FRAÏSSÉ LIMITS OF C*-ALGEBRAS

CHRISTOPHER J. EAGLE, ILIJAS FARAH, BRADD HART, BORIS KADETS, VLADYSLAV KALASHNYK, AND MARTINO LUPINI

ABSTRACT. We realize the Jiang-Su algebra, all UHF algebras, and the hyperfinite II_1 factor as Fraïssé limits of suitable classes of structures. Moreover by means of Fraïssé theory we provide new examples of AF algebras with strong homogeneity properties. As a consequence of our analysis we deduce Ramsey-theoretic results about the class of full-matrix algebras.

1. Introduction

Fraïssé theory lies at the crossroads of combinatorics and model theory. It originates from the seminal work of Fraïssé in [14] for the case of discrete countable structures. Broadly speaking, Fraïssé theory studies the correspondence between homogeneous structures and properties of the classes of their finitely generated substructures. The *age* of a countable structure is the collection of its finitely generated substructures, and the ages of homogeneous structures are precisely the classes of structures known as *Fraïssé classes*. Conversely, given any Fraïssé class one can construct a countable homogeneous structure with the given class as its age. This structure, which is referred to as the *Fraïssé limit* of the class, is unique up to isomorphism, and can be thought of as the structure generically constructed from the class.

Fraïssé theory has been recently generalized to metric structures by Ben Yaacov in [1]. An earlier approach to Fraïssé limits in the metric setting was developed in [35]. Standard examples of metric Fraïssé limits are the Urysohn metric space, its variants, and the Gurarij Banach space (previously construed as a Fraïssé limit in [23]). The Elliott intertwining argument central in classification program for nuclear C*-algebras (see [31]) is closely related to the proof of uniqueness of metric Fraïssé limits.

In this paper we study Fraïssé limits of C*-algebras. In particular we show that several important C*-algebras can be described as Fraïssé limits of suitable classes. As in [26], we work under slightly less general assumptions than [1], and we consider only classes where the interpretation of functional and relational symbols are Lipschitz (see Section 2 below for the precise definitions). In our constructions we consider Fraïssé classes that are not complete (in the sense of [1]) and are not closed under substructures. The reason we do this is that the class of finitely generated substructures of a given C*-algebra tends to be too large. As a matter of fact, conjecturally all simple and separable C*-algebras are singly generated (see

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[36]). As a consequence we only consider classes that are made of suitable "small" subalgebras of the given C*-algebra.

We show that the Jiang-Su algebra $\mathcal{Z}$ [20] and all UHF algebras [16] are limits of suitable Fraïssé classes. Both $\mathcal{Z}$ and UHF algebras are examples of C*-algebras of fundamental importance for the classification program of C*-algebras, a survey of which can be found in [31], [8]. Furthermore we prove that, while the class of finite-dimensional C*-algebras is not Fraïssé, one can obtain a Fraïssé class by adding a distinguished interior trace and imposing a restriction on the number of direct summands. This provides new examples of AF algebras satisfying strong homogeneity properties. Finally we deduce a Ramsey-type result for the class of matrix algebras, either endowed with the operator norm or with the trace-norm. This is obtained from the above mentioned description of (infinite type) UHF algebras as limits, together with a similar characterization of the hyperfinite II_1 factor $\mathcal{R}$. We use the observation that the corresponding automorphism groups are extremely amenable which is a result due to Gromov [17]. The other ingredient is the well known connection between extreme amenability and Ramsey-theoretic properties of a Fraïssé class originally established in [21] and recently generalized to the metric setting in [26].

The paper is divided into seven sections. In Section 2 we recall the basic notions and results of Fraïssé theory, adapted to the framework of C*-algebras. Section 3 contains the results about UHF algebras, AF algebras, and the hyperfinite II_1 factor. The description of the Jiang-Su algebra as a Fraïssé limit is presented in Section 4. We recall the notions of Lévy groups and extremely amenable groups in Section 5, where we observe that the automorphisms groups of the hyperfinite II_1 factor and infinite type UHF algebras are Lévy. This is used in Section 6 to deduce Ramsey-type results about the class of full matrix algebras. We conclude in Section 7 with a discussion of future lines of research and open problems.

2. Fraïssé limits of C*-algebras

In this section we define Fraïssé classes of C*-algebras and their Fraïssé limits. Recall that a *C*-algebra* is a subalgebra of the algebra of bounded linear operators on a Hilbert space which is closed under the adjoint operation * and is closed in the operator norm topology (see [4] for an introduction to C*-algebras). We will often consider *unital* C*-algebras, that is, algebras with a multiplicative identity element, but when we say "C*-algebra" without qualification we mean an algebra which is not necessarily unital. We will consider C*-algebras as examples of metric structures. The literature contains several definitions of metric structures suited to various purposes; the one we present here is the same as in [26].

Definition 2.1. A *language* L consists of a set of *predicate symbols* and a set of *function symbols*. Each predicate symbol P and function symbol f carries an associated *arity* and *Lipschitz constant* C_P and C_f respectively. We assume that every language includes a distinguished symbol d, which will always be interpreted as a metric.

An *L-structure* is a complete metric space (A, d), together with interpretations for the symbols of L: for each

(1) n-ary predicate symbol P, a C_P-Lipschitz function $P^A : A^n \to \mathbb{R}$, and
(2) each n-ary function symbol f, a C_f-Lipschitz function $f^A : A^n \to A$.

We need to say a word about how we will formally see C*-algebras as structures in the sense of the previous definition. For a C*-algebra A, we will consider the unit ball A_1 together with the operator norm as the underlying complete metric space. In terms of the language, for every *-polynomial $p(x_1, \ldots, x_n)$, there will be an n-ary predicate R_p which is interpreted on A_1^n by $\|p(x_1, \ldots, x_n)\|$. This relation is Lipschitz with a constant that is independent of the choice of C*-algebra. If we wish to consider a trace as well then we similarly introduce relations for traces of all *-polynomials on the unit ball; again, all of these relations are Lipschitz. In practice, we will use the usual C*-algebraic notation when we deal with C*-algebras but formally, for the purposes of fitting the continuous Fraïssé context, we will treat them as above.

Since all of our structures fit into the framework described above, we find it convenient to give a presentation of Fraïssé theory which is closer to that of [26] than the more general approach taken in [1]. Our definitions are not identical to those of either [26] or [1]; see Remark 2.7 for discussion of the differences. In particular, [26] do not require their metric spaces to be bounded and therefore their structures are not metric structures in the sense of [2]. In the way that we are viewing C*-algebras (as their unit balls) the underlying metric is bounded and therefore their continuous theory is well-defined.

Definition 2.2. Let A be a C*-algebra, and $\overline{a}$ a tuple from A_1. The *subalgebra generated by* $\overline{a}$ is the smallest C*-subalgebra of A which contains $\overline{a}$, and is denoted $\langle \overline{a} \rangle$. We say A is *finitely generated* if there is a finite tuple $\overline{a}$ such that $A = \langle \overline{a} \rangle$.

Remark 2.3. The condition that a C*-algebra be finitely generated may be weaker than it appears. It is known that a large class of separable unital C*-algebras, including all those which are $\mathcal{Z}$-stable, are generated by single elements (see [36] for this result and further discussion). In particular, some of the C*-algebras we will construct as Fraïssé limits will be singly generated.

Definition 2.4. Let $\mathcal{K}$ be a class of finitely generated structures with distinguished generators.

