TWO $F_{\sigma\delta}$ IDEALS

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ABSTRACT. We find two $F_{\sigma\delta}$ ideals on $\mathbb{N}$ neither of which is F_σ whose quotient Boolean algebras are homogeneous but nonisomorphic. This solves a problem of Just and Krawczyk ([3, Problem C]).

We consider Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$, where $\mathcal{I}$ is an ideal on $\mathbb{N}$ containing the ideal Fin of finite sets. In [3] Just and Krawczyk formulated several conditions on the ideals $\mathcal{I}, \mathcal{J}$ that guarantee their quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ to be isomorphic. By identifying sets of integers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology. We can therefore assign topological complexity to the ideals of sets of integers. In particular, we have F_σ, $F_{\sigma\delta}$, Borel, and so on, ideals on $\mathbb{N}$.

Just and Krawczyk have proved that the Continuum Hypothesis implies that

(1) all quotients over F_σ ideals are pairwise isomorphic, and

(2) the quotient over the ideal of asymptotic density zero sets, $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} |A \cap n|/n = 0\}$, is isomorphic to the quotient over the ideal of logarithmic density zero sets, $\mathcal{Z}_{\log} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty}(\sum_{i\in A\cap n} 1/i)/(\sum_{i<n} 1/i) = 0\}$.

They have also introduced a class of *EU-ideals* that contains both $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ and proved that under CH all quotients over these ideals are homogeneous and pairwise isomorphic. (A Boolean algebra $\mathcal{B}$ is *homogeneous* if it is isomorphic to $\mathcal{B}_A = \{B \in \mathcal{B} : B \leq A\}$, for every $A \in \mathcal{B} \setminus \{0_\mathcal{B}\}$.) Motivated by this result, Just and Krawczyk posed the following problem.

Problem 1 ([3, Problem C]). Is it true that if $\mathcal{I}, \mathcal{J}$ are $F_{\sigma\delta}$ and not F_σ and both $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are homogeneous, then $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$?

We will prove that this problem has a negative answer. We will also prove that there is an $F_{\sigma\delta}$ ideal whose quotient is not isomorphic to a quotient over any P-ideal. (Recal that $\mathcal{I}$ is a *P-ideal* if for every sequence A_n ($n \in \mathbb{N}$) in $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n.)

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Sequential topology. If $\mathcal{B}$ is a σ-complete Boolean algebra, one can define a topology on $\mathcal{B}$ as follows. A sequence A_n ($n \in \mathbb{N}$) *algebraically converges* to A if

$$\bigvee_{m=1}^{\infty} \bigwedge_{n=m}^{\infty} A_n = \bigwedge_{m=1}^{\infty} \bigvee_{n=m}^{\infty} A_n.$$

A subset of $\mathcal{B}$ is closed if it is closed under taking algebraic limits of sequences in it. Open sets are complements of closed sets. See [4] and [1] for more on this topology on complete Boolean algebras.

It is known that quotients over analytic ideals (or more generally, over ideals that have the property of Baire) are never σ-complete (see [3]). If $\mathcal{B}$ is a (not necessarily σ-complete) Boolean algebra, define a topology τ on $\mathcal{B}$ as follows. A sequence A_n ($n \in \mathbb{N}$) *algebraically converges* to A if

(1) For all m, $B_m = \bigwedge_{n=m}^{\infty} A_n$ exists.
(2) For all m, $C_m = \bigvee_{n=m}^{\infty} A_n$ exists.
(3) Both $\bigwedge_{m=1}^{\infty} C_m$ and $\bigvee_{m=1}^{\infty} B_m$ exist and are equal to A.

A subset of $\mathcal{B}$ is τ-closed if it is closed under taking algebraic limits of sequences in it. τ-open sets are the complements of τ-closed sets.

