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Analytic Hausdor gaps

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In [**6**], Hausdor constructed two families $\mathcal{A}$ and $\mathcal{B}$ of sets of integers such that

(a) $A \cap B$ is nite for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$,
(b) for every C $\mathbb{N}$ either $A \setminus C$ is innite for some $A \in \mathcal{A}$ or $B \cap C$ is innite for some $B \in \mathcal{B}$, and
(c) both and $\mathcal{A}$ and $\mathcal{B}$ have order-type equal to ω_1 with respect to the inclusion modulo nite.

Families satisfying (a) are *orthogonal*, those satisfying both (a) and (b) form a *gap* in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (or, they are *not separated* over Fin, the ideal of nite sets), and if they furthermore satisfy (c) they form an (ω_1, ω_1)*-gap*. If (a) and (b) hold and both $\mathcal{A}$ and $\mathcal{B}$ are linearly ordered by the inclusion modulo nite, a gap is *linear*. Before Hausdor, du Bois-Reymond and Hadamard proved that there are no (ω, ω)-gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (see e.g., [**11**]). If (c) is weakened to (recall that a poset is *-directed* if every countable subset has an upper bound)

(c′) Both $\mathcal{A}$ and $\mathcal{B}$ are -directed under the inclusion modulo nite,

then the gap is said to be *Hausdor* (note that a Hausdor gap need not be linear).

We will identify subsets of $\mathbb{N}$ with elements of the Cantor cube $2^{\mathbb{N}}$, and import the compact metric topology, as well as Borel structure, of the latter structure to $\mathcal{P}(\mathbb{N})$. Via this identication, the existence of an (ω_1, ω_1)-gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has some immediate consequences to the classical descriptive set theory—e.g., it gives an example of an uncountable universal measure zero set (see [**11**, §4]) and a partition of the Cantor set into $\aleph_1$ many pairwise disjoint G sets. More recently, Hausdor gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ have played a prominent role in a variety of subjects, from automatic continuity in Banach algebras ([**3**]) to the study of homomorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (see [**12**, §IV], also [**5**, Chapter 3]).

In Hausdor's time the fact that both sides of the gap have length equal to ω_1 probably could have been considered as a support for the Continuum Hypothesis. After the profusion of the method of forcing this specic property of ω_1 is better explained. In ([**7**]) Kunen has proved that, assuming a small amount of reection at the level of $\mathcal{P}(\mathbb{N})$, (ω_1, ω_1) gaps are the only linear Hausdor gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Since this reection principle (later incorporated into Todorcevic's Open Coloring Axiom, see [**16**, §8]) is compatible with Martin's Axiom, ω_1 is the only standard

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cardinal invariant of the continuum such that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ always has (,)- or $(, \omega\ _1)$-gaps. An important consequence of Kunen's proof was that every Hausdor gap can be made *indestructible* under any further $\aleph_1$-preserving forcing; this fact has played a key role in Woodin's consistency proof of 'all homomorphisms of Banach algebras are automatically continuous' (see [**3**]).

Inspired by these results, Todorcevic proved a separation principle implying that there are no analytic Hausdor gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ ([**17**, Corollary 1]; see also [**5**, §5.5]). (Recall that a subset of a compact metric space is *analytic* if it is a continuous image of the irrationals.) His proof shows that in Solovay's model ([**15**]) there are no Hausdor gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ at all, thus indicating that some form of Axiom of Choice is necessary to construct a Hausdor gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. This may be the right place to point out to a rather optimal construction of a Hausdor gap—also due to Todorcevic—given in [**2**, pages 96–98]. The gap is dened by a simple formula whose parameters are an ω_1-chain in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and a ladder system on ω_1.

If $\mathcal{I}$ is an ideal on $\mathbb{N}$ other than Fin, we say that two families $\mathcal{A}, \mathcal{B}$ are $\mathcal{I}$-*orthogonal* (or *orthogonal over* $\mathcal{I}$) if $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. The notions 'separated over $\mathcal{I}$' and '(*Hausdor*) *gap in* $\mathcal{P}(\mathbb{N})/\mathcal{I}$' are dened in a natural way. The question of computing the gap-spectra of quotients other than $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ was rst raised by W. Just (see [**11**]), and some partial answers were obtained by K. Mazur in [**9**].