(1) We say that a structure is a *$\mathcal{K}$-structure* if it is an inductive limit of elements of $\mathcal{K}$.
(2) The class $\mathcal{K}$ has the *near amalgamation property (NAP)* if whenever $A, B_0, B_1 \in \mathcal{K}$, and $\varphi_i : A \to B_i$ are morphisms, then for every $\epsilon > 0$ there is a $C \in \mathcal{K}$ and morphisms $\psi_i : B_i \to C$ such that $d(\psi_0\varphi_0(\overline{a}), \psi_1\varphi_1(\overline{a})) < \epsilon$, where $\overline{a}$ is the distinguished generating set of A.
(3) The class $\mathcal{K}$ has the *amalgamation property (AP)* if, in the definition of NAP, we may take $\epsilon = 0$.
(4) The class $\mathcal{K}$ has the *joint embedding property (JEP)* if for all $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ such that A and B embed into C.

The properties defined above have clear analogues in classical Fraïssé theory. In the classical setting one works with *countable* classes of finite structures, in order to ensure that the resulting limit object is also countable. In the metric setting it is necessary to replace countability by separability in a suitably chosen topology, which we now describe. As in [1, Definition 2.10], if $\mathcal{K}$ is a class of finitely generated structures, we denote by $\mathcal{K}_n$ the subclass of $\mathcal{K}$ consisting of all members of $\mathcal{K}$ whose distinguished generating sets have size n. If $\mathcal{K}$ has JEP and NAP, we can define a pseudo-metric on $\mathcal{K}_n$ by defining

$$d^{\mathcal{K}}(\overline{a},\overline{b}) = \inf\{d_C(\overline{a},\overline{b}) : \overline{a},\overline{b} \in C, C \in \mathcal{K}\}$$

where d_C is the distance computed in C (see [1, Definition 2.11]) and $\bar{a}$ and $\bar{b}$ are the distinguished generators of elements of $\mathcal{K}_n$.

Definition 2.5. A class $\mathcal{K}$ of finitely generated structures with JEP and NAP has the *weak Polish Property (WPP)* if for each n the pseudo-metric space $(\mathcal{K}_n, d^{\mathcal{K}})$ is separable.

Finally, we come to the central definitions of Fraïssé classes and Fraïssé limits.

Definition 2.6. A class $\mathcal{K}$ of finitely generated structures is a *Fraïssé class* if it satisfies JEP, NAP and WPP.

A $\mathcal{K}$-structure M is a *Fraïssé limit* of the Fraïssé class $\mathcal{K}$ if:

(1) M is $\mathcal{K}$-*universal*: For every $A \in \mathcal{K}$ there is an embedding of A into M,
(2) M is *approximately* $\mathcal{K}$-*homogeneous*: for all $A, B \subseteq M$ such that $A \cong B$, $A, B \in \mathcal{K}$ and for every $\epsilon > 0$ there is an automorphism σ of M such that if $\bar{a}$ and $\bar{b}$ are the generators of A and B then $d(\bar{a}, \sigma(\bar{b})) < \epsilon$.

Remark 2.7. The classes that we are considering are *incomplete* in the sense of [1, Definition 2.12]. The completions of our classes will include their Fraïssé limits. The classes we consider also fail to be *hereditary*, that is, we will have classes $\mathcal{K}$, and members $A \in \mathcal{K}$ with finitely generated substructures $B \subseteq A$ and $B \notin \mathcal{K}$. As a consequence, we do not have the usual correspondence between Fraïssé classes and ages of homogeneous structures. Nevertheless, our definitions do allow us to construct limits of Fraïssé classes, and hence obtain interesting information about the limit objects.

Theorem 2.8. *Every Fraïssé class has a Fraïssé limit which is unique up to isomorphism.*

The proof is a straightforward adaptation of the proofs of Lemma 2.17 and Theorem 2.19 from [1].

In the discrete setting many (though not all) well-known Fraïssé limits have theories with quantifier elimination. The main results of [5] show that quantifier elimination is a rare phenomenon for C*-algebras; in particular, it is shown in [5] that the only noncommutative C*-algebra with quantifier elimination is $M_2(\mathbb{C})$, so none of the noncommutative C*-algebras we construct as Fraïssé limits in the subsequent sections have quantifier elimination. In Section 3.2 we show that the hyperfinite II$_1$ factor $\mathcal{R}$ is the Fraïssé limit of matrix algebras viewed as von Neumann algebras. The theory of $\mathcal{R}$ also does not have quantifier elimination, as shown in [18].

We do have one example of a C*-algebra which is a Fraïssé limit whose theory has quantifier elimination, namely the algebra $C(2^{\mathbb{N}})$ of continuous functions on the Cantor set. It is straightforward to see that this algebra is the Fraïssé limit of the class of finite-dimensional commutative C*-algebras (i.e., the algebras of the form $\mathbb{C}^n$). Quantifier elimination for the theory of $C(2^{\mathbb{N}})$ is proved in [7]. In fact, by the results of [6], the theory of $C(2^{\mathbb{N}})$ is the only theory of infinite-dimensional commutative unital C*-algebras which has quantifier elimination.

3. AF algebras

We now turn to describing several examples of Fraïssé classes of finite-dimensional C*-algebras. Throughout this section, when we discuss $M_n(\mathbb{C})$ we are considering it as being n^2-generated by the standard matrix units. Recall that a *(normalized) trace* on a unital C*-algebra A is a continuous linear functional $\tau : A \to \mathbb{C}$ such that $\tau(1) = 1$, it is positive (i.e., $\tau(a^*a) \geq 0$ for all $a \in A$), and $\tau(ab) = \tau(ba)$ for all $a, b \in A$. The space of traces of A, $T(A)$, is a weak*-compact and convex subset of the unit ball of the dual of A. Every unital *-homomorphism between tracial algebras $\varphi\colon A \to B$ gives rise to the continuous affine map $\varphi_*\colon T(B) \to T(A)$: if $\tau \in T(B)$ and $a \in A$, define

$$\varphi_*(\tau)(a) = \tau(\varphi(a)).$$

This contravariant functor will also play a role in the proof of Lemma 4.4. It is a well-known fact from linear algebra that each matrix algebra $M_n(\mathbb{C})$ has a unique trace, and that trace τ is given by $\tau([a_{i,j}]) = \frac{1}{n}\sum_{j=1}^{n} a_{j,j}$. We will make frequent and unmentioned use of the following well-known properties of finite-dimensional C*-algebras.

Fact 3.1. (1) *Every finite-dimensional C*-algebra is isomorphic to a finite direct sum of matrix algebras.*

(2) *If $A = M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$, then every trace on A is a convex combination of the (unique) traces on $M_{k_1}(\mathbb{C}), \ldots, M_{k_n}(\mathbb{C})$.*

(3) *There is a unital embedding of $M_n(\mathbb{C})$ into $M_m(\mathbb{C})$ if and only if n divides m. A unital embedding of finite-dimensional algebras A and B is characterized up to unitary conjugacy by the multiplicities with which it maps each direct summand of A into each direct summand of B (that is, by its Bratteli diagram; see* [4, Section III.2] *or* [10, Section 4.4]*).*

When we consider a finite-dimensional algebra $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$, we always consider it as being generated by elements of the form $a_1 \oplus \cdots \oplus a_n$, where the a_i's vary over the distinguished generators of the $M_{k_i}(\mathbb{C})$'s.

We begin by observing that when we consider classes of finite-dimensional C*-algebras near amalgamation can be replaced by actual amalgamation.