Proposition 2. *If $\mathcal{I}$ is an analytic P-ideal, then there is a complete metric on $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that induces τ.*

Proof. Let ϕ be a lower semicontinuous submeasure such that $\mathcal{I} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \phi(A \setminus n) = 0\}$, as guaranteed by [5]. Define a metric d_ϕ on $\mathcal{P}(\mathbb{N})/\mathcal{I}$ by

$$d_\phi([A]_{\mathcal{I}}, [B]_{\mathcal{I}}) = \limsup_n \phi((A\Delta B) \setminus n).$$

This metric is complete (see [2, Lemma 1.3.3]). It is easily checked that a sequence is d_ϕ-convergent if and only if it is τ-convergent. □

Theorem 3. *There are two ideals $\mathcal{I}$ and $\mathcal{J}$ such that*

(1) *both $\mathcal{I}$ and $\mathcal{J}$ are $F_{\sigma\delta}$ and neither $\mathcal{I}$ nor $\mathcal{J}$ is F_σ,*
(2) *both quotient algebras over $\mathcal{I}$ and $\mathcal{J}$ are homogeneous,*
(3) *these quotient algebras are not isomorphic.*

Proof. We will take $\mathcal{I}$ and $\mathcal{J}$ to be the following ideals on $\mathbb{Q} \cap [0,1]$:

$$\text{NWD}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : A \text{ is nowhere dense}\}$$

$$\text{NULL}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \text{ is of Lebesgue measure } 0\}.$$

(The closure $\overline{A}$ is taken in $\mathbb{R}$.) To see that NWD($\mathbb{Q}$) is $F_{\sigma\delta}$, enumerate the basis of $\mathbb{Q}$ as $\{U_n\}$, $\mathbb{Q}$ as $\{q_n\}$, and the basis of $\mathbb{Q} \cap U_m$ as $\{V_{mn}\}$. The set

$$K_m = \{A \subseteq \mathbb{Q} : (\exists n) A \cap V_{mn} \subseteq \{q_1, \dots, a_n\}\}$$

is hereditary and F_σ, and $A \in K_m$ if and only if $A \cap U_m$ is nowhere dense. Therefore NWD($\mathbb{Q}$) $= \bigcap_m K_m$.

To see that NULL($\mathbb{Q}$) is $F_{\sigma\delta}$, for each n enumerate all finite unions of rational intervals of measure $\leq 1/n$ and proceed as above, using the compactness of $[0,1]$.

Neither of these ideals is F_σ.

Ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic* if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection h between $\mathbb{N} \setminus B$ and $\mathbb{N} \setminus A$ such that for all $X \subseteq \mathbb{N} \setminus A$ we have

$$X \in \mathcal{I} \qquad \Leftrightarrow \qquad h^{-1}(X) \in \mathcal{J}.$$

Claim 1. *The quotient* $\mathcal{P}(\mathbb{Q})/\operatorname{NWD}(\mathbb{Q})$ *is homogeneous.*

Proof. Let A be a positive set. Then the interior B of $\overline{A}$ is nonempty, hence $B \cap A$ is dense in itself. Thus $B \cap A$ is homeomorphic to $\mathbb{Q}$, and $A \setminus B$ is nowhere dense. The homeomorphism is a Rudin–Keisler isomorphism between $\operatorname{NWD}(\mathbb{Q})$ and $\operatorname{NWD}(\mathbb{Q}) \restriction A$, and it induces an isomorphism between $\mathcal{P}(\mathbb{Q})/\operatorname{NWD}(\mathbb{Q})$ and $\mathcal{P}(A)/\operatorname{NWD}(Q) \restriction A$. □

Claim 2. *The quotient* $\mathcal{P}(\mathbb{Q})/\operatorname{NULL}(\mathbb{Q})$ *is homogeneous.*

Proof. Like in the proof of Claim 1, we need to prove that for every positive A the ideals $\operatorname{NULL}(\mathbb{Q})$ and $\operatorname{NULL}(\mathbb{Q}) \restriction A$ are Rudin–Keisler isomorphic. We shall prove this in two steps.

If A, B are two subsets of $\mathbb{Q} \cap [0,1]$ with the same closure, there is a bijection $f\colon A \to B$ such that $\lambda(\overline{X}) = \lambda(\overline{f''X})$ for all $X \subseteq A$. Let $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$ be 1-1 enumerations. Find a bijection f so that $\lim_i d(a_i, f(a_i)) = 0$, making sure that every isolated point is fixed by f. Such f satisfies the requirements because $\overline{f''X}\Delta\overline{X}$ is countable for every X.

In the second step we prove that for every $K \subseteq [0,1]$ of positive measure there is $g\colon K \to [0,1]$ such that $\lambda(X) = 0$ if and only if $\lambda(g''X) = 0$ for every closed $X \subseteq K$. The function defined by

$$g(a) = \frac{\lambda([0,a] \cap K)}{\lambda(K)}.$$

has the property that $\lambda(g''U) = \lambda(U)/\lambda(K)$ for every interval U. Therefore this equality holds for all Lebesgue-measurable sets, and g is as required.

