RIGIDITY CONJECTURES

ILIJAS FARAH

Abstract. If $\mathcal{B}$ is a Boolean algebra and $\mathcal{I}, \mathcal{J}$ are two of its ideals, when are their quotients, $\mathcal{B}/\mathcal{I}$ and $\mathcal{B}/\mathcal{J}$, isomorphic? A sufficient condition is the existence of an automorphism of $\mathcal{B}$ that sends $\mathcal{I}$ onto $\mathcal{J}$. We conjecture that in the case when $\mathcal{B}$ is the power-set of the natural numbers, $\mathcal{P}(\mathbb{N})$, and $\mathcal{I}, \mathcal{J}$ are its 'simply definable' ideals (see §1) other than the Fréchet ideal, Fin, this condition is also necessary. This conjecture is an attempt to subsume the results of [28, 40, 41, 16, 17, 15, 39, 7, 6, 20, 21], where some of its instances have already been proved. Similar phenomena occur in other categories, for example for quotient groups (see [8]).

§1. Introduction. As a Boolean algebra, $\mathcal{P}(\mathbb{N})$ is well understood: it is completely generated by countably many atoms (the singletons). We will concentrate on quotients of this algebra. An *ideal* $\mathcal{I}$ on $\mathbb{N}$ is an ideal of the Boolean algebra $\mathcal{P}(\mathbb{N})$, i.e., it is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements.

We consider $\mathcal{P}(\mathbb{N})$ with its Cantor-set topology, obtained by identifying subsets of $\mathbb{N}$ with their characteristic functions. This enables us to import the Borel structure and Lebesgue measure in the study of subsets of $\mathcal{P}(\mathbb{N})$. We can therefore talk about F_σ, Borel, analytc, and so on ideals on $\mathbb{N}$. (A set is *analytic* if it is a continuous image of a Borel set of the reals.)

Instead of $\mathbb{N}$, we may consider ideals on $\mathbb{Q}$, some countable ordinal, or any countable set, since some ideals can be visualized more easily in this way. To simplify the notation, in our arguments we will pretend that all of the ideals live on $\mathbb{N}$. Some examples of ideals:

$$P_1 = \{A \subseteq \mathbb{N} : 1 \notin A\}$$
$$\mathrm{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$$
$$\mathcal{I}_{1/n} = \left\{A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty\right\}$$
$$\mathcal{Z}_0 = \left\{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\right\}$$

Received by the editors February 24, 2001.
The author acknowledges support received from the National Science Foundation via grant DMS-0070798 and from the PSC-CUNY grant #62785-00-31.

$$\mathrm{NWD}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense}\}$$
$$\mathcal{I}_{\omega^2} = \{A \subseteq \omega^2 : \text{the order-type of } A \text{ is less than } \omega^2\}.$$

Definitions of some of these ideals depend on a parameter, changing of which leads to a whole class of ideals with similar properties (see §2). All of the above ideals have simple definitions. In the Cantor-set topology, the first one is both closed and open, the next two are F_σ, the next two are $F_{\sigma\delta}$, and the last one is $G_{\delta\sigma\delta}$.

To $\mathcal{I}$ we associate an equivalence relation: $AE_{\mathcal{I}}B$ if and only if $A\Delta B \in \mathcal{I}$. We will denote the $E_{\mathcal{I}}$-equivalence class of X by $[X]_{\mathcal{I}}$. This relation is a congruence, and $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is the *quotient algebra over* $\mathcal{I}$.

Basic Question. *How does a change of the ideal $\mathcal{I}$ effect the change of its quotient algebra, $\mathcal{P}(\mathbb{N})/\mathcal{I}$?*

First of all, this question needs to be put in a proper context. It is clear what we mean by saying that two Boolean algebras are isomorphic.

Definition 1.1. Ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic*, $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$, if there are sets $A \in \mathcal{I}$ and $B \in \mathcal{J}$ and a bijection $h\colon (\mathbb{N} \setminus B) \to (\mathbb{N} \setminus A)$ such that for all $X \subseteq \mathbb{N}$ we have $X \in \mathcal{I}$ if and only if $h^{-1}(X) \in \mathcal{J}$.

If $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$ and both $\mathcal{I}$ and $\mathcal{J}$ contain an infinite set, then we may assume that $A = B = \emptyset$, and in this case h is a permutation of the natural numbers. Thus the map $A \mapsto h^{-1}(A)$, used in the lemma below, is an automorphism of $\mathcal{P}(\mathbb{N})$.

Lemma 1.2. *For any two ideals $\mathcal{I}, \mathcal{J}$, if $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$ then $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$.*

Proof Define $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ by
$$\Phi([X]_{\mathcal{I}}) = [h^{-1}(X)]_{\mathcal{J}}.$$
Since $h^{-1}(Y) \in \mathcal{J}$ if and only if $Y \in \mathcal{I}$, Φ is an isomorphic embedding. Since the restriction of h to $\mathbb{N} \setminus B$ is one-to-one, Φ is an isomorphism. ⊣

Does Lemma 1.2 have a converse? A positive answer, that $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$ if and only if $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$, would also be the strongest possible answer to Basic Question. It is therefore not surprising that the answer to an unrestricted version of this question is negative. If $\mathcal{I}$ and $\mathcal{J}$ are maximal proper ideals, then both $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are two-element Boolean algebras, and therefore isomorphic. But if $\mathcal{J} \supseteq \mathrm{Fin}$ is a maximal proper ideal, it is not Rudin–Keisler isomorphic to P_1 (consider $h^{-1}(\{1\})$).

Problem 1.3. *Isolate the optimal conditions under which Lemma 1.2 has a converse.*

By the above example, it is necessary to exclude the possibility of having a nonprincipal maximal ideal. Our approach is to restrict our attention to the ideals that are 'simply definable' in some way. This approach was taken in [7] where we have considered only the ideals that are analytic subsets of $\mathcal{P}(\mathbb{N})$. (Recall that a subset of $\mathcal{P}(\mathbb{N})$ is *analytic* if it is a continuous image of a Borel set of reals.) In §4.1 we will consider (arguably) the largest class of reasonably definable ideals.

§2. Ideals on $\mathbb{N}$. Before we proceed with the analysis of Problem 1.3, let us get acquainted with some examples of ideals. From now on we will consider only the ideals that contain all finite subsets of $\mathbb{N}$, since the principal ideals do not give rise to interesting quotients.

An $A \subseteq \mathbb{N}$ is *$\mathcal{I}$-positive* if $A \notin \mathcal{I}$. We write $\mathcal{I} \restriction A$ for $\mathcal{I} \cap \mathcal{P}(A)$ and consider this to be an ideal on A. An ideal $\mathcal{I}$ is *dense* if every $\mathcal{I}$-positive set A has an infinite subset belonging to $\mathcal{I}$.

2.1. Summable ideals. If $f \colon \mathbb{N} \to \mathbb{R}^+$ then

$$\mathcal{I}_f = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} f(n) < \infty \right\}$$

is an example of a *summable* ideal. Note that this ideal is *proper* (i.e., $\mathbb{N} \notin \mathcal{I}_f$) if and only if $\sum_{n=1}^{\infty} f(n) = \infty$ and that it is dense if and only if $\lim_n f(n) = 0$. Since $\mathcal{I}_f = \bigcup_{m \in \mathbb{N}} \{A : \sum_{n \in A} f(n) \leq m\}$, each $\mathcal{I}_f$ is an F_σ ideal.

A simple (even F_σ) characterization of all pairs (f, g) such that $\mathcal{I}_f \approx_{\mathrm{RK}} \mathcal{I}_g$ can be obtained from [7, Lemma 1.12.5].

