarily trivial ([8]). On the other hand, TA implies that all automorphisms of the Calkin algebra are trivial ([12]). Note that 'trivial' as defined here is equivalent to 'inner' for automorphisms of the Calkin algebra, but this is not true for arbitrary corona algebras since in some cases the relevant multiplier algebra has outer automorphisms, unlike $\mathcal{B}(H)$.

Problem 7.3. In what categories can one prove that the consistency of the assertion that all isomorphisms between quotient structures based on standard Borel spaces are trivial?

References

[1] J. L. Alperin, J. Covington, and D. Macpherson, *Automorphisms of quotients of symmetric groups*, Ordered groups and infinite permutation groups, Math. Appl., vol. 354, Kluwer Acad. Publ., Dordrecht, 1996, pp. 231–247.

[2] T. Bartoszynski and H. Judah, *Set theory: on the structure of the real line*, A.K. Peters, 1995.

[3] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al., eds.), London Math. Soc. Lecture Notes Series, no. 350, Cambridge University Press, 2008, pp. 315–427.

[4] C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990.

[5] S. Coskey and I. Farah, *Group cohomology and corona automorphisms*, in preparation, 2011.

[6] I. Farah, *Completely additive liftings*, The Bulletin of Symbolic Logic **4** (1998), 37–54.

[7] ———, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Memoirs of the American Mathematical Society, vol. 148, no. 702, 2000.

[8] ———, *Liftings of homomorphisms between quotient structures and Ulam stability*, Logic Colloquium '98, Lecture notes in logic, vol. 13, A.K. Peters, 2000, pp. 173–196.

[9] ———, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?*, Illinois Journal of Mathematics **46** (2003), 999–1033.

[10] ———, *Luzin gaps*, Trans. Amer. Math. Soc. **356** (2004), 2197–2239.

[11] ———, *Rigidity conjectures*, Logic Colloquium 2000, Lect. Notes Log., vol. 19, Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 252–271.

[12] ———, *All automorphisms of the Calkin algebra are inner*, Annals of Mathematics **173** (2011), 619–661.

[13] Q. Feng, M. Magidor, and W.H. Woodin, *Universally baire sets of reals*, Set Theory of the Continuum (H. Judah, W. Just, and W.H. Woodin, eds.), Springer-Verlag, 1992, pp. 203–242.

[14] W. Just, *Repercussions on a problem of Erdös and Ulam about density ideals*, Canadian Journal of Mathematics **42** (1990), 902–914.

[15] ———, *A modification of Shelah's oracle chain condition with applications*, Transactions of the American Mathematical Society **329** (1992), 325–341.

[16] ———, *A weak version of AT from OCA*, MSRI Publications **26** (1992), 281–291.

[17] W. Just and A. Krawczyk, *On certain Boolean algebras $\mathcal{P}(\omega)/I$*, Transactions of the American Mathematical Society **285** (1984), 411–429.

[18] V. Kanovei and M. Reeken, *On Ulam's problem concerning the stability of approximate homomorphisms*, Tr. Mat. Inst. Steklova **231** (2000), 249–283.

[19] ———, *New Radon–Nikodym ideals*, Mathematika **47** (2002), 219–227.

[20] P.B. Larson, *The stationary tower*, University Lecture Series, vol. 32, American Mathematical Society, Providence, RI, 2004, Notes on a course by W. Hugh Woodin.

[21] ———, *Forcing over models of determinacy*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 2121–2177.

[22] D.A. Martin and R.M. Solovay, *A basis theorem for $\mathbf{\Sigma}^1_3$ sets of reals*, Ann. of Math. (2) **89** (1969), 138–159.

[23] N.C. Phillips and N. Weaver, *The Calkin algebra has outer automorphisms*, Duke Math. Journal **139** (2007), 185–202.

[24] Andrzej Roslanowski and Saharon Shelah, *Norms on possibilities i: forcing with trees and creatures*, Memoirs of the American Mathematical Society **141** (1999), xii + 167, math.LO/9807172.

[25] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Mathematics Journal **23** (1956), 409–419.

[26] S. Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[27] S. Shelah and J. Steprāns, *PFA implies all automorphisms are trivial*, Proceedings of the American Mathematical Society **104** (1988), 1220–1225.

[28] ———, *Non-trivial homeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ without the continuum hypothesis*, Fundamenta Mathematicae **132** (1989), 135–141.

[29] ———, *Martin's axiom is consistent with the existence of nowhere trivial automorphisms*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 2097–2106.

[30] Saharon Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, xxix+496 pp, 1982.

[31] ———, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[32] ———, *Properness without elementaricity*, Journal of Applied Analysis **10** (2004), 168–289, math.LO/9712283.

[33] Juris Steprāns, *The autohomeomorphism group of the Čech-Stone compactification of the integers*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4223–4240.

[34] B. Veličković, *OCA and automorphisms of $\mathcal{P}(\omega)/\,$Fin*, Top. Appl. **49** (1992), 1–13.

[35] W.H. Woodin, *Beyond Σ^2_1 absoluteness*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 515–524.

[36] ———, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, revised ed., de Gruyter Series in Logic and its Applications, vol. 1, Walter de Gruyter GmbH & Co. KG, Berlin, 2010.

[37] J. Zapletal, *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc. **167** (2004), no. 793, viii+141.

[38] ———, *Forcing idealized*, Cambridge University Press, 2008.

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