To conclude the proof, let $A \subseteq \mathbb{Q}$ be positive. By the above, we can find maps $g\colon A \to [0,1]$ and $f\colon (g''A) \to \mathbb{Q}$ such that $f \circ g$ is a Rudin–Keisler isomorphism. □

Lemma 4. *The sequential topology on* $\mathcal{P}(\mathbb{Q})/\operatorname{NWD}(\mathbb{Q})$ *is not Hausdorff.*

Proof. In this proof, by open we mean relatively open in $\mathbb{Q}$ unless otherwise stated. Let us write $\mathcal{I}$ for $\operatorname{NWD}(\mathbb{Q})$. We claim that each open in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$ set containing $[\emptyset]_{\mathcal{I}}$ contains $[\mathbb{Q}]_{\mathcal{I}}$ in its closure. Let $\mathcal{D}$ be an open neighborhood of $[\emptyset]_{\mathcal{I}}$ in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$.

It is straightforward to verify the following two facts about convergence in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$. (The second of these facts is of rather general nature while the first one in characteristic to $\operatorname{NWD}(\mathbb{Q})$.)

(1) If (U_n) is an increasing sequence of open sets, then $[U_n]_{\mathcal{I}} \to [\bigcup_n U_n]_{\mathcal{I}}$
(2) Let U be open (perhaps empty), $q \in \mathbb{Q}$, and let V_n be an open ball around q of radius $1/n$. Then $[U \cup V_n]_{\mathcal{I}} \to [U]_{\mathcal{I}}$.

List elements of $\mathbb{Q}$: $q_0, q_1, q_2, \dots$ By induction, using (2), we construct a sequence of open sets (U_n) with $[U_n]_{\mathcal{I}} \in \mathcal{D}$ and with U_{n+1} being the union of U_n and an open ball around q_{n+1}. Then by (1), $[U_n]_{\mathcal{I}} \to [\bigcup_n U_n]_{\mathcal{I}} = [\mathbb{Q}]_{\mathcal{I}}$. □

Lemma 5. *The sequential topology on* $\mathcal{P}(\mathbb{Q})/\operatorname{NULL}(\mathbb{Q})$ *is Hausdorff.*

Proof. Let $\lambda(A)$ be the Lebesgue measure of A. Let us write $\mathcal{J}$ for $\operatorname{NULL}(\mathbb{Q})$, and let $X = [x]_{\mathcal{J}}$, $Y = [y]_{\mathcal{J}}$, etc. We claim that whenever $\lim_i X_i = Y$ in $\mathcal{P}(\mathbb{Q})/\operatorname{NULL}(\mathbb{Q})$, we have $\lim_i \lambda(\overline{x_i \Delta y}) = 0$. Assume the contrary, and fix a sequence X_i converging to Y such that $\liminf_i \lambda(\overline{x_i \Delta y}) = \delta > 0$. Let $B_n = \bigvee_{i \geq n} X_i$ and $C_n = \bigwedge_{i \geq n} X_i$. Since $B_n \geq X_n \geq C_n$ and $B_n \geq Y \geq C_n$ for all n, for every n we have either $\lambda(\overline{b_n \setminus y}) \geq \delta/2$ or $\lambda(\overline{c_n \setminus y}) \geq \delta/2$.

Let us assume that $\lambda(\overline{c_n \setminus y}) \geq \delta/2$ for infinitely many n.

By making small changes to these sets, we may assume $c_1 \subseteq c_2 \subseteq c_3 \subseteq \cdots \subseteq y$. Therefore we have $\lambda(\overline{y \setminus c_n}) \geq \delta/2$ for all n. The set $F = \bigcap_{n=1}^{\infty} \overline{y \setminus c_n}$ has measure at least $\delta/2$, since $\lambda(\overline{y \setminus c_n}) \geq \delta/2$ for all n and this is a decreasing sequence of closed subsets of $[0,1]$. For each n find $s_n \in y \setminus (c_1 \cup \cdots \cup c_n)$ such that $\inf_{a \in F} d(s_n, a) \leq 1/n$, assuring that the closure of $x = \{s_n : n \in \mathbb{N}\}$ includes F. Then $x \cap c_n$ is finite for all n, moreover $x \subseteq y$, and $[x]_{\mathcal{J}} \neq [\emptyset]_{\mathcal{J}}$. Therefore the sequence C_n does not converge to Y, contrary to our assumption.