2.2. Density ideals. Let $\mathbb{N} = \bigcup_{n=0}^{\infty} I_n$ be a partition of $\mathbb{N}$ into pairwise disjoint finite sets, and let $\mu = \{\mu_n\}_{n=0}^{\infty}$ be a sequence of measures such that μ_n concentrates on I_n. Then

$$\mathcal{Z}_\mu = \{A \subseteq \mathbb{N} : \lim_{n \to \infty} \mu_n(A \cap I_n) = 0\}$$

is an $F_{\sigma\delta}$ ideal. Such ideals are called *density ideals* and they were introduced and studied in [7, §13]. Both Fin and $\mathcal{Z}_0$ are density ideals (see [7, Theorem 1.13.3] for the latter). Dense density ideals are very different from the summable ideals (see [7, Proposition 1.13.14]).

There are many pairs (μ, ν) such that $\mathcal{Z}_\mu \not\approx_{\mathrm{RK}} \mathcal{Z}_\nu$ ([7, Theorem 1.13.12]), but no simple characterization of such pairs is known.

2.3. Ideals induced by submeasures. A map $\phi \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a *submeasure* if $\phi(\emptyset) = 0$, and ϕ is monotonic and subadditive. It is moreover *lower semicontinuous* if $\phi(A) = \lim_{n \to \infty} \phi(A \cap n)$. To a lower semicontinuous submeasure ϕ we associate an ideal

$$\mathrm{Exh}(\phi) = \{A \subseteq \mathbb{N} : \lim_{n \to \infty} \phi(A \setminus n) = 0\}.$$

Such ideals are always P-ideals. (An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence A_n ($n \in \mathbb{N}$) of elements of $\mathcal{I}$ there is an $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite, for all n.) For example, Fin, all summable ideals, and all density ideals are P-ideals. Each summable ideal can be written in the form $\mathrm{Exh}(\phi)$: let $\phi(A) = \sum_{n \in A} f(n)$. Similarly, $\mathcal{Z}_\mu = \mathrm{Exh}(\sup_n \mu_n)$. The ideal $\mathrm{Exh}(\phi)$ is dense if and only if $\lim_n \phi(\{n\}) = 0$. By a result of Solecki ([32, 33], every analytic P-ideal is of the form $\mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ. In particular, all analytic P-ideals are automatically $F_{\sigma\delta}$.

2.4. F_σ ideals. A G_δ ideal that contains Fin has to be equal to $\mathcal{P}(\mathbb{N})$, by the Baire Category Theorem. Hence F_σ ideals are the simplest nontrivial ideals on $\mathbb{N}$. By a result of K. Mazur ([26, Lemma 1.2]) an ideal $\mathcal{I}$ is F_σ if and only if

$$\mathcal{I} = \mathrm{Fin}(\phi) = \{A \subseteq \mathbb{N} : \phi(A) < \infty\}.$$

for some lower semicontinuous submeasure ϕ. In spite of their simplicity, F_σ-ideals form a very rich and well-studied structure (see [23]).

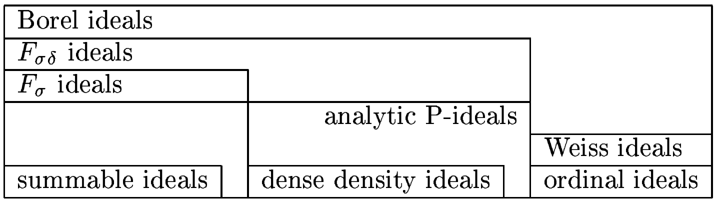

FIGURE 1. Classes of Borel ideals

2.5. Nonpathological ideals. A submeasure ϕ is *nonpathological* if for all A in the domain of ϕ we have $\phi(A) = \sup_{\mu \leq \phi} \mu(A)$, where the supremum is taken over all measures μ pointwise dominated by ϕ. (This definition of a nonpathological submeasure was introduced in [7] and it differs somewhat from the standard one; what we call pathological is sometimes called *weakly pathological.*) An ideal is *nonpathological* if it is of the form $\mathrm{Exh}(\phi)$ or $\mathrm{Fin}(\phi)$ for some lower semicontinuous nonpathological submeasure ϕ. All summable and all density ideals are obviously nonpathological. Nonpathological ideals were introduced in [7, §1] in an attempt to describe a class of ideals for which Todorcevic's conjecture (Conjecture 4.1) is true. It turns out that essentially all ideals occurring in the literature are nonpathological. In §4 we will see that the nonpathological ideals have a very interesting property related to Basic Question and Problem 1.3.

2.6. Other $F_{\sigma\delta}$ ideals. While the structure of F_σ ideals and analytic P-ideals is well-understood, some very natural questions on the structure of $F_{\sigma\delta}$ ideals are still open (see [34]). An interesting example of an $F_{\sigma\delta}$ ideal, suggested to us by Michal Hrušák, is $\mathrm{NWD}(\mathbb{Q})$. It is $F_{\sigma\delta}$: If $\mathcal{B}$ is a countable basis for $\mathbb{Q}$ consisting of nonempty sets, then $A \in \mathrm{NWD}(\mathbb{Q})$ if and only if $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(V \subseteq U$ and $V \cap A = \emptyset\}$. By [11], this ideal is moreover homogeneous (an ideal $\mathcal{I}$ is *homogeneous* if $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{I} \restriction A$ for every $\mathcal{I}$-positive set A). This answers [7, Question 3.7.6], where it was asked whether there are any homogeneous analytic ideals other than Fin and the ideals in §§2.7–2.8 below.

2.7. Ordinal ideals. For a countable ordinal α let $\mathcal{I}_\alpha$ be the family of all subsets of α of strictly smaller order type. This family is an ideal if and only if α is an *indecomposable ordinal*, i.e., if α is not equal to the sum of two strictly smaller ordinals; equivalently, if $\alpha = \omega^\beta$ for some ordinal β. These ideals can also be considered as iterated Fubini products of the ideal Fin (see [21]). All the ordinal ideals are clearly homogeneous, and $\mathcal{I}_\omega = \mathrm{Fin}$ is the only P-ideal among them. All $\mathcal{I}_\alpha$ are Borel ideals, and each $\mathcal{I}_{\omega^\alpha}$ is by a result of Zafrany $\Sigma^0_{2\alpha}$-complete. In particular, these ideals have arbitrarily high Borel complexity, and they are pairwise Rudin–Keisler nonisomorphic.

2.8. Cantor–Bendixson ideals. Let X be a countable topological space, and let α be a countable ordinal. Let

$$\mathrm{CB}_\alpha(X) = \{Y \subseteq X : \text{ the Cantor–Bendixson rank of } Y \text{ is } < \alpha\}.$$

Then $\mathrm{CB}_\alpha(X)$ is an ideal if and only if α is additively indecomposable. A special case of Cantor–Bendixson ideals are the *Weiss ideals*, $\mathcal{W}_{\omega^\alpha} = \mathrm{CB}_\alpha(\omega^\alpha)$. This is the ideal of all subsets of ω^α that do not contain a closed copy of ω^α, and it was suggested by W. Weiss ([42]). The ideal $\mathcal{W}_\omega = \mathrm{Fin}$ is the only P-ideal among these ideals. These ideals are also homogeneous, have arbitrarily high Borel complexities, and they are pairwise Rudin–Keisler nonisomorphic ([21]). They are also Rudin–Keisler nonisomorphic to all ordinal ideals except Fin.

§3. Liftings. We begin our analysis of Problem 1.3 by looking at the connecting maps between quotients. Let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be a homomorphism. A map $\Phi_*\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $[\Phi_*(X)]_\mathcal{J} = \Phi([X]_\mathcal{I})$ for all X, or equivalently, such that the diagram ($\pi_\mathcal{I}$ is the natural projection of $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})/\mathcal{I}$)

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{\Phi_*} & \mathcal{P}(\mathbb{N}) \\ \downarrow{\scriptstyle \pi_\mathcal{I}} & & \downarrow{\scriptstyle \pi_\mathcal{J}} \\ \mathcal{P}(\mathbb{N})/\mathcal{I} & \xrightarrow[\Phi]{} & \mathcal{P}(\mathbb{N})/\mathcal{J} \end{array}$$

commutes, is a *lifting* (or *representation*) of Φ. The reader should be warned that we **do not** require Φ_* to have any algebraic properties; in particular it need not be a homomorphism.