The interest in the structure of analytic ideals and their quotients was intensi-ed by [**17**] and its sequels [**19, 18**]. In [**19**] it was proved that every embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into a quotient over an analytic ideal that has a continuous (equivalently, Baire-measurable) lifting (see [**5**, §1.5]) preserves all Hausdor gaps. Since by a result of Mathias ([**8**]) there is always such an embedding, this implies that the Hausdor-gap spectrum of any analytic quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is at least as rich as the Hausdor-gap spectrum of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. In particular, this implies the existence of (ω_1, ω_1)-gaps in every analytic quotient, answering the original Just's question. (For more on preservation of gaps see [**19**] and [**5**, §5.9].)

The analysis of gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has played an important role in Shelah's consistency proof of the statement that all automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial ([**12**, §IV]). Later on, when obtaining this conclusion from PFA ([**13**]) one of the key parts of the proof consisted in making a given gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ indestructible. Although the gaps occurring in this proof were nonlinear, the connection with Kunen's result is evident (it is worth noting that 'freezing' the gap in both cases depends on the same technique; see also [**1**]).

After the author has extended Shelah's result to all quotients over analytic P-ideals ([**5**, §3.3]) it was natural to expect that Kunen's and Todorcevic's results will soon be extended to all quotients over analytic P-ideals. It therefore came as a surprise when in January 1998 (see [**5**, Theorem 5.10.2]) the author has constructed an analytic Hausdor gap $\mathcal{A}, \mathcal{B}$ in a quotient over a certain F P-ideal $\mathcal{I}$. (Shortly after, this object was used to construct a topological space with interesting cellularity properties, see [**10**].) Such a gap cannot be made indestructible, since by a result of Todorcevic increasing the additivity of Lebesgue measure separates such gaps (see the proof of Lemma 7).

The proof of [**5**, Theorem 5.10.2] proceeded by constructing suitable families $\mathcal{A}, \mathcal{B}$ and then dening $\mathcal{I} = \mathcal{A} \cap \mathcal{B}$. Therefore we had no control over the choice of $\mathcal{I}$, and it was rather unclear whether such a gap can be constructed in a quotient over some more natural analytic P-ideal. In [**5**, Question 5.13.9] and [**5**, Question 5.13.7] it was asked whether the quotient over the summable ideal $\mathcal{I}_{1/n} = \{A : \sum_{n \in A} 1/n < \infty\}$ satises some natural separation principle or has analytic Hausdor gaps. The following implies that there are analytic Hausdor gaps in the quotient over $\mathcal{I}_{1/n}$.

Theorem 1. *Let $\mathcal{I}$ be any F P-ideal. Then there is an analytic Hausdor gap in its quotient if and only if $\mathcal{I}$ is not generated by a single set over* Fin.

However, it turns out that quotients over all analytic P-ideals satisfy a separation principle that implies Todorcevic's separation principle in the case of Fin. This result will appear in [**4**].

Theorem 1 completely describes the class of all F P-ideals that allow an analytic Hausdor gap in their quotient, and hence gives a partial solution to [**5**, Problem 5.13.5], that asked for a complete description of all analytic ideals that allow an analytic Hausdor gap in their quotient.

The rest of this note is devoted to the proof of Theorem 1. Although this paper is self-contained, familiarity with parts of [**5**, Chapter 1 and Chapter 5] would be useful, and we will refer to [**5**] for proofs of some of the results used here.

If $\mathcal{A}$ and $\mathcal{B}$ form an analytic Hausdor gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$, and $\mathcal{I}$ is an analytic P-ideal, then $\mathcal{A}$ and $\mathcal{B}$ are analytic P-ideals as well, and $\mathcal{A} \cap \mathcal{B}\ \ \mathcal{I}$. By a result of Solecki (see [**14**]) every analytic P-ideal $\mathcal{I}$ is of the form

$$\mathcal{I} = \mathrm{Exh}(\ ) = \Big\{A \quad \mathbb{N} : \lim_{n \to \infty} \ (A \setminus [1, n)) = 0\Big\}$$

for some lower semicontinuous submeasure on $\mathbb{N}$. In particular, every such ideal is F . Moreover, $\mathcal{I}$ is F if and only if can be chosen to be *exhaustive*, i.e., so that the ideal $\mathrm{Fin}(\ ) = \{A : \ (A) < \infty\}$ coincides with $\mathrm{Exh}(\ )$. Hence the existence of an analytic Hausdor gap in a quotient over an analytic P-ideal is equivalent to the existence of a triple of lower semicontinuous submeasures with certain properties (see [**5**, Proposition 5.10.1]).