Lemma 3.2. *Let $\mathcal{K}$ be a subclass of the class of finite-dimensional C*-algebras. The following are equivalent:*

(1) *$\mathcal{K}$ has NAP,*
(2) *$\mathcal{K}$ has AP.*

Proof. The direction (2) $\implies$ (1) is obvious. For the other direction, suppose that $\mathcal{K}$ has NAP. Take $A, B_1, B_2 \in \mathcal{K}$, and let $\varphi_i : A \to B_i$ be morphisms. Write $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Let

$$\overline{a} = ((I_{n_1}, 0, \ldots, 0), (0, I_{n_2}, 0, \ldots, 0), \ldots, (0, \ldots, 0, I_{n_k}))$$

By definition of NAP, with $\epsilon = \frac{1}{2}$, there is a $\mathcal{K}$-structure C, and maps $\psi_i : B_i \to C$ such that $d(\psi_0\varphi_0(\overline{a}), \psi_1\varphi_1(\overline{a})) < \epsilon$.

We claim that C satisfies the definition of AP. Consider the Bratteli diagrams of the embeddings $\psi_0\varphi_0$ and $\psi_1\varphi_1$ of A into C. If these Bratteli diagrams are the same, then after conjugating by a unitary we have $\psi_0\varphi_0 = \psi_1\varphi_1$, so C exactly amalgamates B_0 and B_1 over A. If the Bratteli diagrams are not the same, then for some i the ranks of the matrices $\psi_0\varphi_0(0,\dots,I_{n_i},\dots,0))$ and $\psi_1\varphi_1(0,\dots,I_{n_i},\dots,0)$ are not equal. These images are then projections of different ranks, so

$$\|\psi_0\varphi_0(0,\dots,I_{n_i},\dots,0)) - \psi_1\varphi_1(0,\dots,I_{n_i},\dots,0)\| = 1,$$

which contradicts our choice of C. □

In the setting of Banach spaces, the class of all finite-dimensional Banach spaces is a Fraïssé class, with the Gurarij space as its limit (see [1, Section 3.3]). By contrast, the class of all finite-dimensional C*-algebras is *not* a Fraïssé class. The obstacle to amalgamation comes from considering traces.

Proposition 3.3. *The class of finite-dimensional C*-algebras is not a Fraïssé class.*

Proof. We show that this class does not have AP. Let $A = \mathbb{C}\oplus\mathbb{C}$, $B = M_2(\mathbb{C})$, and $C = M_3(\mathbb{C})$. Consider the following embeddings $\iota_{A,C} : A \to C$ and $\iota_{B,C} : B \to C$:

$$\iota_{A,C}(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad \iota_{B,C}(a,b) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}.$$

Suppose that D is a finite-dimensional C*-algebra which amalgamates B and C over A with respect to these embeddings, via embeddings $\iota_{B,D}$ and $\iota_{C,D}$. Let $x = \iota_{B,D} \circ \iota_{A,B}(1,0)$, and note that $x = \iota_{C,D} \circ \iota_{A,C}(1,0)$ by definition of amalgamation.

Let τ_D be a trace on D. On the image of B in D the trace τ_D restricts to a trace, which must be the unique trace τ_B from B. Therefore,

$$\tau_D(x) = \tau_B(\iota_{A,B}(1,0)) = \tau_B\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}.$$

Similarly, τ_D restricts to the unique trace τ_C on the image of C in D. Then we have

$$\tau_D(x) = \tau_C(\iota_{A,C}(1,0)) = \tau_C\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \frac{1}{3}.$$

This contradiction finishes the proof. □

3.1. **UHF algebras.** If we restrict our attention to subclasses of the class of matrix algebras, we can obtain UHF algebras as Fraïssé limits. Recall that a separable unital C*-algebra which arises as the direct limit of unital embeddings of matrix algebras is called a *uniformly hyperfinite (UHF)* algebra. It is well-known that UHF algebras are classified by *supernatural numbers*, that is, formal products $\prod_{p \text{ prime}} p^{n_p}$, where each $n_p \in \mathbb{N}\cup\{\infty\}$; given a UHF algebra A, which is the direct limit of $M_{k_1}(\mathbb{C}) \to M_{k_2}(\mathbb{C}) \to \cdots$, the associated supernatural number is given by $n_p = \sup\{r : p^r \mid k_i \text{ for some } i\}$. See [10, Chapter 4] for more details.

Theorem 3.4. *Every UHF algebra is a Fraïssé limit.*

Proof. Let A be a UHF algebra, and write A as the direct limit of matrix algebras $M_{n_1}(\mathbb{C}), M_{n_2}(\mathbb{C}), \dots$. As usual, we view each $M_{n_i}(\mathbb{C})$ with its standard matrix units as generators. Let $\mathcal{K} = \{M_{n_i}(\mathbb{C}) : i \in \mathbb{N}\}$. We then have

$$\mathcal{K}_n = \begin{cases} \{M_{n_i}(\mathbb{C})\} & \text{if } n = n_i^2 \text{ for some } i \\ \emptyset & \text{if } n \neq n_i^2 \text{ for all } i \end{cases}.$$

In particular, it is clear that WPP holds. The class $\mathcal{K}$ has a minimal element $M_{n_1}(\mathbb{C})$, so JEP will be a consequence of AP. To see AP, note that if $M_{n_i}(\mathbb{C})$ is embedded in $M_{n_j}(\mathbb{C})$ and $M_{n_k}(\mathbb{C})$, and (without loss of generality) $n_j \leq n_k$, then $M_{n_j}(\mathbb{C})$ embeds in $M_{n_k}(\mathbb{C})$ in a way which (up to unitary equivalence) respects the embedding of $M_{n_i}(\mathbb{C})$. Therefore $M_{n_k}(\mathbb{C})$ itself serves to amalgamate $M_{n_j}(\mathbb{C})$ and $M_{n_k}(\mathbb{C})$ over $M_{n_i}(\mathbb{C})$.

It is clear from the construction of the Fraïssé limit of $\mathcal{K}$ that this limit is a UHF algebra with the same supernatural number as A, and hence is isomorphic to A. □

An argument similar to the one in Theorem 3.4 shows that the class of full matrix algebras with injective (not necessarily unital) *-homomorphisms as morphisms is a Fraïssé class. The corresponding limit is the (non-unital) C*-algebra of compact operators on the separable infinite-dimensional Hilbert space [3, Section I.8].

3.2. **The hyperfinite II_1 factor.** In Theorem 3.4 matrix algebras were regarded as finite-dimensional C*-algebras, but we can also regard them as tracial von Neumann algebras. A *tracial von Neumann* algebra is a unital C*-algebra M endowed with a distinguished trace τ such that the unit ball of M is complete with respect to the trace-norm $\|x\|_\tau = \tau(x^*x)^{\frac{1}{2}}$. As was shown in [12], tracial von Neumann algebras can be regarded as metric structures in the language of unital C*-algebras with the additional predicate symbol for a distinguished trace, where the symbol for the metric is interpreted as the distance associated with the trace-norm. Since the operator norm is not uniformly continuous with respect to the trace-norm, it is no longer part of the structure. A tracial von Neumann algebra is *separable* if it is separable with respect to the trace-norm.

A *finite factor* is a tracial von Neumann algebra (M, τ) such that the center of M consists only of the scalar multiples of the identity. When M is a finite factor the trace τ on M is uniquely determined. Full matrix algebras are examples of finite factors. A finite factor that is not isomorphic to a full matrix algebra is called a II_1 factor. Equivalently, a finite factor is a II_1 factor when the trace assumes all values in $[0, 1]$ on projections.