The simplest way to describe a homomorphism is via one of its liftings. The homomorphism defined in the proof of Lemma 1.2 has a particularly simple lifting; it is a map of the form

$$\Phi_h(A) = h^{-1}(A)$$

for some $h\colon \mathbb{N} \to \mathbb{N}$. We say that such a lifting is *completely additive*, since it preserves the infinitary Boolean operations, namely it satisfies the formulas

$$\Phi_*(A^\complement) = (\Phi_*(A))^\complement,$$
$$\Phi_*(A) = \bigcup_{n\in A} \Phi_*(\{n\}).$$

If Φ is an isomorphism and it has a completely additive lifting Φ_h, then h is a Rudin–Keisler isomorphism between the underlying ideals. Thus Problem 1.3 is tightly associated with the question which homomorphisms have completely additive liftings. We also consider *additive liftings*, the liftings that preserve the finitary Boolean operations of $\mathcal{P}(\mathbb{N})$.

3.1. Topological liftings. Since we consider $\mathcal{P}(\mathbb{N})$ with its Cantor space topology, the methods of set theory can be applied to liftings that are simple in a topological way, for example continuous, Lebesgue-measurable or Baire-measurable.

A subset of $\mathcal{P}(\mathbb{N})$ is *meager* if it is a countable union of nowhere dense sets. A set $X \subseteq \mathcal{P}(\mathbb{N})$ has the *Property of Baire* if the symmetric difference $X\Delta U$ is meager for some open U. A function is *Baire-measurable* if the preimage of every open set has the property of Baire.

Relationship between simple liftings is given in Figure 2. The fact that every Baire-measurable lifting can be turned into a continuous one was proved in [40, p.

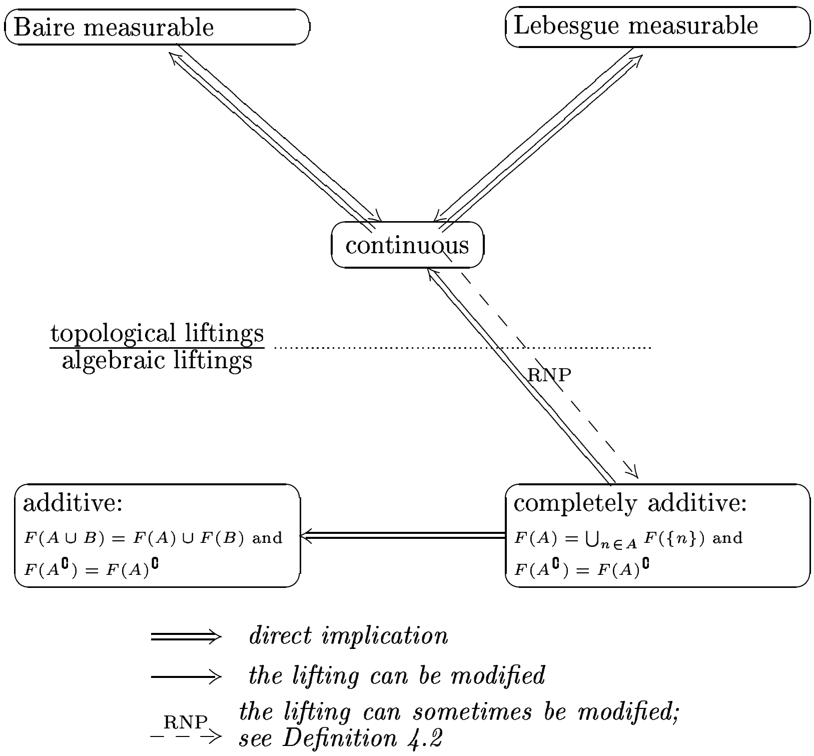

FIGURE 2. Types of liftings

132] and [39, Theorem 3]. An analogous result for Lebesgue-measurable liftings was proved independently in [20] and [13, Proposition 1C]. See [8] for more on transformations between simple types of liftings.

Finitely additive liftings that are not completely additive did not attract much attention, but they may turn out to be interesting. Every known homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself that can be constructed without using additional Set-theoretic axioms has an additive lifting (see the definition of $F_{\mathcal{U}}$ in [7, Example 3.2.3]). In a situation when all endomorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ have additive liftings every complete Boolean algebra embeddable into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ would have to be σ-centered (see [7, Proposition 4.11.7]). This conclusion was conjectured to be consistent by A. Dow. It is not known whether it is consistent with the Axiom of Choice (see [7, Question 3.14.2 and Question 4.11.4]). However, every endomorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ that has a Baire-measurable lifting has a (completely) additive lifting (essentially by [40], see also Theorem 4.3).

§4. Quotients over Borel ideals. The example given before Problem 1.3 involves a maximal nonprincipal ideal on $\mathbb{N}$. Such ideals are never Borel, as a matter of fact they are not 'definable in a reasonable way.' (We will return to this in §4.1.) In this paper we will restrict our attention to simply definable ideals. Let us first state a conjecture of Todorcevic that has initially inspired the research that resulted in [7].

CONJECTURE 4.1 (Todorcevic's conjecture, [39, Problem 1]). *Suppose $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ and that Φ is a homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire lifting. Then Φ has a completely additive lifting.*

DEFINITION 4.2 ([7, §1]). An ideal $\mathcal{J}$ has the *Radon–Nikodym property* if every homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ with a Baire-measurable lifting has a completely additive lifting.

Thus Todorcevic's conjecture can be restated as 'all analytic P-ideals have the Radon–Nikodym property.' In [7, §1.7] this conjecture was proved to be equivalent to a finite combinatorial statement (see also [8]). This was used to prove the following (the nonpathological ideals were defined in §2.5).

THEOREM 4.3 (Farah, [7, Theorem 1.9.1]). *Every nonpathological analytic P-ideal has the Radon–Nikodym property. Hence Todorcevic's Conjecture is true for all nonpathological analytic P-ideals.* ⊣

This result relied on Ulam-stability of 'approximate homomorphisms.' For more on this, see [8] where it was also shown that similar results can be proved for quotient groups instead of quotient Boolean algebras.

The class of nonpathological ideals contains virtually all examples of analytic P-ideals previously appearing in the literature. In particular, all summable and all density ideals are nonpathological. The proof of Theorem 4.3 used the finitization of Todorcevic's conjecture mentioned before. The observation that this finitization can be avoided has led to the following definition.

DEFINITION 4.4 (Kanovei–Reeken, [20, 21]). An ideal $\mathcal{I}$ is an *F-ideal* if for every $\varepsilon > 0$ and $X \subseteq \mathbb{N}\times 2^{\mathbb{N}}$ such that each vertical section $X_n = \{x : (n,x) \in X\}$ is Haar-measurable and has measure at least ε the set (X^y is the vertical section, $\{n : (n,y) \in X\}$)

$$X^+ = \{y \in 2^{\mathbb{N}} : X^y \notin \mathcal{I}\}$$

is Haar-measurable and has measure at least ε.

A minor modification of the proof of Theorem 4.3 shows that all F-ideals have the Radon–Nikodym property. Also, all nonpathological ideals are F-ideals. This is essentially a consequence of a result of Christensen [4], to the effect that nonpathological submeasures satisfy a variant of Fubini's theorem. But much more is true.

THEOREM 4.5 (Kanovei–Reeken, [20, 21]). *All nonpathological F_σ ideals, as well as all ordinal ideals and all Weiss ideals are F-ideals. In particular, they all have the Radon–Nikodym property.* ⊣

In [7, Theorem 1.9.1] it was shown that some restriction in Todorcevic's conjecture is necessary by constructing certain 'pathological' ideals which violate this conjecture. It should be noted that the status of this conjecture is still unknown if one adds the requirement that Φ is onto, or equivalently, an isomorphism between two analytic quotients. In fact, the following variation on Todorcevic's conjecture is still open (two quotients are *Baire-isomorphic* if there is an isomorphism that has a Baire-measurable—equivalently, continuous—lifting).