The following lemma gives relatively general conditions on a lower semicontinuous submeasure implying the existence of an analytic Hausdor gap in the quotient over $\mathrm{Exh}(\ )$.

Lemma 2. *Assume is a lower semicontinuous submeasure and there are J_n, $k_n \in \mathbb{N}$, and I_{ni} ($i \quad k_n$) for $n \in \mathbb{N}$ such that*

1. *$J_n \quad \mathbb{N}$ are pairwise disjoint,*
2. *$\max_{k \in J_n}(\ (\{k\})) < 2^{\ \ n}$,*
3. *I_{ni} ($i \quad k_n$) are pairwise disjoint subsets of J_n,*
4. *$\ (I_{ni}) \quad 1$ for all $i \quad k_n$, and*
5. *$(\forall A \quad J_n)(((\forall i \quad k_n)\ \ (A \cap I_{ni}) > 1/(n+1)^2) \to \ (A \cap J_n) \quad 1)$.*

Then there is an analytic Hausdor gap in the quotient over $\mathrm{Exh}(\ )$.

Proof. For $A \quad \mathbb{N}$ and $n \in \mathbb{N}$ dene submeasures

$$_n(A) = |\{j \in \mathbb{N} : A \cap I_{nj} \neq \emptyset\}|$$
$$_n(A) = \sup_{j \quad k_n} \ (A \cap I_{nj}).$$

Since both values $_n(A)$ and $_n(A)$ depend only on a nite set $A \cap J_n$, both submeasures

$$(A) = \sum_{n=1}^{\infty} \frac{_n(A)}{n},$$
$$(A) = \sup_{j \in \mathbb{N}} n \quad _n(A)$$

are lower semicontinuous. Let $J = \bigcup_{n=1}^{\infty} J_n$; then

$$\mathcal{A} = \mathrm{Exh}(\ ) \cap \mathcal{P}(J),$$
$$\mathcal{B} = \mathrm{Exh}(\ ) \cap \mathcal{P}(J)$$

are analytic P-ideals. In particular, $\mathcal{A}/\mathrm{Exh}(\ )$ and $\mathcal{B}/\mathrm{Exh}(\ )$ are -directed. To prove that $\mathcal{A}, \mathcal{B}$ form a Hausdor gap in $\mathcal{P}(\mathbb{N})/\mathrm{Exh}(\ )$, we need only prove that $\mathcal{A}$ and $\mathcal{B}$ are $\mathrm{Exh}(\ )$-orthogonal and that they are not separated by a single set over $\mathrm{Exh}(\ )$.

Claim 1. *$\mathcal{A}$ and $\mathcal{B}$ are* $\mathrm{Exh}(\ )$*-orthogonal.*

Proof. Pick $A, B \quad J$. Then $(A \cap B \cap J_n) \quad _n(A) \quad _n(B)$, thus

$$(A \cap B) \quad \sum_{n=1}^{\infty} (A \cap B \cap J_n) \quad \sum_{n=1}^{\infty} \frac{_n(A)}{n} \quad (n \quad _n(B)) \quad (A) \quad (B).$$

If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $(A) < \infty$ and $\lim_{m \to \infty} \quad B \setminus \bigcup_{n=1}^{m} J_n \quad = 0$, thus by the above $\lim_{l \to \infty} \quad ((A \cap B) \setminus [1, l)) = 0$, and $A \cap B \in \mathrm{Exh}(\ )$, as required. □

Claim 2. *$\mathcal{A}$ and $\mathcal{B}$ are not separated over* $\mathrm{Exh}(\ )$.

Proof. Assume the contrary, that for some $C \quad \mathbb{N}$ we have $A \setminus C \in \mathrm{Exh}(\ )$ and $B \cap C \in \mathrm{Exh}(\ )$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. We claim that

(6) $\lim_{n \to \infty} \sup_{m \quad n, j \quad k_m} \quad (I_{mj} \setminus C) = 0$.