A II_1 factor is *hyperfinite* if it can be locally approximated (in trace-norm) by full matrix algebras. The unique hyperfinite II_1 factor is traditionally denoted by $\mathcal{R}$, and can be concretely realized as the direct limit of the direct sequence $(M_{2^n}(\mathbb{C}))_{n\in\mathbb{N}}$ in the category of tracial von Neumann algebras. The same proof as Theorem 3.4 shows that the class of full matrix algebras regarded as finite factors is a Fraïssé class. Since a direct limit of finite factors is a finite factor, the Fraïssé limit of the class of full matrix algebras is the hyperfinite II_1 factor $\mathcal{R}$. That is, we have:

Theorem 3.5. *The hyperfinite II_1 factor $\mathcal{R}$ is the Fraïssé limit of the class of full matrix algebras (regarded as finite factors).*

3.3. **Finite width algebras.** We now return to considering C*-algebras. Throughout this section we consider finite-dimensional C*-algebras as being unital, so in particular all the embeddings we consider will be unital *-homomorphisms. To progress beyond UHF algebras, we need to consider more general classes of finite-dimensional algebras than just matrix algebras. With the obstacles encountered in Proposition 3.3 in mind, we make the following definitions.

Definition 3.6. (1) A finite-dimensional C*-algebra A has *width* n if A can be written as a direct sum of exactly n matrix algebras.
(2) A trace τ on a finite-dimensional C*-algebra A is *interior* if, when τ is written as a convex combination of the unique traces on the matrix algebras which appear as direct summands of A, none of the coefficients are 0. The trace τ is *rational* if all of these coefficients are rational.

Lemma 3.7. *Let A, B, C be finite-dimensional C*-algebras of width n, and let α, β, γ be rational interior traces on A, B, C, respectively. Let $\Phi : A \to B$ and $\Psi : A \to C$ be trace-preserving embeddings. Then there exists $N \in \mathbb{N}$ such that B and C can be amalgamated into $M_N(\mathbb{C})$ over A by trace-preserving embeddings.*

Proof. Write $A = M_{h_1}(\mathbb{C}) \oplus \cdots \oplus M_{h_n}(\mathbb{C})$. For each i, let α_i be the unique trace on $M_{h_i}(\mathbb{C})$, and let $a_i \in \mathbb{Q}$ be such that $\alpha = \sum_{i=1}^n a_i \alpha_i$. Write $B = M_{l_1}(\mathbb{C}) \oplus \cdots \oplus M_{l_n}(\mathbb{C})$, and $C = M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$, and denote the traces on B and C by $\beta = \sum_{i=1}^n b_i \beta_i$ and $\gamma = \sum_{i=1}^n c_i \gamma_i$, respectively. For each $i, j \leq n$, let $t_{i,j}$ be the multiplicity with which A_i is embedded by Φ in B_j; similarly, let $q_{i,j}$ be multiplicity with which A_i is embedded by Ψ in C_j.

A direct computation from the definition of Φ (respectively, Ψ) being trace-preserving shows that for all $1 \leq j \leq n$,

$$\sum_{i=1}^n \frac{b_i}{l_i} t_{j,i} = \frac{a_j}{h_j} = \sum_{i=1}^n \frac{c_i}{k_i} q_{j,i}. \tag{3.1}$$

We consider the conditions necessary to create a trace-preserving amalgamation of B and C into $M_N(\mathbb{C})$. For each $1 \leq i \leq n$, let s_i be the multiplicity with which $M_{l_i}(\mathbb{C})$ is embedded in $M_N(\mathbb{C})$ by this hypothetical embedding, and let r_i be similarly the multiplicity of the embedding of $M_{k_i}(\mathbb{C})$. We immediately see that we must have

$$\sum_{i=1}^n l_i s_i = N = \sum_{i=1}^n k_i r_i. \tag{3.2}$$

For the traces β and γ to be preserved (with respect to the unique trace δ on $M_N(\mathbb{C})$), we must additionally have, for each $1 \leq j \leq n$,

$$b_j \sum_{i=1}^n l_i s_i = l_j s_j, \tag{3.3}$$

and

$$c_j \sum_{i=1}^n k_i r_i = k_j r_j. \tag{3.4}$$

Finally, we must make our amalgamation respect Φ and Ψ. It is sufficient to ensure that each $M_{h_i}(\mathbb{C})$ from A embeds into $M_N(\mathbb{C})$ via B and C with the same multiplicities. That is, we must satisfy the following for all $1 \leq j \leq n$:

$$\sum_{i=1}^n t_{j,i} s_i = \sum_{i=1}^n q_{j,i} r_i. \tag{3.5}$$

Finding any positive integers $s_1, \ldots, s_n, r_1, \ldots, r_n$ satisfying 3.2, 3.3, 3.4, and 3.5 will complete the proof.

If we view Equation 3.3 as a linear system in variables s_i then the facts that $\sum_{i=1}^n b_i = \sum_{i=1}^n c_i = 1$ and all $b_i, c_i \neq 0$ imply that the system of equations 3.3 is equivalent to

$$s_i = \frac{b_i l_n}{b_n l_i} s_n \qquad \text{for all } i < n,$$

and similarly Equation 3.4 is equivalent to

$$r_i = \frac{c_i k_n}{c_n k_i} r_n \qquad \text{for all } i < n.$$

Given these conditions, Equation 3.2 reduces to

$$r_n = \frac{l_n c_n}{b_n k_n} s_n.$$

If we choose any s_n and define the remaining r_i, s_i as above, straightforward substitution shows that Equation 3.5 follows from Equation 3.1. Therefore if $s_n \in \mathbb{N}$ is chosen so that the above formulas for the s_i, r_i all yield integer values, then Equations 3.2 - 3.5 will be satisfied. □

Proposition 3.8. *The class of finite-dimensional algebras of width $n \geq 2$ with a distinguished interior trace has AP. Moreover, we can always choose the amalgam to have a rational trace.*

Proof. Let A, B, C be algebras of width n with distinguished traces α, β, γ, and let $\Phi : A \to B$ and $\Psi : A \to C$ be morphisms which each preserve α. By continuity, and the fact that α, β, γ are interior, the maps Φ and Ψ each preserve an open neighbourhood of traces around α. Let U be the intersection of these neighbourhoods, so Φ and Ψ both preserve U.

Let $\tau_1, \ldots, \tau_n$ be rational traces on A which form the vertices of an $(n-1)$-simplex contained in U. Apply Lemma 3.7 to each τ_i to produce matrix algebras $M_{N_1}(\mathbb{C}), \ldots, M_{N_n}(\mathbb{C})$ which embed B and C over A with trace-preserving embeddings. Let $D = M_{N_1}(\mathbb{C}) \oplus \cdots \oplus M_{N_n}(\mathbb{C})$, and embed B and C into D by taking the direct sum of the embeddings into each $M_{n_i}(\mathbb{C})$; let Θ be the resulting embedding of A into D. The extremal traces on D are mapped by Θ to the τ_i, so by convexity there is some interior rational trace δ on D which is mapped by Θ to α. Then (D, δ) is the required amalgam of B and C over A. □

We can now show that certain classes of finite-dimensional algebras are Fraïssé classes. To obtain information about the Fraïssé limits we will use the K_0 functor. To each unital C*-algebra A is associated an abelian group $K_0(A)$, and to each embedding $f : A \to B$ an injective group homomorphism $K_0(f) : K_0(A) \to K_0(B)$. Since we will not explicitly need the construction of K_0, we refer the reader to [4], [30], or [10] for the definition.