CONJECTURE 4.6. *The following are equivalent for any two Borel ideals $\mathcal{I}, \mathcal{J}$:*
(1) *The quotients over $\mathcal{I}$ and $\mathcal{J}$ are Baire-isomorphic.*
(2) $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$.
Moreover, every isomorphism as in (1) has a completely additive lifting.

LEMMA 4.7. *Conjecture 4.6 is true for isomorphisms between quotients over ideals that have the Radon–Nikodym property.*

PROOF Let $\mathcal{I}$ and $\mathcal{J}$ be ideals with the RNP and let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be an isomorphism. There are partial functions h and g from $\mathbb{N}$ into $\mathbb{N}$ such that $A \mapsto h^{-1}(A)$ is a lifting of Φ and $A \mapsto g^{-1}(A)$ is a lifting of Φ^{-1}. A standard argument shows that the set $X = \{n : g(h(n)) \neq n\}$ belongs to $\mathcal{J}$ (see e.g., the proof of [7, Proposion 1.4.6]). Since h is one-to-one on $\mathbb{N}\setminus X$, it is a Rudin–Keisler isomorphism between $\mathcal{I}$ and $\mathcal{J}$. ⊣

PROPOSITION 4.8. *If $\mathcal{I}$ is an F-ideal and that there is an embedding of $\mathcal{P}(\mathbb{N})/\mathcal{J}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Baire-measurable lifting. Then $\mathcal{J}$ is an F-ideal.*

PROOF This is very similar to the proof of [7, Proposition 1.10.3]. By the Radon–Nikodym property of F-ideals, the isomorphism has a completely additive lifting, or the form $A \mapsto h^{-1}(A)$ for some $h\colon \mathbb{N} \to \mathbb{N}$. Then h is a Rudin–Keisler reduction between $\mathcal{I}$ and $\mathcal{J}$, i.e., we have $A \in \mathcal{J}$ if and only if $h^{-1}(A) \in \mathcal{I}$. Fix $X \subseteq \mathbb{N} \times 2^{\mathbb{N}}$ such that every vertical section has measure at least ε, for some fixed $\varepsilon > 0$. We need to prove that the set $\{z : X^z \notin \mathcal{J}\}$ has measure at least ε.

Define $Y \subseteq \mathbb{N} \times 2^{\mathbb{N}}$ so that $Y_n = X_{h(n)}$. Since $\mathcal{I}$ is an F-ideal, the set

$$Z = \{z \in Y : Y^z \notin \mathcal{I}\}$$

has measure at least ε. Fix $z \in Z$. Since $(n, z) \in Y$ implies $(h(n), z) \in X$, we conclude that $Y^z \notin \mathcal{I}$ implies $X^z \notin \mathcal{J}$, hence $Z \subseteq \{z \in X : X^z \notin \mathcal{J}\}$, and therefore $\mathcal{J}$ is an F-ideal. ⊣

By Proposition 4.8 and Lemma 4.7 we have a partial answer to Conjecture 4.6.

PROPOSITION 4.9. *Conjecture 4.6 is true for every pair of ideals $\mathcal{I}, \mathcal{J}$ such that at least one of them is an F-ideal.* ⊣

At present it is unclear which analytic ideals have the Radon–Nikodym property, but it is known that not all ideals with the RNP are F-ideals. By [10, Proposition 9.4], $\mathrm{NWD}(\mathbb{Q})$ has the RNP but it is not an F-ideal.

Several classes of ideals, such as summable ideals or nonpathological ideals, share the property of F-ideals proved in Proposition 4.8 (see [7, §1.10]).

QUESTION 4.10. *Assume $\mathcal{I}$ is a Borel ideal that has the Radon–Nikodym property and that there is an embedding of $\mathcal{P}(\mathbb{N})/\mathcal{J}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that has a Baire-measurable lifting. Does $\mathcal{J}$ necessarily have the Radon–Nikodym property?*

Similarly to Todorcevic's conjecture, Conjecture 4.6 has an equivalent reformulation in terms of Ulam-stability of *approximate isomorphisms* between finite Boolean algebras (see [7, Question 1.14.4]).

4.1. Quotients over projective ideals and beyond. A reader not interested in ideals whose complexity is beyond Borel (or analytic) may wish to go straight to §5. We will consider all ideals $\mathcal{I}$ of the form $\mathcal{I} = \{A \subseteq \mathbb{N} : \phi(A)\}$ for a first-order formula ϕ with ordinals and reals as parameters. Under a suitable large cardinal assumption, these are exactly the those ideals that belong to $L(\mathbb{R})$, the smallest class containing all reals and all ordinals and closed under the primitively recursive set functions. Following [37], we say that ideals (or, subsets of $\mathcal{P}(\mathbb{N})$) that belong to $L(\mathbb{R})$ are *definable*. By a result of Solovay [35], if there is an inaccessible cardinal κ then after the Levy collapse of all cardinals strictly less than κ to ω_1 all definable subsets of $\mathcal{P}(\mathbb{N})$ have the Property of Baire and are Lebesgue-measurable. In particular, there are no nonprincipal ultrafilters on $\mathbb{N}$ hence the example stated before Problem 1.3 does not apply. We say that $L(\mathbb{R})$ of such forcing extension is a *Solovay model*. Moreover, the existence of more substantial large cardinals (like a weakly compact Woodin cardinal) implies that $L(\mathbb{R})$ is elementarily equivalent to some Solovay's model ([31, 12]). Hence if sufficiently large cardinals exist the theory of $L(\mathbb{R})$ is unchangeable by forcing and canonical: all definable sets of reals are Lebesgue measurable, have the Property of Baire, are determined, every definable $X \subseteq \mathbb{R}^2$ can be uniformized on a dense G_δ set, and so on (see [31], [2]).

In the following conjecture and elsewhere by 'some Solovay's model' we mean 'a Solovay's model obtained from sufficiently large cardinals.'

CONJECTURE 4.11. *In some Solovay's model the following are equivalent for any two definable ideals $\mathcal{I}, \mathcal{J}$:*

(1) $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$,
(2) $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$.

Moreover, every isomorphism as in (1) has a completely additive lifting.

It is possible that the assumptions of Lemma 4.12 below imply the conclusion of Conjecture 4.11. By Lemma 1.2, the interesting direction is (1) implies (2). Since in Gödels constructible universe and other canonical inner models for moderately large cardinals there is a projective well-ordering of $\mathcal{P}(\mathbb{N})$ ([25]), and therefore a projective nonprincipal ultrafilter on $\mathbb{N}$, in these models (1) does not imply (2).

The assumptions of the following lemma are true in Solovay's model.

LEMMA 4.12. *Assume that every definable relation can be uniformized on a dense G_δ set and that all definable sets of reals have the property of Baire. Then every definable homomorphism between quotients over definable ideals has a continuous lifting.*

PROOF Assume Φ is in $L(\mathbb{R})$, and let

$$X = \{(A, B) \in \mathcal{P}(\mathbb{N})^2 : \Phi([A]_{\mathcal{I}}) = [B]_{\mathcal{J}}\}.$$

Let F be a function that uniformizes X on a dense G_δ set. We may assume F is Baire-measurable. Hence Φ has a Baire-measurable lifting, and therefore a continuous lifting as well. $\dashv$

By Lemma 4.12, Conjecture 4.11 is equivalent to:

CONJECTURE 4.13. *In some Solovay's model every isomorphism between quotients over ideals on $\mathbb{N}$ that has a continuous lifting has a completely additive lifting.*

As an immediate consequence of Lemma 4.12, under its assumptions an isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ has a continuous lifting if and only if its inverse has a continuous lifting.

§5. The influence of the Continuum Hypothesis. The discussion in §4.1 suggests that a 'definable' version of Problem 1.3 may have a satisfactory answer. While our decision to consider only the definable ideals is justified by the example given before Problem 1.3, considering only the definable isomorphisms seems less natural.