Otherwise, we may nd an innite $X \quad \mathbb{N}$, $\varepsilon > 0$, and a 'choice function' $f \in \prod_{n \in X} [1, k_n]$ such that

$$(I_{nf(n)} \setminus C) > \varepsilon$$

for all $n \in X$. We may furthermore shrink X so that $\sum_{n \in X} 1/n < \infty$. Let $A = \bigcup_{n \in X} I_{nf(n)}$; then $(A) \quad \sum_{n \in X} \varepsilon/n < \infty$, thus $A \in \mathcal{A}$. Note that $A \cap C = \emptyset$. However, for $n \in X$ we have $(A \cap J_n) \quad (A \cap I_{nf(n)}) \quad \varepsilon$, therefore $A \notin \mathrm{Exh}(\ )$, contradicting the assumption on C.

By (6) for all but nitely many n we have $\sup_{j \quad k_n} \quad (I_{nj} \setminus C) < 1/2$, and therefore

(7) $(I_{nj} \cap C) > 1/2$ for all but nitely many n and all $j \quad k_n$.

Let Y be the set of such n. We shall now nd $B_n \quad J_n \cap C$ $(n \in Y)$ so that for all $n \in Y$ we have

(8) $(B_n) \quad 1$,
(9) $_n(B_n) \quad 1/n^2$.

Once these sets are constructed, let $B = \bigcup_{n \in Y} B_n$. Then $B \quad C$ and $n \quad _n(B) \quad 1/n$, thus $B \in \mathcal{B}$, yet $B \notin \mathrm{Exh}(\ )$ by (8). Since this contradicts the assumption that C separates $\mathcal{A}$ and $\mathcal{B}$, it will suce to describe the construction of B_n $(n \in Y)$.

Fix $n \in Y$. By (7) and (2) we may nd $B_n \quad J_n \cap C$ such that

(10) $1/n^2 \quad 2^{\ n} \quad (B_n \cap I_{nj}) \quad 1/n^2$.

for all j k_n, thus (9) holds. By (5), $(B_n \cap J_n)$ 1, thus (8) holds as well. This completes the proof. □

Claim 1 and Claim 2 complete the proof of lemma. □

LEMMA 3. *Assume that is an exhaustive lower semicontinuous submeasure such that* $\lim_{n\to\infty}$ $(\{n\}) = 0$ *and* Exh( ) *is a proper ideal. Then satises the conditions of Lemma 2.*

PROOF. Since Fin( ) = Exh( ) is a proper ideal, we have $(\mathbb{N}) = \infty$. We describe the recursive construction of J_n $(n \in \mathbb{N})$. Suppose J_i $(i$ $n)$ have already been constructed. Since $\lim_{k\to\infty}$ $(\{k\}) = 0$ and is lower semicontinuous, we may nd m such that

$$\bigcup_{i=1}^{n\ \ 1} J_i \quad [1, m) \qquad \text{and} \qquad \sup_{j\ \ m} (\{j\}) < 2^{\ \ n}.$$

We will have $\min(J_n)$ m, assuring (1) and (2). Since $(\mathbb{N}) = \infty$ and is lower semicontinuous, we can nd $m = m_0 < m_1 < m_2 < \dots$ such that 1 $([m_i, m_{i+1})$ for all i. Let T be the family of all B $[m, \infty)$ such that for some $j \in \mathbb{N}$ and all i j we have

(11) $1/n^2 <$ $(B \cap [m_i, m_{i+1}))$, and

(12) (B) 1.

We claim that T $\mathcal{P}$ $([m_0, m_k])$ for some $k \in \mathbb{N}$. Otherwise T is innite, and by Knig's lemma there is an innite sequence of sets B_i $[m_i, m_{i+1})$ whose union satises (11) for all i yet has submeasure at most 1. But this contradicts the exhaustivity of . Thus T $\mathcal{P}$ $([m_0, m_k])$ for some k. Let $k_n = m_{k+1}$, and $I_{ni} = [m_{i\ \ 1}, m_i)$ for i k_n. Then clearly (3) and (4) hold. Let us check (5). If A J_n is such that $(A \cap I_{ni}) > 1/n^2$ for all i k_n, then since A / $[m_0, m_k)$ we have $A \notin T$, and therefore $(A) > 1$, as required.

This describes the recursive construction of sets satisfying the assumptions of Lemma 2. □

I do not know whether the assumption that is exhaustive is necessary in Lemma 3, but a satisfying all assumptions of Lemma 3 with exhaustivity replaced by $(\mathbb{N}) = \infty$ and failing the conclusion of Lemma 3 would have to be rather peculiar.