Theorem 3.9. *For each $n \geq 2$, and each interior trace τ on $\mathbb{C}^n$, the class $\mathcal{K}(n, \tau)$ of finite-dimensional C*-algebras A of width n with a distinguished interior trace α such that there is an embedding of $\mathbb{C}^n$ into A which preserves τ, is a Fraïssé class.*

The Fraïšsé limit of $\mathcal{K}(n, \tau)$ is simple, has a unique trace, and is not self-absorbing. As an abelian group, the K_0 group of the Fraïssé limit is divisible and of rank n. Hence when $n \neq m$, the limits of $\mathcal{K}(n, \tau)$ and $\mathcal{K}(m, \sigma)$ are non-isomorphic.

Proof. It follows from Proposition 3.8 that this class has AP, and since this class has a minimal element, JEP is a consequence of AP. By Proposition 3.8 we have countably members of $\mathcal{K}(n,\tau)$ (namely, finite-dimensional algebras with distinguished rational traces) such that every other member of $\mathcal{K}(n,\tau)$ embeds into one of them. Since the space of substructures of a fixed member of $\mathcal{K}(n,\tau)$ is separable in $d^{\mathcal{K}}$, we conclude that $\mathcal{K}(n,\tau)$ has WPP.

Let A denote the Fraïssé limit of $\mathcal{K}(n,\tau)$. It is clear from the proof of Proposition 3.8 that whenever a finite-dimensional algebra B appears in the construction of A, at some future stage there is a finite-dimensional algebra C such that each direct summand of B embeds into each direct summand of C. By [4, Corollary III.4.3] the limit A is simple.

At each stage of the amalgamation in the proof of Proposition 3.8 we have a $(B,\rho)\in\mathcal{K}(n,\tau)$, and we choose an open set around ρ which is preserved by the relevant embeddings. Given any trace σ on B other than ρ, in a future stage we may amalgamate with (B,ρ) again, this time choosing an open set around ρ which does not include σ. So only the trace ρ is preserved to the limit algebra A, and hence A has a unique trace.

For the remaining claims, we consider $K_0(A)$. For any choice of sequence A_k from $\mathcal{K}(n,\tau)$ such that $A=\overline{\bigcup_{k\geq 1}A_k}$, we have $K_0(A)=\varinjlim K_0(A_k)$ (see [4, Theorem IV.3.3]). Each A_k is a direct sum of exactly n matrix algebras, so as abelian groups, $K_0(A_k)\cong\mathbb{Z}^n$. The maps in $\mathcal{K}(n,\tau)$ are embeddings, and so the maps in the direct limit of K_0 groups are injective. For torsion-free groups rank can be defined directly in terms of linear independence, and it follows that the direct limit of rank n torsion-free abelian groups via injective maps has rank n; therefore we have $\operatorname{rank}(K_0(A))=n$.

Finally, we show that A is not self-absorbing. By the Kunneth formula for C*-algebras [34] there is an injective map $K_0(A)\otimes K_0(A)\to K_0(A\otimes A)$. As $K_0(A)$ has rank n, we have that $K_0(A)\otimes K_0(A)$ has rank n^2, and hence cannot be injected into the rank n group $K_0(A)$. Therefore $K_0(A\otimes A)\not\cong K_0(A)$, and also $A\not\cong A\otimes A$. □

4. The Jiang-Su algebra

The Jiang-Su algebra $\mathcal{Z}$ was constructed by Jiang and Su in [20]. This infinite-dimensional algebra is K-theoretically indistinguishable from the one-dimensional algebra $\mathbb{C}$. The tensorial absorption of the $\mathcal{Z}$ plays a central role in Elliott's classification program of nuclear C*-algebras (see e.g. [8] and the introduction to [33]). $\mathcal{Z}$ exhibits many of the properties of a Fraïssé limit. In this section we show that $\mathcal{Z}$ is indeed a Fraïssé limit. We begin with some basic definitions and properties.

Definition 4.1. Fix $p,q\in\mathbb{N}$. The *dimension drop algebra* $\mathcal{Z}_{p,q}$ is defined to be (we identify $M_p(\mathbb{C})\otimes M_q(\mathbb{C})$ and $M_{pq}(\mathbb{C})$)

$$\mathcal{Z}_{p,q}=\left\{\, f\in C\big([0,1],M_{pq}(\mathbb{C})\big): f(0)\in M_p(\mathbb{C})\otimes 1_q \text{ and } f(1)\in 1_p\otimes M_q(\mathbb{C})\,\right\},$$

considered as a C*-algebra with the operations inherited from $C([0,1],M_{pq}(\mathbb{C}))$.

A dimension drop algebra $\mathcal{Z}_{p,q}$ is *prime* if p and q are co-prime.

Prime dimension drop algebras are projectionless (i.e., do not have projections other than 0 and 1). As an inductive limit of projectionless algebras, $\mathcal{Z}$ is projectionless as well, and moreover its K_0 coincides with K_0 of $\mathbb{C}$.

Given a probability measure μ on $[0,1]$ there is a natural trace τ_μ on $\mathcal{Z}_{p,q}$ given by

$$\tau_\mu(f) = \int_0^1 \tau(f(t))d\mu$$

where τ is the unique trace on $M_{pq}(\mathbb{C})$. By using Riesz representation theorem for bounded linear functionals on $C([0,1])$ and the uniqueness of traces on fibres of $\mathcal{Z}_{p,q}$ one shows that all traces of $\mathcal{Z}_{p,q}$ are of this form, hence $T(\mathcal{Z}_{p,q})$ is affinely homeomorphic to the space of probability measures on $[0,1]$.

We need to remind the reader of a number of facts about measures before we can define the class $\mathcal{K}$ for which $\mathcal{Z}$ is a Fraïssé limit. We say that a probability measure μ on $[0,1]$ is faithful and diffuse if the function $u(t) = \mu([0,t])$ is a strictly increasing and continuous. This will imply that the trace defined above as τ_μ is faithful and μ is diffuse as a measure i.e. for every $F \subseteq [0,1]$ with $\mu(F) > 0$ there is $E \subset F$ such that $\mu(E) < \mu(F)$.

Fact 4.2. *If μ is a faithful and diffuse probability measure on $[0,1]$ and $u(t) = \mu([0,t])$ then for any $f \in C([0,1])$,*

$$\int_0^1 f d\mu = \int_0^1 f(u(t))dt$$

where dt is Lebesgue measure on $[0,1]$.

We will say that a trace τ_μ on $\mathcal{Z}_{p,q}$ is faithful and diffuse if the associated measure is.

Fact 4.3. *Suppose that τ_μ and τ_λ are two faithful and diffuse traces on a prime dimension drop algebra $\mathcal{Z}_{p,q}$ then there is an automorphism σ of $\mathcal{Z}_{p,q}$ such that $\tau_\mu = \tau_\lambda \circ \sigma$.*

Proof. It suffices to prove this when λ is Lebesgue measure on $[0,1]$. Let $u(t) = \mu([0,t])$. Then the map $\sigma : \mathcal{Z}_{p,q} \to \mathcal{Z}_{p,q}$ given by $\sigma(f) = f(u)$ is easily seen to be the desired automorphism. □

The class $\mathcal{K}$ that we will consider is the class of all pairs $(\mathcal{Z}_{p,q}, \tau)$ where p and q are co-prime and τ is a faithful and diffuse trace on $\mathcal{Z}_{p,q}$. The language for this class will contain the usual language of C*-algebras together with a relation for a trace.