5.1. Saturatedness of quotients. The early study of quotients related to $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ has concentrated on $\mathbb{R}^{\mathbb{N}}/\mathrm{Fin}$, the structure of real-valued sequences partially ordered by $f \prec g$ if and only if $\lim_{n\to\infty} g(n) - f(n) = \infty$. One of the important features of this structure, discovered by P. du Bois-Reymond and J. Hadamard, is that it has no countable *gaps*. Namely, if $\mathcal{A}, \mathcal{B}$ are two countable subsets of $\mathbb{R}^{\mathbb{N}}/\mathrm{Fin}$ such that for every $f \in \mathcal{A}$ and every $g \in \mathcal{B}$ we have $f \prec g$, then there is h such that $f \prec h \prec g$ for all $f \in \mathcal{A}$ and all $g \in \mathcal{B}$.

Recall that a Boolean algebra $\mathcal{B}$ is *atomless* if for every positive $b \in \mathcal{B}$ there is a positive $c \leq b$ such that $b \setminus c$ is positive. If $\mathcal{I}$ is a proper ideal on $\mathbb{N}$ including Fin, the quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless. Since the theory of atomless Boolean algebras is $\aleph_0$-categorical, all the quotients that we consider in this note are elementarily equivalent (see [3]).

We will use the term *saturated* in the model-theoretic sense (see e.g., [3]). Hence '$\mathcal{B}$ is *countably saturated*' (or '$\mathcal{B}$ is $\aleph_1$*-saturated*') means that every countable finitely satisfiable type with parameters in $\mathcal{B}$ is realized in $\mathcal{B}$. Continuum Hypothesis implies that all countably saturated Boolean algebras of size $2^{\aleph_0}$ (in particular, all quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as above) are saturated, and therefore isomorphic.

THEOREM 5.1 (Just–Krawczyk, [18]). *Every quotient over an F_σ ideal is countably saturated.* ⊣

Thus we have the following result, first proved for summable ideals by Erdös and Monk.

COROLLARY 5.2 (Just–Krawczyk, [18]). *CH implies that all quotients over F_σ ideals are pairwise isomorphic.* ⊣

Note that an isomorphism between two saturated models is constructed via the back-and-forth argument of transfinite length, and an isomorphism constructed in this way is unlikely to have a continuous lifting.

But quotients over many natural ideals are not countably saturated. For example, A_i ($i \in \mathbb{N}$) such that $\lim_n |A_i \cap n|/n = 1/i$ and $A_i \supseteq A_{i+1}$ for all i form a sequence of $\mathcal{Z}_0$-positive sets with no $\mathcal{Z}_0$-positive lower bound. Moreover, a quotient over an analytic P-ideal $\mathcal{I}$ is countably saturated if and only if $\mathcal{I}$ is F_σ (essentially proved in [18], see [7, the end of §1.3]). Still, in [18, Theorem 3] it was shown that the quotients over many different density ideals, including the ideals $\mathcal{Z}_0$ and $\mathcal{Z}_{\log} = \{A \subseteq \mathbb{N} : \limsup_n (\sum_{i \in A \cap n} 1/i)/\log n = 0\}$, are pairwise isomorphic under CH. It is not difficult to see that $\mathcal{Z}_0 \not\approx_{\mathrm{RK}} \mathcal{Z}_{\log}$ (see [7, Proposition 1.13.13] for a more general result).

Using results of [18], in [7, §3.14] it was shown that there are at least six isomorphism types of analytic quotients in any model of set theory.

QUESTION 5.3 ([7, Question 3.14.3]). *Are there infinitely (or even uncountably) many analytic P-ideals whose quotients are, provably in ZFC, pairwise nonisomorphic?*

This question, as well as the corresponding question for quotients over arbitrary analytic (or definable) ideals is still open. At the moment when [7] was finished, there was a large supply of potential candidates for finding infinitely many pairwise nonisomorphic analytic quotients. For example, it seemed likely that quotients over Weiss ideals and ordinal ideals will turn out to be pairwise nonisomorphic. Also, a large class of analytic P-ideals was constructed by Louveau and Velickovic in [24]. They have constructed a family $\mathcal{J}_A$ $(A \subseteq \mathbb{N})$ of analytic P-ideals such that $\mathcal{P}(\mathbb{N})/\mathcal{J}_A$ is Baire-isomorphic with $\mathcal{P}(\mathbb{N})/\mathcal{J}_B$ (in a rather weak sense) if and only if $A \Delta B$ is finite.

THEOREM 5.4 (Farah, [9]). *Assume the Continuum Hypothesis.*

1. *The quotients over all Louveau–Velickovic ideals are isomorphic.*
2. *Quotients over all ordinal ideals and all Weiss ideals are countably saturated, and therefore isomorphic to $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.*
3. *If $\mathcal{Z}_\mu$ and $\mathcal{Z}_{\mu'}$ are dense and such that*

$$\limsup_{n\to\infty} \mu_n(I_n) = \limsup_{n\to\infty} \mu'_n(I'_n) = \infty,$$

then their quotients are isomorphic. ⊣

Therefore (1) and (2) show that under CH all of the analytic quotients mentioned in this paper or in [7] fall into one of finitely many isomorphism classes. It is curious that (3) implies that under CH there are only two isomorphism classes of dense density ideals.

While the Conjecture 7.3 implies that a quotient over an analytic P-ideal cannot be isomorphic to a quotient over an analytic ideal that is not a P-ideal, Theorem 5.4 shows that under CH the situation is quite different. On the other hand, in [11] it was proved that the quotient over $\mathrm{NWD}(\mathbb{Q})$ cannot be isomorphic to a quotient over any analytic P-ideal.

It is worth noting that the Continuum Hypothesis provides the optimal ambient for testing Question 5.3 (see [19] for the definitions).

THEOREM 5.5 (Woodin, [43]). *Assume there are class many measurable Woodin cardinals. If $\phi(X)$ is a first-order statement of $L(\mathbb{R})$ and in some forcing extension there is $X \subseteq \mathbb{R}$ such that $\phi(X)$ holds, then $(\exists X)\phi(X)$ holds in every forcing extension that satisfies CH.* ⊣

This theorem applies in the case when $\phi(X)$ is saying 'X is a lifting of an isomorphism between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$.' Therefore if two Borel ideals have isomorphic quotients in some forcing extension, then they have isomorphic quotients in every forcing extension that satisfies CH, at least assuming large cardinals.

§6. When CH fails. In §5 we have seen that the existence of a well-ordering of the reals all of whose proper initial segments are countable can be used to construct isomorphisms between rather different ideals. But can such isomorphisms be constructed by using a weaker assumption? In general, if the size of the continuum is bigger than $\aleph_1$, an attempt to construct an isomorphism using the back-and-forth method runs into a problem. The following is a generalization of the classical result of Hausdorff ([14]).

THEOREM 6.1 (Todorcevic, [39]). *Let $\mathcal{I}$ be an analytic ideal. Then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not $\aleph_2$-saturated. More precisely, there are two families $\mathcal{A}, \mathcal{B}$ of size $\aleph_1$ such that*

(1) *$A \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, yet*
(2) *there is no C such that $A \setminus C \in \mathcal{I}$ and $C \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$.* ⊣

We say that families $\mathcal{A}, \mathcal{B}$ satisfing (1) are *orthogonal over $\mathcal{I}$*, or that they form a *pregap in* $\mathcal{P}(\mathbb{N})/\mathcal{I}$. If (2) fails and C is a witness, then C *separates* $\mathcal{A}$ from $\mathcal{B}$ over $\mathcal{I}$. If both (1) and (2) hold, then $(\mathcal{A}, \mathcal{B})$ is a *gap* in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ (for more on gaps in analytic quotients see [39], [37], or [7, Chapter 5]).