An ideal $\mathcal{I}$ is *dense* if every innite set has an innite subset in $\mathcal{I}$. It is *dense below* A (for some A $\mathbb{N}$) if every innite subset of A has an innite subset in $\mathcal{I}$. Note that Exh( ) is dense if and only if $\lim_{n\to\infty}$ $(\{n\}) = 0$. An ideal has the *Frechet property* if it is not dense below any positive set. The following special case of a result of Todorcevic shows that it suces to prove Theorem 1 for dense ideals.

LEMMA 4. *An F P-ideal has the Frechet property if and only if it is generated by a single set over* Fin.

PROOF. This follows immediately from [**18**, Theorem 2]; for a proof see [**5**, Corollary 1.2.11]. □

PROOF OF THEOREM 1. The direct implication follows from [**17**, Corollary 1], since an ideal generated by a single set over Fin is isomorphic to Fin.

Let $\mathcal{I}$ be an F P-ideal not generated by a single set over Fin. By Solecki's theorem, let be a lower semicontinuous exhaustive submeasure such that $\mathcal{I} =$

Exh( ). By Lemma 4, there is an $\mathcal{I}$-positive set A such that $\mathcal{I}$ is dense below A; we may assume that $A = \mathbb{N}$, i.e., that $\mathcal{I}$ is dense. This is easily equivalent to $\lim_{k\to\infty} \ (\{k\}) = 0$. By the exhaustivity of , $(\mathbb{N}) = \infty$, thus by Lemma 3 the assumptions of Lemma 2 are satised and there is an analytic Hausdor gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as required. □

Recall that the *summable ideal* is dened by

$$\mathcal{I}_{1/n} = \left\{ A \quad \mathbb{N} : \sum_{n\in A} \frac{1}{n} < \infty \right\}.$$

For some time Fin and $\mathcal{I}_{1/n}$ (and its obvious variations) were essentially the only known examples of F P-ideals, until a number of radically dierent examples was found (see [**5**, §1.11]), but $\mathcal{I}_{1/n}$ still stands out as the simplest nontrivial F P-ideal.

COROLLARY 5. *There is an analytic Hausdor gap in* $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$. □

Corollary 6 below gives partial answers to [**19**, Problem 2], [**5**, Problem 5.13.16], and [**5**, Question 5.13.17]. Together with the Kunen–Todorcevic result, it also shows that the gap-spectra of quotients over F P-ideals other than Fin radically dier from the gap-spectrum of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. Recall that the ideals $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$ and a bijection h between $\mathbb{N}\setminus A$ and $\mathbb{N}\setminus B$ such that $C \in \mathcal{I}$ if and only if $h^{-1}(C) \in \mathcal{J}$. Clearly, this implies that the quotients over $\mathcal{I}$ and $\mathcal{J}$ are isomorphic.

COROLLARY 6. *Assume Martin's Axiom. Then there is a* $(\mathfrak{c},\mathfrak{c})$*-gap in the quotient over every* F *P-ideal not isomorphic to* Fin.

PROOF. Since Martins Axiom implies that $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$, this follows immediately from Theorem 1 and Lemma 7 below. □

Recall that $\mathrm{add}(\mathcal{N})$ is the additivity of the Lebesgue measure and that $\mathrm{cof}(\mathcal{N})$ is the conality of the ideal of Lebesgue null sets.

LEMMA 7. *Assume* $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$. *Then there is an* $(\mathrm{add}(\mathcal{N}),\mathrm{add}(\mathcal{N}))$*-gap in the quotient over every analytic P-ideal whose quotient contains an analytic Hausdor gap.*

PROOF. This proof is identical to the proof of [**5**, Lemma 5.6.3], but we recall it for the reader's convenience. Let $\mathcal{I}$ be an analytic P-ideal and let $\mathcal{A}_0, \mathcal{B}_0$ be an analytic Hausdor gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Let $\mathcal{A}$ be the ideal generated by $\mathcal{A}_0$ and $\mathcal{I}$, and let $\mathcal{B}$ be the ideal generated by $\mathcal{B}_0$ and $\mathcal{I}$. Both $\mathcal{A}$ and $\mathcal{B}$ are analytic P-ideals.