The original construction of the Jiang-Su algebra was as an inductive limit of a sequence of prime dimension drop algebras. It has a unique (definable) trace which when we refer to it, we will call τ. When we consider $\mathcal{Z}$ as a structure in our language with a relation for the trace, we will mean that $\mathcal{Z}$ is expanded by this unique trace. The key properties of $\mathcal{Z}$ that we will need are contained in the following lemmas.

Lemma 4.4. *Every $(A,\tau) \in \mathcal{K}$ embeds in a trace-preserving manner into $\mathcal{Z}$. In fact, $\mathcal{Z}$ is an inductive limit of a chain (A_n, τ_n) from $\mathcal{K}$ where $(A_0, \tau_0) = (A, \tau)$. In particular, $\mathcal{Z}$ is a $\mathcal{K}$-structure.*

Proof. These facts follow immediately from the main construction in [20]; see Propositions 2.5 and 2.8 □

The following result is implicit in §3 of [20]; we give a proof for completeness.

Lemma 4.5. *$\mathcal{K}$ has the joint embedding property.*

Proof. Suppose $(p,q)=1$. We will show that $A=\mathcal{Z}_{p,q}$ embeds into $B=\mathcal{Z}_{pq,k}$ for any prime $k>pq$. Because of this inequality, we can write $k=ap+bq$ for some positve a and b. Define a *-homomorphism $\tilde{\varphi}:\mathcal{Z}_{p,q}\to C([0,1],M_{pqk}(\mathbb{C}))$ as follows, for $t\in[0,1]$:

$$\tilde{\varphi}(f)(t)=\begin{pmatrix} f(0) & \dots & 0 & & & \\ \vdots & \ddots & \vdots & & & \\ 0 & \dots & f(0) & & & \\ & & & f(t) & \dots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \dots & f(t) \end{pmatrix}$$

where there are ap copies of $f(0)$ and bq copies of $f(t)$ on the diagonal. $\tilde{\varphi}(f)(0)=f(0)\otimes id$ and we can find a unitary $u(1)$ such that $u^*(1)\tilde{\varphi}(f)u(1)\in id\otimes M_k(\mathbb{C})$. If we choose a continuous path of unitaries u on $[0,1]$ from id to $u(1)$ then $\varphi(f)=u^*\tilde{\varphi}(f)u$ is our desired map. We now want to see that φ can be chosen to be trace-preserving in our class $\mathcal{K}$. In light of Fact 4.3, if τ is the trace induced on B by Lebesgue measure, we need to show that τ restricted to the image of A under φ is a faithful and diffuse trace on A. But from the form of $\tilde{\varphi}$, this is clear.

Finally, suppose $(p,q)=1$ and $(p',q')=1$. Let n be a common multiple of pq and $p'q'$ and k some prime bigger than n. Then from above, both $\mathcal{Z}_{p,q}$ and $\mathcal{Z}_{p',q'}$ embed into $\mathcal{Z}_{n,k}$ preserving any faithful and diffuse trace. □

The following lemma will be critical for establishing that $\mathcal{K}$ has the near amalgamation property. Here, if u is a unitary, $\mathrm{Ad}\,(u):x\mapsto uxu^*$ denotes the inner automorphism associated with u.

Lemma 4.6. *Suppose that $A\in\mathcal{K}$ and $\varphi,\psi:A\to\mathcal{Z}$ are trace-preserving embeddings. If $\bar{a}\in A$ and $\epsilon>0$, then there is a unitary $u\in\mathcal{Z}$ such that*

$$\|(\mathrm{Ad}(u)\circ\varphi)(\bar{a})-\psi(\bar{a})\|<\epsilon$$

Proof. This is an immediate consequence of Robert's [29, Theorem 1.0.1] once we make some observations. Theorem 1.0.1 proves a result about algebras that are noncommutative CW (NCCW) complexes and ones which have stable rank one. Dimension-drop algebras are examples of NCCW complexes. In order for an algebra to be stable rank one, the invertible elements of that algebra must be dense. We shall check the assumptions of Robert's theorem hold for $\mathcal{Z}$. Every invertible in $\mathcal{Z}_{p,q}$ is a continuous function from $[0,1]$ into the set of invertible elements of $M_{pq}(\mathbb{C})$. Since invertible elements are dense in $M_{pq}(\mathbb{C})$, it is an exercise in topology of $[0,1]$ to show that the invertible elements are dense in $\mathcal{Z}_{p,q}$. Since every unitary in $M_{pq}(\mathbb{C})$ is of the form $\exp(ia)$ for a self-adjoint a, a similar exercise shows that every unitary in $\mathcal{Z}_{p,q}$ is of the form $\exp(ia)$ for a self-adjoint a and in particular that the unitary group of $\mathcal{Z}_{p,q}$ is connected. This shows that the group K_1 of $\mathcal{Z}_{p,q}$ is trivial (this is the only fact about K_1 that we will need; we refer the reader to [30] for more information). In particular $\mathcal{Z}$ is an inductive limit of NCCW complexes with trivial K_1 and is stable rank one.

Since prime dimension drop algebras are projectionless so is their inductive limit, $\mathcal{Z}$. Additionally, $\mathcal{Z}$ has a unique trace τ and two positive elements a and b in $\mathcal{Z}$ are approximately unitarily equivalent if and only if $\tau(a^n)=\tau(b^n)$ for all n.

A very special case of Robert's theorem [29, Theorem 1.0.1] implies that if A has stable rank 1 and B is an inductive limit of NCCW complexes with trivial K_1, unique trace, and the above property of $\mathcal{Z}$, then the following hold (see §3 for the definition of Φ_*).

(1) For every trace σ of A there is a unital *-homomorphism $\varphi\colon A \to B$ such that $\varphi_*(\tau) = \sigma$.
(2) Two homomorphism $\varphi, \psi\colon A \to B$ are approximately unitarily equivalent if and only if $\varphi_*(\tau) = \psi_*(\tau)$.

The lemma now follows. □

We can now prove the main result.

Theorem 4.7. *The Jiang-Su algebra $\mathcal{Z}$ with its distinguished trace is the Fraïssé limit of the Fraïssé class $\mathcal{K}$.*

Proof. This is automatic by Lemma 4.6 if we can see that $\mathcal{K}$ is a Fraïssé class. Lemma 4.5 directly shows that $\mathcal{K}$ has the joint embedding property. Since every element of $\mathcal{K}$ embeds into $\mathcal{Z}$, $\mathcal{K}$ has the weak Polish property. We are left to show that $\mathcal{K}$ satisfies the near amalgamation property. Towards this end, suppose that A, B and C are in $\mathcal{K}$ and that $\varphi : A \to B$ and $\psi : A \to C$. By Lemma 4.5, we can choose $D \in \mathcal{K}$ and maps $\varphi' : B \to D$ and $\psi' : C \to D$. Now by Lemma 4.4, we can assume that $\mathcal{Z}$ is an inductive limit of D_n from $\mathcal{K}$ such that $D_0 = D$. So resetting the notation, we have maps φ, ψ from A into D and D begins an inductive chain $\langle D_n : n \in \mathbb{N}\rangle$ leading to $\mathcal{Z}$. By Lemma 4.6, for a fixed $\epsilon > 0$ there is a unitary $u \in Z$ such that

$$\|(\mathrm{Ad}(u) \circ \varphi)(\bar{a}) - \psi(\bar{a})\| < \epsilon/3$$

where $\bar{a}$ are generators for A. By the definability of unitaries, there is some $n \in \mathbb{N}$ and some unitary $u' \in D_n$ so that $\|u - u'\| < \epsilon/3$. D_n will now work as the near amalgam of φ and ψ. □

Remark 4.8. Although this proof shows that the Jiang-Su algebra is a Fraïssé limit, it is a bit unsatisfactory in that it uses the existence of the algebra itself to establish the key properties of the Fraïssé class. Additionally, it relies heavily on [29] in order to prove near amalgamation. In an earlier version of the present paper we asked whether there was a self-contained proof that K is a Fraisse class. Such a proof was found by Masumoto in [25].