Methods of [30] can be used to show that in certain situations when CH fails (in particular, in Cohen's original model for $2^{\aleph_0} = \aleph_2$) all countably saturated quotients over Borel ideals are still isomorphic. The paper [30] was motivated by the study of automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. An automorphism of a quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is *trivial* if and only if it has a completely additive lifting (equivalently, if and only if it is induced by a Rudin–Keisler automorphism of the ideal $\mathcal{I}$). The saturatedness of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ implies that it has the maximal number of (mostly) nontrivial automorphisms (a result first proved by W. Rudin, [27]).

The following was the first result in the direction considered in this paper.

THEOREM 6.2 (Shelah, [28]). *There is a forcing extension in which all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are trivial.* ⊣

Once the Shelah's approach was systematized ([40, 16]) it became clear that the proof of Theorem 6.2 naturally splits into two parts:

(a) In some forcing extension, all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ have continuous liftings.
(b) An automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ has a continuous lifting if and only if it has a completely additive lifting.

Since part (b) belongs to the discussion of §4, let us concentrate on (a). Roughly, its proof proceeds as follows. If Φ is an automorphism with no continuous lifting, then there is a forcing $\mathcal{P}_\Phi$ that adds a set $X \subseteq \mathbb{N}$ such that the following families of ground-model sets (Φ_* is a lifting of Φ):

$$\mathcal{A}_{\Phi,X} = \{\Phi_*(A) : A \in (\mathcal{P}(\mathbb{N}))^V, A \setminus X \in \mathrm{Fin}\}$$
$$\mathcal{B}_{\Phi,X} = \{\Phi_*(B) : B \in (\mathcal{P}(\mathbb{N}))^V, B \cap X \in \mathrm{Fin}\}$$

are forced to form a gap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$; hence in the extension Φ cannot be extended to a homomorphism. Such forcings are iterated, taking care (i) that every nontrivial automorphism Φ is destroyed and (ii) that the gap $\mathcal{A}_{\Phi,X}, \mathcal{B}_{\Phi,X}$ remains a gap in the final model. The first task is accomplished by using a standard bookkeeping device; the second one required inventing a new method called *oracle-chain condition* (see [28, §V]).

CONJECTURE 6.3. *Assume there is a weakly compact cardinal. Then there is a forcing extension in which all isomorphisms between quotients over definable ideals have continuous liftings.*

§7. Axiomatic approach. At the end of §5 we have seen that CH implies an extreme answer to Problem 1.3—roughly, any two quotients that could possibly be isomorphic are isomorphic under CH. In this section we attempt to isolate the assumptions under which the situation is the opposite, i.e., two (simply definable) ideals have isomorphic quotients if and only if the ideals are isomorphic. This would be an instance of a frequently encountered phenomenon in mathematics, that connecting maps between definable structures are usually definable themselves.

A natural ambient for this program seems to be the Proper Forcing Axiom, PFA. A fragment of the second-order theory of the uncountable turns out to be rather canonical under PFA (see [36, §8], [38]). This phenomenon is explained by Woodin ([44]) who defines a model in which the Π_2-theory of $\mathcal{P}(\omega_1)$ is maximal. Hence in this model many peculiar objects on ω_1 (like a Suslin tree or a well-ordering of the reals in type ω_1) do not exist. It is not surprising that a large fragment of PFA holds in his model. We are looking for a similar situation for a fragment of the second-order theory of $\mathcal{P}(\mathbb{N})$.

Although the forcing $\mathcal{P}_\Phi$ that destroys a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ used in the proof of Theorem 6.2 is proper, for a while it was unclear whether the PFA implies that all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ are trivial. This was finally proved in [29]. One of the novelties introduced in [29] was Lemma 7.1 below. A gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is *indestructible* if it remains a gap in every further $\aleph_1$-preserving extension.

LEMMA 7.1 (Shelah–Steprāns, [29]). *Assume Φ is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. Then there is a proper forcing that adds $X \subseteq \mathbb{N}$ such that*

$$\mathcal{A}_{\Phi,X} = \{\Phi_*(A) : A \in (\mathcal{P}(\mathbb{N}))^V, A \setminus X \in \mathrm{Fin}\}$$
$$\mathcal{B}_{\Phi,X} = \{\Phi_*(B) : B \in (\mathcal{P}(\mathbb{N}))^V, B \cap X \in \mathrm{Fin}\}$$

form an indestructible gap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. ⊣

The proof of Lemma 7.1 relied on results of [1]. The phenomenon that gaps in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ can be made indestructible was discovered by Kunen ([22]) in the case when both sides of the gap are σ-directed under the inclusion mod Fin (see also [36, p. 74]).

Our motivation for assuming Martin's Maximum (see [12]) instead of the weaker PFA in the following Conjecture is that MM implies that $L(\mathbb{R})$ is elementarily equivalent to some Solovay's model ([12], [31]), and it therefore provides an ambient for Conjecture 4.11 as well.

Conjecture 7.2. *Martin's Maximum implies that every isomorphism between quotients over definable ideals has a continuous lifting.*

We can summarize Conjectures 4.11 and 7.2 as follows.

Conjecture 7.3. *Assume Martin's Maximum.*

(a) *The following are equivalent for any two definable ideals $\mathcal{I}, \mathcal{J}$:*
   (1) $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$.
   (2) $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$.
(b) *Moreover, every isomorphism between such quotients has a completely additive lifting.*

In the present paper we have been mainly concerned with isomorphisms between quotients, since both Conjecture 4.6 and Conjecture 7.2 fail for homomorphisms (see [7, Theorem 1.9.5] and [7, §3.2], respectively). The former fails because Todorcevic's conjecture fails outside of class of nonpathological ideals (see §2.5). The reason why the latter fails is much simpler: if $\mathcal{K}$ is a maximal nonprincipal ideal on $\mathbb{N}$, then $\mathcal{P}(\mathbb{N})/\mathcal{K}$ can be embedded into any analytic quotient, and the embedding cannot have a continuous lifting since $\mathcal{K}$ does not have the Property of Baire. However, Conjecture 7.2 does have a plausible generalization for arbitrary homomorphisms.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and let $\mathbb{N} = A \dot\cup B$ be a partition of $\mathbb{N}$ into two disjoint $\mathcal{I}$-positive sets. Let Φ_1 and Φ_2 be homomorphisms of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(A)/\mathcal{I}$ and $\mathcal{P}(B)/\mathcal{I}$, respectively. Their *amalgamation* $\Phi = \Phi_1 \oplus \Phi_2$ is defined by the following diagram (id stands for the identity map):

$$\begin{array}{ccccc} & & \mathcal{P}(A)/\mathcal{I} & & \\ & {\scriptstyle\Phi_1}\nearrow & & \searrow {\scriptstyle\mathrm{id}} & \\ \mathcal{P}(\mathbb{N})/\mathrm{Fin} & & \xrightarrow{\Phi=\Phi_1\oplus\Phi_2} & & \mathcal{P}(\mathbb{N})/\mathcal{I}\ . \\ & {\scriptstyle\Phi_2}\searrow & & \nearrow {\scriptstyle\mathrm{id}} & \\ & & \mathcal{P}(B)/\mathcal{I} & & \end{array}$$

In this situation we write $\Phi_1 = \Phi^A$ and $\Phi_2 = \Phi^B$.

Definition 7.4. The ideal $\mathcal{I}$ has the *continuous lifting property* if for every homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is an amalgamation of a homomorphism with a continuous lifting and a homomorphism whose kernel is a nonmeager ideal on $\mathbb{N}$.

Lemma 7.5. *If $\mathcal{I}$ has the continuous lifting property and $\mathcal{J}$ is an ideal that has the Property of Baire, then every isomorphism between the quotients over $\mathcal{I}$ and $\mathcal{J}$ has a continuous lifting.*

Proof Recall that by a result of Jalali-Naini and Talagrand an ideal that contains Fin and has the Property of Baire is nonmeager if and only if it is not proper (see also [7, §3.10]). Therefore if $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{J} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is an isomorphism and $\ker(\Phi^B)$ is nonmeager, then $B \in \mathcal{I}$. $\dashv$

Hence the following is a stronger version of Conjecture 7.3.