By a result of Todorcevic ([**18**, Theorem 5]), for every analytic P-ideal $\mathcal{A}$ the partially ordered set $(\mathcal{A}, \ )$ (i.e., $\mathcal{A}$ ordered by the inclusion modulo nite) is Tukey-reducible to Lebesgue null sets ordered by the inclusion (see also [**5**, Theorem 5.6.2]). In particular, every subset X of $\mathcal{A}$ of size less than $\mathrm{add}(\mathcal{N})$ is bounded in $(\mathcal{A}, \ )$, and there is a $Y \ \mathcal{I}$ of size $\mathrm{cof}(\mathcal{N})$ such that Y is -conal. Since $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$, we can recursively construct two chains, $\mathcal{A}' \ \mathcal{A}$ and $\mathcal{B}' \ \mathcal{B}$, increasing modulo Fin and each of length $\mathrm{add}(\mathcal{N})$ such that $\mathcal{A}'$ is conal in $\mathcal{A}$ and $\mathcal{B}'$ is conal in $\mathcal{B}$. Hence $\mathcal{A}', \mathcal{B}'$ form an $(\mathrm{add}(\mathcal{N}),\mathrm{add}(\mathcal{N}))$ gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. □

Note that Lemma 7 shows that an analytic Hausdor gap in a quotient over an analytic P-ideal can always be lled by a ccc forcing, contrasting the Kunen's result on the indestructibility of gaps in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. It should be noted that the statement '$\mathcal{P}(\mathbb{N})/\,$Fin contains a $(\mathfrak{c}, \mathfrak{c})$ Hausdorff gap' is independent from Martin's Axiom (Kunen, [**7**]).

The only two analytic P-ideals (up to the isomorphism) that are known not to have analytic Hausdorff gaps in their quotients are Fin (by [**19**, Corollary 1]) and $\emptyset \times$ Fin (by [**5**, Proposition 5.3.2] and Lemma 7). By a result of Todorcevic (a special case of which is Lemma 4), these are also the only two analytic P-ideals that have the Fréchet property (see [**5**, Corollary 1.2.11]).

QUESTION 8. (a) *Assume $\mathcal{I}$ is an analytic P-ideal that does not have the Fréchet property. Is there an analytic Hausdorff gap in its quotient?*

(b) *Assume $\mathcal{I}$ is an F_σ ideal that does not have the Fréchet property. Is there an analytic Hausdorff gap in its quotient?*

(Note that Fin is not the only F_σ ideal with the Fréchet property; e.g., the ideal Ib of [**9**, §1.4] generated by the branches of the tree $\mathbb{N}^{<\mathbb{N}}$ has the Fréchet property.) Let us explicitly state a special case of Question 8(a).

QUESTION 9. *Assume a dense analytic P-ideal is equal to* Exh(φ) *for a lower semicontinuous submeasure φ such that $\varphi(\mathbb{N}) = \infty$. Is there an analytic Hausdorff gap in its quotient?*

Lemma 2 can be used to give a positive answer to this question for many 'natural' submeasures φ. While a positive solution to Question 9 may not require much beyond what is given in the proof of Theorem 1, the following question most likely requires a different approach.

QUESTION 10. *Is there an analytic Hausdorff gap in the quotient over the ideal of asymptotic density zero sets, $\mathcal{Z}_0 = \{A : \limsup_n |A \cap n|/n = 0\}$?*

The existence of analytic Hausdorff gaps does not immediately give new linear gaps, unless MA or a similar axiom is assumed (see Lemma 7; to see why linear Hausdorff gaps are of some interest, see e.g., [**10**]). The following is a stronger version of [**5**, Conjecture 5.13.4].

CONJECTURE 11. *There is an $\langle \mathrm{add}(\mathcal{N}), \mathrm{add}(\mathcal{N})\rangle$-gap in the quotient over every F_σ P-ideal that is not generated by a single set over* Fin.

A confirmation of Conjecture 11 would give a positive answer to [**19**, Problem 3], where it was asked whether some standard cardinal invariants of the continuum other than ω, ω_1 and the bounding number for $\mathbb{N}^{\mathbb{N}}/\,$Fin naturally occur in the (linear) gap spectra of some analytic quotient.

At the end, let us point out that analytic Hausdorff gaps exist in quotients over analytic ideals that are not P-ideals. For example, the quotient over the following ideal (otp stands for the order-type of a subset of an ordinal)

$$\mathcal{I}_{\omega^2} = \{A \subseteq \omega^2 : \mathrm{otp}(A) < \omega^2\}$$

(also known as Fin $\times$ Fin) has such a gap (see [**5**, Proposition 5.8.3]).

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