5. Lévy automorphism groups

A Polish group G is *extremely amenable* if every continuous action of G on a compact space has a fixed point (see [27]). Suppose that $(H_n, d_n)_{n\in\mathbb{N}}$ is a sequence of compact metric groups equipped with their normalized Haar measures μ_{H_n}. The sequence $(H_n)_{n\in\mathbb{N}}$ has the *Lévy concentration property* if for any sequence $A_n \subset H_n$ of Borel subsets such that $\liminf_n \mu_{H_n}(A_n) > 0$ and for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu_{H_n} \{x \in H_n : \exists a \in A_n, d(a, x) \leq \varepsilon\} = 1;$$

see also [27, Definition 1.2.6 and Remark 1.2.9]. A Polish group is *Lévy* if it admits an increasing sequence $(H_n)_{n\in\mathbb{N}}$ of compact subgroups with dense union with the Lévy concentration property with respect to the metrics induced by a compatible metric on G. Every Lévy group is extremely amenable [27, Theorem 4.1.3].

If M is a II_1 factor then the automorphism group $\mathrm{Aut}(M)$ of M is a Polish group with respect to the topology of pointwise convergence in trace-norm. Similarly if A is a separable C*-algebra then the automorphism group $\mathrm{Aut}(A)$ of A is a Polish group with respect to the topology of pointwise convergence in norm.

Let U_n denote the unitary group of $M_n(\mathbb{C})$. It can be naturally identified with a subgroup of the unitary group of both the hyperfinite II_1 factor $\mathcal{R}$, as well as of the unitary group of a UHF algebra whose supernatural number is divisible by n. The metrics induced by these embeddings correspond to the trace-norm and to the operator norm, respectively. The groups $\mathrm{SU}_n = \{u \in U_n : \det(u) = 1\}$ form a Lévy sequence with respect to either metric ([27, Theorem 4.1.14]). We note that all automorphisms of $M_n(\mathbb{C})$ are inner and that $\mathrm{Aut}(M_n(\mathbb{C}))$ is naturally isomorphic to SU_n, via $u \mapsto \mathrm{Ad}\, u$.

The proof of the first two parts of the following Proposition are well-known (cf. [15]), but for the convenience of the reader we include outlines of their proofs, as well as a more detailed proof of the third claim.

Proposition 5.1. *The automorphism groups of*

(1) *the hyperfinite II_1 factor,*
(2) *UHF algebras, and*
(3) *the AF algebras obtained in Theorem 3.9*

are Lévy and, in particular, extremely amenable.

Proof. By the above, for $n \in \mathbb{N}$ the group $H_n := \{\mathrm{Ad}\, u : u \in \mathrm{SU}_n\}$ can be identified with a compact subgroup of $\mathrm{Aut}(\mathcal{R})$. These groups have the Lévy approximation property, and since $\bigcup_{n\in\mathbb{N}} U_n$ is dense $U(\mathcal{R})$ and all automorphisms of $\mathcal{R}$ are approximately inner, $\bigcup_n H_n$ is dense in $\mathrm{Aut}(\mathcal{R})$. Therefore (1) follows.

The proof of (2) is identical, although the groups H_n are now considered with a different metric.

In order to prove (3), fix $m \geq 2$ and let A be one of the AF algebras constructed in Theorem 3.9 with Bratteli diagram of width m. Since $K_0(A)$ is linearly ordered, all automorphisms of A are approximately inner by Elliott's classification of AF algebras [4, Theorem IV.4.3]. Writing A as an inductive limit of finite-dimensional algebras A_n, we represent $U(A)$ as an inductive limit of $U(A_n)$ and $\mathrm{Aut}(A)$ as an inductive limit of $\mathrm{Aut}(A_n)$. The algebra A_n is a direct sum of matrix algebras $M_{k(i)}(\mathbb{C})$ for $1 \leq i \leq m$ and therefore $\mathrm{Aut}(A_n) \cong \prod_{i\leq m} \mathrm{SU}_{k(i)}$. An inspection of the proof of Theorem 3.9 shows that $\lim_n \min_{i\leq m} k(i) = \infty$. It is now an easy exercise to show that the sequence $\mathrm{Aut}(A_n)$ has the Lévy property, and (3) follows. □

6. A Ramsey theorem for matrix algebras

In this section we deduce from Proposition 5.1 Ramsey-type results for matrix algebras. We will use the correspondence between extreme amenability of a Fraïssé limit and the Ramsey property of the corresponding Fraïssé class established in [26, Theorem 3.10] building on a previous results in the discrete case from [21].

Suppose that $\mathcal{K}$ is a Fraïssé class in the sense of Definition 2.4. If A, B are elements of $\mathcal{K}$ with distinguished set of generators $\bar{a}$ for A, denote by ${}^{A}B$ space of embeddings of A inside B endowed with the metric

$$\rho_{\bar{a}}(\varphi, \psi) = \max_i d(\varphi(a_i), \psi(a_i)).$$

A *coloring* of ${}^{A}B$ is a 1-Lipschitz map $\gamma : {}^{A}B \to [0,1]$.

Suppose that $\mathcal{K}$ satisfies the property that ${}^{A}B$ is compact for every $A, B \in \mathcal{K}$. In this case, the definition of the *approximate Ramsey property* ([26], Def. 3.3) is equivalent to: for every $A, B \in \mathcal{K}$ and every $\varepsilon > 0$, there is $C \in \mathcal{K}$ such that for any coloring γ of ${}^{A}C$ there is $\beta \in {}^{B}C$ such that $\gamma(\beta \circ -)$ varies by at most ε on ${}^{A}B$.

In [26], a version of the following is proved as Proposition 3.4.

Proposition 6.1. *Suppose that $\mathcal{K}$ is a Fraïssé class with limit M and for all $A, B \in \mathcal{K}$, ${}^{A}B$ is compact then the following are equivalent:*

(1) *$\mathcal{K}$ has the approximate Ramsey property.*
(2) *For every $A, B \in \mathcal{K}$, $\varepsilon > 0$, and every coloring γ of ${}^{A}M$, there is $\beta \in {}^{B}M$ such that $\gamma(\beta \circ -)$ varies by at most ε on ${}^{A}B$; we say M has the approximate Ramsey property.*

The following result can be proved with the same methods as [26, Theorem 3.10].