Therefore we have $\lambda(\overline{b_n \setminus y}) \geq \delta/2$ for every n. The proof that this case leads to the contradiction is identical to the above.

An easy induction on the sequential rank shows that every τ-closed set is closed in the metric topology induced by λ. Therefore for $y \subseteq \mathbb{Q}$ and $\varepsilon > 0$ the set

$$\{[a]_{\mathcal{J}} : \lambda(\overline{a \cap y}) < \varepsilon\}$$

includes an open neighborhood of $[y]_{\mathcal{J}}$, in turn implying the space is Hausdorff. □

Since the sequential topology is defined in algebraic terms, an isomorphism between Boolean algebras is automatically a homeomorphism. Therefore the two quotients are not isomorphic, and this concludes the proof. □

Note that Lemma 4 and Proposition 2 together imply

Proposition 6. *The quotient $\mathcal{P}(\mathbb{Q})/\operatorname{NWD}(\mathbb{Q})$ is not isomorphic to $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for any analytic P-ideal $\mathcal{I}$.* □

During the course of proving Lemma 5 we have proved that the sequential topology on $\mathcal{P}(\mathbb{Q})/\operatorname{NULL}(\mathbb{Q})$ is stronger than a metric topology. It is not difficult to see that two topologies differ, but even more is true. If I is an ideal on $[0,1]$ that contains all singletons, define the ideal $I(\mathbb{Q})$ on $\mathbb{Q} \cap [0,1]$ by

$$I(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \in I\}.$$

(The closure $\overline{A}$ is taken in $\mathbb{R}$).

Theorem 7. *If I is a σ-ideal on $\mathbb{Q} \cap [0,1]$ containing all singletons, then the sequential topology on $\mathcal{P}(\mathbb{Q})/I(\mathbb{Q})$ is not metric. Therefore the quotient $\mathcal{P}(\mathbb{Q})/I(\mathbb{Q})$ is not isomorphic to $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for any analytic P-ideal $\mathcal{I}$.*

Proof. Define a sequence a_n $(n \in \mathbb{N})$ of subsets of $\mathbb{Q} \cap [0,1]$ by

$$a_{\frac{n(n+1)}{2}+i} = [i/n, (i+1)/n],$$

if $0 \leq i < n$. Then $\lim_{i\to\infty} \lambda(\overline{a_i}) = 0$. However, the sequence $A_i = [a_i]_{I(\mathbb{Q})}$ does not converge to $[\emptyset]_{I(\mathbb{Q})}$ algebraically. This is because $C_n = \bigwedge_{i \geq n} [a_n]_{I(\mathbb{Q})} = [\emptyset]_{I(\mathbb{Q})}$ and $B_n = \bigvee_{i \geq n} [a_n]_{I(\mathbb{Q})} = [\mathbb{Q}]_{I(\mathbb{Q})}$ for all n.

We claim that every subsequence of $\{a_n\}$ has a further subsequence that converges to $[\emptyset]_{I(\mathbb{Q})}$. Once proved, this will imply that the topology is not metric.

For $i \in \mathbb{N}$ let x_i and y_i be the left and right endpoints of the interval a_i. For a subsequence a_{n_i} $(i \in \mathbb{N})$ we can find a subsequence a_{m_i} such that $\lambda(a_{m_i}) < 2^{-i}$ and $\lim_i x_{m_i} = x$ and $\lim_i y_{m_i} = y$ for some x, y. We necessarily have $x = y$, and therefore for $k \in \mathbb{N}$ we have

$$b_k = \overline{\bigcup_{i \geq k} a_{m_i}} = \bigcup_{i \geq k} a_{m_i} \cup \{x\}.$$

Therefore $\lambda(\overline{b_k}) < 2^{-k+1}$. Since $\overline{b_k} \supseteq \overline{b_{k+1}}$ for all k and I is a σ-ideal, the sequence $B_k = \bigvee_{i\geq k}[a_{m_i}]_{I(\mathbb{Q})}$ converges to $[\emptyset]_{I(\mathbb{Q})}$. This proves our claim and concludes the proof. □

There are analytic ideals that are not P-ideals whose quotients are metrizable. For example, all F_σ ideals are of this form, because their quotients are discrete in the sequential topology (this follows from [3]).

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