CONJECTURE 7.6. *Martin's Maximum implies that every definable ideal has the continuous lifting property.*

It should be pointed out that the emphasis of the above conjectures is on the existence of continuous liftings, and not on Martin's Maximum. If there is a model of MM in which there is an isomorphism between quotients over definable ideals with no continuous lifting, this would merely be a suggestion that MM is not the sufficient assumption for Conjecture 7.3. The real refutation of Conjecture 7.2 would be a construction of two definable ideals and an isomorphism between their quotients that has no continuous lifting without using CH or some other additional set-theoretic axiom compatible with large cardinals.

§8. Positive results.

8.1. Quotients over analytic P-ideals. Theorem 8.1 below was proved before the above conjectures were stated; it was actually the starting point for formulating these conjectures. The assumption used in this result is the Open Coloring Axiom, as defined by Todorcevic in [36, §8], taken together with familiar Martin's Axiom, MA. Recall that $[A]^2$ is the family of all two-elements subsets of A and that $B \subseteq A$ is *K-homogeneous* if $[B]^2 \subseteq K$. It is *σ-K-homogeneous* if it can be covered by countably many K-homogeneous sets. A subset K of $[X]^2$ is *open* if $\{(x,y) : \{x,y\} \in K\}$ is an open subset of X^2.

OCA. If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is a partition such that K_0 is an open subset of $[X]^2 = \{\{x,y\} : x \neq y,\ x,y \in X\}$ then one of the following applies:

(a) X is σ-K_1-homogeneous, or
(b) X has an uncountable K_0-homogeneous subset.

Both OCA and MA are consequences of PFA (for OCA see [36, §8]).

THEOREM 8.1 (Farah, [7, Theorem 3.3.5]). *OCA and MA imply that all analytic P-ideals have the continuous lifting property. Therefore Conjecture 7.2 is true for all analytic P-ideals, and Conjecture 7.3 is true for all nonpathological analytic P-ideals.* ⊣

An axiom called OCA was introduced earlier in [1]. This axiom, even if amplified with MA, is not a sufficient assumption for Theorem 8.1, since it holds in Velickovic's model of MA in which there is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ ([41]).

COROLLARY 8.2. *Conjecture 7.2 is true for all P-ideals. Conjecture 7.3 is true for all nonpathological P-ideals.*

PROOF By [37], in some Solovay's model all P-ideals are analytic, hence the result is an immediate consequence of Theorem 8.1. ⊣

8.2. Quotients over F_σ ideals. OCA_∞ is a strengthening of OCA, introduced in [5], that deals with infinitely many open colorings simultaneously. It was extracted from an earlier unpublished work of Steprāns.

OCA_∞. If X is a separable metric space and $[X]^2 = K_0^n \cup K_1^n$, $n \in \mathbb{N}$, is a sequence of partitions such that each K_0^n is open and that $K_0^{n+1} \subseteq K_0^n$ for all n, then one of the following applies:

(a) $X = \bigcup_{n\in\mathbb{N}} F_n$, where each F_n is K_1^n-homogeneous, or
(b) There is an uncountable $Y \subseteq X$ that is σ-K_0^n-homogeneous for all n.

This axiom was shown to follow from PFA in [5] by following the proof of [36, Theorem 8.0] that PFA implies OCA.

THEOREM 8.3 (Farah, 2000). *OCA_∞ and MA together imply that all F_σ ideals have the continuous lifting property. Hence Conjecture 7.2 is true for all F_σ ideals and Conjecture 7.3 is true for all nonpathological F_σ ideals.* $\dashv$

It should be pointed out that at this point Theorem 8.3 is the only result proved using OCA_∞ that is still not known to follow from OCA. A 'local' version of this theorem (see §8.4) was proved earlier from OCA and MA by Just ([17]).

8.3. Using stronger axioms. A more direct and simpler proof of both Theorem 8.1 and Theorem 8.3 from PFA instead of OCA (or OCA_∞) and MA can be given using the following generalization of Shelah–Steprāns Lemma (Lemma 7.1).

LEMMA 8.4 (Farah, [10, §6]). *If $\mathcal{I}$ is an analytic P-ideal or an F_σ ideal, $\mathcal{J}$ is an analytic ideal, and $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{J} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is an isomorphism with no continuous lifting, then there is a proper forcing that adds $X \subseteq \mathbb{N}$ such that*

$$\mathcal{A}_{\Phi,X} = \{\Phi_*(A) : A \in (\mathcal{P}(\mathbb{N}))^V, A \setminus X \in \mathcal{I}\}$$
$$\mathcal{B}_{\Phi,X} = \{\Phi_*(B) : B \in (\mathcal{P}(\mathbb{N}))^V, B \cap X \in \mathcal{I}\}$$

form an indestructible gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. $\dashv$

A positive answer to the following question would be a major step toward confirming Conjecture 7.2.

QUESTION 8.5. *Can Lemma 8.4 be proved assuming only that $\mathcal{I}$ is an analytic ideal? Or that $\mathcal{I}$ and $\mathcal{J}$ are definable, under a suitable large cardinal assumption?*

In [10] the author was able to isolate a class of *strongly countably determined by closed approximations* ideals and to verify Conjecture 7.2, and moreover Conjecture 7.6, for the ideals in this class. This class includes all analytic P-ideals, all F_σ ideals, and $\mathrm{NWD}(\mathbb{Q})$. Every ideal that is strongly countably determined by closed approximations is $F_{\sigma\delta}$, and at present it is unknown whether this class of ideals coincides with $F_{\sigma\delta}$ ideals.

The methods used in [10] do not seem to apply to quotients over ideals more complex than $F_{\sigma\delta}$. and at this point we have only partial results on such quotients.

8.4. Quotients over ideals beyond $F_{\sigma\delta}$. An ideal $\mathcal{I}$ is *ccc over* Fin if every family of $\mathcal{I}$-positive sets whose pairwise intersections are finite is countable. Every such ideal is rather large, in particular it is nonmeager (see [7, §3.3]). A homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *has continuous liftings locally* if the ideal

$$\mathcal{J}_\Phi = \{A \subseteq \mathbb{N} : \Phi \restriction \mathcal{P}(A)/\mathcal{I} \text{ has a continuous lifting}\}$$

is ccc over Fin.

The first step in the proof of Theorem 8.1, Theorem 8.3, as well as all other similar results, is to prove that the homomorphism has continuous liftings locally. In the final step of the proof these liftings are amalgamated into a single continuous lifting.

THEOREM 8.6 (Farah, 2000). *Assume there exists a weakly compact cardinal. Then there is a forcing extension in which every homomorphism between quotients over definable ideals has continuous liftings locally. Moreover, in this model all P-ideals and all F_σ-ideals have the continuous lifting property.* ⊣

The proof of this result, probably relevant to Conjecture 6.3, will appear elsewhere. Corollary 8.7 below shows that a weaker version of Conjecture 7.3 is true for a rather rich class of quotients. Let $\mathcal{C}$ be the class consisting of all nonpathological P-ideals, all nonpathological F_σ-ideals, all ordinal ideals, and all Weiss ideals.

COROLLARY 8.7. *Assume there exists a weakly compact cardinal. Then there is a forcing extension in which for all $\mathcal{I}, \mathcal{J}$ in $\mathcal{C}$ we have $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$ if and only if $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$.*

PROOF Go to the model of Theorem 8.6. For nonpathological ideals this follows by Proposition 4.9. Recall that ordinal and Weiss ideals are all homogeneous. If $\mathcal{I}$ is homogeneous and an isomorphism between $\mathcal{I}$ and $\mathcal{J}$ has a locally continuous lifting, then $\mathcal{I}$ is isomorphism to $\mathcal{J} \upharpoonright A$ for some $\mathcal{J}$-positive set A. By Theorem 4.5, if $\mathcal{I}$ is an ordinal ideal or a Weiss ideal this implies $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J} \upharpoonright A$; but by homogeneity this is possible only if $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$. ⊣

REFERENCES

[1] U. ABRAHAM, M. RUBIN, and S. SHELAH, *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*, ***Annals of Pure and Applied Logic***, vol. 29 (1985), pp. 123–206.