Theorem 6.2. *Suppose that M is the limit of a Fraïssé class $\mathcal{K}$. The following statements are equivalent:*

(1) $\mathrm{Aut}(M)$ *is extremely amenable.*
(2) *$\mathcal{K}$ has the approximate Ramsey property.*

Suppose that B is a unital subalgebra of the hyperfinite II_1 factor $\mathcal{R}$. Endow the space ${}^{M_k(\mathbb{C})}B$ of unital embeddings of $M_k(\mathbb{C})$ into B with the metric

$$d_2(\alpha, \alpha') = \sup_{\|x\| \leq 1} \left\| (\alpha - \alpha')(x) \right\|_2 .$$

The following is an immediate corollary of Proposition 6.1, Theorem 6.2 and the extreme amenability of $\mathrm{Aut}(\mathcal{R})$.

Theorem 6.3. *The class of matrix algebras equipped with the metric d_2 and its Fraïssé limit, $\mathcal{R}$, have the approximate Ramsey property.*

Using the extreme amenability of the automorphism groups of infinite type UHF algebras one can obtain similar results for matrix algebras with respect to the operator norm. If $q = \prod_p p^{n_p}$ for $n_p \in \{0, \infty\}$, then we denote by $\mathbb{M}_q$ the infinite type UHF algebras with associated supernatural number q. For $A \subset \mathbb{M}_q$ define ${}^{M_k(\mathbb{C})}A$ to be the set of embeddings of $M_k(\mathbb{C})$ into A endowed with the metric

$$d(\alpha, \alpha') = \sup_{\|x\| \leq 1} \left\| (\alpha - \alpha')(x) \right\| .$$

Theorem 6.4. *For any supernatural number q, both $\mathbb{M}_q$ and its associated Fraïssé class have the approximate Ramsey property.*

Finally one can use the fact that the algebra $\mathcal{K}(H)$ of compact operators is the Fraïssé limit of the class of full matrix algebras, and that $\mathrm{Aut}(\mathcal{K}(H))$ is extremely amenable to obtain the analogues of the above results where one considers not necessarily unital injective *-homomorphisms as embeddings. The same results hold for the finite width AF algebras and their associated Fraïssé classes as described in section 3.

7. Future work

Both the Jiang-Su algebra and the infinite type UHF algebras are examples of strongly self-absorbing C*-algebras ([37]). A unital C*-algebra D is *strongly self-absorbing* if there is a sequence of unitaries u_n in $D \otimes D$ such that

$$\Phi(a) = \lim_n u_n(a \otimes 1)u_n^*$$

is well-defined for all $a \in D$ and Φ is an isomorphism between D and $D \otimes D$.

In addition to $\mathcal{Z}$ and the infinite type UHF algebras, the only other currently known examples of strongly self-absorbing algebras are the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ together with the tensor products with $\mathcal{O}_\infty$ and infinite type UHF algebras. Strongly self-absorbing algebras play a pivotal role in Elliott's classification program for nuclear, simple, separable, unital C*-algebras (see [31, Chapters 5 and 7] for the role of $\mathcal{O}_2$ and $\mathcal{O}_\infty$ and the more recent [8] and [33] for the role of $\mathcal{Z}$) These algebras also have remarkable model-theoretic properties (see [9, §2.2 and §4.5] and [11]). Every strongly self-absorbing C*-algebra is an atomic model of its theory, and all atomic models can be viewed as Fraïssé limits of their type space. Nevertheless, strongly self-absorbing C*-algebras share a number of properties with the Fraïssé limits not common to all atomic models, and it is natural to conjecture that all known, and perhaps all, strongly self-absorbing algebras can be construed as Fraïssé limits of Fraïssé classes from which information about their automorphism group may be extracted.

Problem 7.1. Let A be a strongly self-absorbing C*-algebra. Is A a nontrivial Fraïssé limit?

Since all strongly self-absorbing algebras are singly generated, and $\mathcal{O}_2$ is moreover the universal algebra with two generators satisfying particularly simple relations, it may be necessary to consider Fraïssé categories other than C*-algebras, such as (unital) operator spaces (see [24]).

The important first step in proving that a nuclear algebra A is strongly self-absorbing is to prove that it is tensorially self-absorbing, i.e., that $A \otimes A \cong A$. Proofs that $\mathcal{O}_2$ and $\mathcal{Z}$ enjoy this property are nontrivial, and Elliott's proof that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ in particular precipitated remarkable progress (see [31], [8]).

In the case of $\mathcal{Z}$, we note that if one considers the class $\mathcal{K}$ of dimension-drop algebras with distinguished traces as used in Section 4, we could modify the construction by considering a new class $\mathcal{K}'$ which is just the closure of $\mathcal{K}$ under taking finite tensor products together with the induced traces. It would be interesting to know if this class is a Fraïssé class. If so, this would give a direct proof that $\mathcal{Z}$ is self-absorbing.

It is possible that viewing other strongly self-absorbing algebras as Fraïssé limits may result in new proofs of tensorial self-absorbtion. Such proofs would give information about these algebras, and this technique may also be useful in understanding Jacelon's non-unital analogue of $\mathcal{Z}$ ([19]).

Problem 7.2. Is Jacelon's simple, monotracial, stably projectionless C*-algebra $\mathcal{W}$ a nontrivial Fraïssé limit? Is $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{W}$?

The construction of $\mathcal{W}$ resembles the construction of $\mathcal{Z}$, with the role of dimension-drop algebras being played by the so-called Razak building blocks ([28]).

Another goal of this research is to shed new light on the automorphisms groups of strongly self-absorbing C*-algebras such as $\mathcal{Z}$, $\mathcal{O}_2$, and $\mathcal{O}_\infty$. For example an affirmative answer to Problem 7.1 would be a first step towards the solution of the following problem.

Problem 7.3. Suppose A is strongly self-absorbing. Is $\mathrm{Aut}(A)$ extremely amenable?

Problem 7.4 ([32, Question 9.1])**.** Is $\mathrm{Aut}(\mathcal{O}_2)$ a universal Polish group?

Note that [13, Theorem 7.4] and the main result of [32] together imply that $\mathrm{Aut}(\mathcal{O}_2)$ induces the universal orbit equivalence relation for Polish group actions. Moreover by Kirchberg's $\mathcal{O}_2$-absorption theorem [22] every simple, separable, nuclear and unital C*-algebra A satisfies $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. In particular the automorphism group of A embeds into the automorphism group of $\mathcal{O}_2$ via the map $\alpha \mapsto \alpha \otimes id_{\mathcal{O}_2}$.

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Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, Canada, M5S 2E4
E-mail address: eaglec@uvic.ca
Current address: Department of Mathematics and Statistics, University of Victoria, P.O. Box 1700 STN CSC, Victoria, British Columbia, Canada, V8W 2Y2
URL: http://www.math.toronto.edu/cjeagle/

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 34, Belgrade, Serbia
E-mail address: ifarah@mathstat.yorku.ca
URL: http://www.math.yorku.ca/∼ifarah

Department of Mathematics and Statistics, McMaster University, 1280 Main Street, West Hamilton, Ontario, Canada L8S 4K1
E-mail address: hartb@mcmaster.ca
URL: http://www.math.mcmaster.ca/∼bradd/

School of Mathematics and Mechanical Engineering, Kharkiv V.N. Karazin National University, Kharkiv, Ukraine
E-mail address: borja.kadets@gmail.com

Department of Mathematics, 340 Rowland Hall, University of California, Irvine, Irvine, CA 92697-3875
E-mail address: vkalashn@uci.edu

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3
E-mail address: lupini@caltech.edu
Current address: Mathematics Department, California Institute of Technology, 1200 E. California Blvd, MC 253-37, Pasadena, CA 91125
URL: http://www.lupini.org