[2] A. ANDRETTA, *Notes on descriptive set theory*, in preparation, 2000.

[3] C.C. CHANG and H.J. KEISLER, ***Model theory***, North–Holland, 1973.

[4] J.P.R. CHRISTENSEN, *Some results with relation to the control measure problem*, ***Vector space measures and applications II*** (R.M. Aron and S. Dineen, editors), Lecture Notes in Mathematics, vol. 645, Springer, 1978, pp. 27–34.

[5] I. FARAH, *Cauchy nets and open colorings*, ***Publ. Inst. Math. (Beograd) (N.S.)***, vol. 64(78) (1998), pp. 146–152, 50th anniversary of the Mathematical Institute, Serbian Academy of Sciences and Arts (Belgrade, 1996).

[6] ———, *Completely additive liftings*, ***The Bulletin of Symbolic Logic***, vol. 4 (1998), pp. 37–54.

[7] ———, ***Analytic quotients: theory of liftings for quotients over analytic ideals on the integers***, Memoirs of the American Mathematical Society, vol. 148, no. 702, 2000, 177 pp.

[8] ———, *Liftings of homomorphisms between quotient structures and Ulam stability*, ***Logic colloquium '98*** (S. Buss, P. Hájek, and P. Pudlák, editors), Lecture notes in logic, vol. 13, A.K. Peters, 2000, pp. 173–196.

[9] ———, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?*, preprint, available at http://www.math.csi.cuny.edu/~farah, 2001.

[10] ———, *Luzin gaps*, preprint, available at http://www.math.csi.cuny.edu/~farah, 2001.

[11] I. FARAH and S. SOLECKI, *Two $F_{\sigma\delta}$ ideals*, ***Proceedings of the American Mathematical Society***, (to appear, available at http://www.math.csi.cuny.edu/~farah).

[12] M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals and nonregular ultrafilters, I*, ***Annals of Mathematics***, vol. 127 (1988), pp. 1–47.

[13] D.H. Fremlin, *Notes on* FARAH *P99*, preprint, University of Essex, June 1999.

[14] F. Hausdorff, *Die Graduierung nach dem Endverlauf*, ***Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften; Mathematisch–Physiche Klasse***, vol. 31 (1909), pp. 296–334.

[15] W. Just, *Repercussions on a problem of Erdös and Ulam about density ideals*, ***Canadian Journal of Mathematics***, vol. 42 (1990), pp. 902–914.

[16] ———, *A modification of Shelah's oracle chain condition with applications*, ***Transactions of the American Mathematical Society***, vol. 329 (1992), pp. 325–341.

[17] ———, *A weak version of AT from OCA*, ***Mathematical Science Research Institute Publications***, vol. 26 (1992), pp. 281–291.

[18] W. Just and A. Krawczyk, *On certain Boolean algebras $\mathcal{P}(\omega)/I$*, ***Transactions of the American Mathematical Society***, vol. 285 (1984), pp. 411–429.

[19] A. Kanamori, ***The higher infinite: large cardinals in set theory from their beginnings***, Perspectives in Mathematical Logic, Springer–Verlag, Berlin–Heidelberg–New York, 1995.

[20] V. Kanovei and M. Reeken, *On Ulam's problem concerning the stability of approximate homomorphisms*, ***Tr. Mat. Inst. Steklova***, vol. 231 (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, pp. 249–283.

[21] ———, *New Radon–Nikodym ideals*, ***Mathematika***, (to appear).

[22] K. Kunen, *$\langle\kappa, \lambda^*\rangle$-gaps under MA*, preprint, 1976.

[23] C. Laflamme, *Combinatorial aspects of F_σ filters with an application to $\mathcal{N}$-sets*, ***Proceedings of the American Mathematical Society***, vol. 125 (1997), pp. 3019–3025.

[24] A. Louveau and B. Velickovic, *A note on Borel equivalence relations*, ***Proceedings of the American Mathematical Society***, vol. 120 (1994), pp. 255–259.

[25] B. Löwe and J. Steel, *An introduction to core model theory*, ***Sets and proofs, Logic Colloquium 1997, volume 1***, London Mathematical Society lecture Notes, no. 258, Cambridge University Press, 1999.

[26] K. Mazur, *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebra $\mathcal{P}(\omega)/I$*, ***Fundamenta Mathematicae***, vol. 138 (1991), pp. 103–111.

[27] W. Rudin, *Homogeneity problems in the theory of čech compactifications*, ***Duke Mathematics Journal***, vol. 23 (1956), pp. 409–419.

[28] S. Shelah, ***Proper forcing***, Lecture Notes in Mathematics 940, Springer, 1982.

[29] S. Shelah and J. Steprāns, *PFA implies all automorphisms are trivial*, ***Proceedings of the American Mathematical Society***, vol. 104 (1988), pp. 1220–1225.

[30] ———, *Non-trivial homeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ without the continuum hypothesis*, ***Fundamenta Mathematicae***, vol. 132 (1989), pp. 135–141.

[31] S. Shelah and W.H. Woodin, *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable*, ***Israel Journal of Mathematics***, vol. 70 (1990), pp. 381–394.

[32] S. Solecki, *Analytic ideals*, ***The Bulletin of Symbolic Logic***, vol. 2 (1996), pp. 339–348.

[33] ———, *Analytic ideals and their applications*, ***Annals of Pure and Applied Logic***, vol. 99 (1999), pp. 51–72.

[34] ———, *Filters and sequences*, ***Fundamenta Mathematicae***, vol. 163 (2000), pp. 215–228.

[35] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, ***Annals of Mathematics***, vol. 92 (1970), pp. 1–56.

[36] S. Todorcevic, ***Partition problems in topology***, Contemporary mathematics, vol. 84, American Mathematical Society, Providence, Rhode Island, 1989.

[37] ———, *Definable ideals and gaps in their quotients*, ***Set theory: Techniques and applications*** (C.A. DiPrisco et al., editors), Kluwer Academic Press, 1997, pp. 213–226.

[38] ———, *Basis problems in combinatorial set theory*, ***Proceedings of the international congress of mathematicians, vol. II (Berlin, 1998)***, Documenta Mathematica, 1998, pp. 43–52.

[39] ———, *Gaps in analytic quotients*, ***Fundamenta Mathematicae***, vol. 156 (1998), pp. 85–97.

[40] B. VELICKOVIC, *Definable automorphisms of* $\mathcal{P}(\omega)/\mathrm{Fin}$, ***Proceedings of the American Mathematical Society***, vol. 96 (1986), pp. 130–135.

[41] ———, *OCA and automorphisms of* $\mathcal{P}(\omega)/\mathrm{Fin}$, ***Topology and its Applications***, vol. 49 (1992), pp. 1–12.

[42] W. WEISS, *Partitioning topological spaces*, ***Mathematics of Ramsey Theory*** (J. Nešetřil and V. Rödl, editors), Algorithms and Combinatorics, vol. 5, Springer–Verlag, Berlin, 1990, pp. 154–171.

[43] W.H. WOODIN, Σ^2_1*-absoluteness*, handwritten note of May 1985.

[44] ———, ***The axiom of determinacy, forcing axioms and the nonstationary ideal***, de Gruyter Series in Logic and Its Applications, vol. 1, de Gruyter, 1999.

DEPARTMENT OF MATHEMATICS
CUNY, COLLEGE OF STATEN ISLAND AND GRADUATE CENTER, NEW YORK
AND MATEMATICKI INSTITUT, KNEZA MIHAILA 35, BELGRADE
E-mail: ifarah@gc.cuny.edu
URL: http://www.math.csi.cuny.edu/~farah