Analytic Hausdor gaps

Ilijas Farah

In [6], Hausdor constructed two families \( \mathcal{A} \) and \( \mathcal{B} \) of sets of integers such that

(a) \( \mathcal{A} \cap \mathcal{B} \) is infinite for all \( \mathcal{A} \in \mathcal{A} \) and all \( \mathcal{B} \in \mathcal{B} \),

(b) for every \( \mathcal{C} \in \mathbb{N} \) either \( \mathcal{A} \setminus \mathcal{C} \) is infinite for some \( \mathcal{A} \in \mathcal{A} \) or \( \mathcal{B} \cap \mathcal{C} \) is infinite for some \( \mathcal{B} \in \mathcal{B} \), and

(c) both \( \mathcal{A} \) and \( \mathcal{B} \) have order-type equal to \( \omega_1 \) with respect to the inclusion modulo finite.

Families satisfying (a) are orthogonal, those satisfying both (a) and (b) form a gap in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) (or, they are not separated over \( \text{Fin} \), the ideal of finite sets), and if they furthermore satisfy (c) they form a \( (\omega_1, \omega_1) \)-gap. If (a) and (b) hold and both \( \mathcal{A} \) and \( \mathcal{B} \) are linearly ordered by the inclusion modulo finite, a gap is linear. Before Hausdor, Du Bois-Reymond and Hadamard proved that there are no \( (\omega, \omega) \)-gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) (see e.g., [11]). If (c) is weakened to (recall that a poset is directed if every countable subset has an upper bound)

(c') Both \( \mathcal{A} \) and \( \mathcal{B} \) are directed under the inclusion modulo finite,

then the gap is said to be Hausdor (note that a Hausdor gap need not be linear).

We will identify subsets of \( \mathbb{N} \) with elements of the Cantor cube \( 2^{\mathbb{N}} \), and import the compact metric topology, as well as Borel structure, of the latter structure to \( \mathcal{P}(\mathbb{N}) \). Via this identification, the existence of an \( (\omega_1, \omega_1) \)-gap in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) has some immediate consequences to the classical descriptive set theory—e.g., it gives an example of an uncountable universal measure zero set (see [11, §4]) and a partition of the Cantor set into \( \aleph_1 \) many pairwise disjoint \( G \) sets. More recently, Hausdor gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) have played a prominent role in a variety of subjects, from automatic continuity in Banach algebras ([3]) to the study of homomorphisms of \( \mathcal{P}(\mathbb{N})/\text{Fin} \) (see [12, §IV], also [5, Chapter 3]).

In Hausdor's time the fact that both sides of the gap have length equal to \( \omega_1 \) probably could have been considered as a support for the Continuum Hypothesis. After the profusion of the method of forcing this specic property of \( \omega_1 \) is better explained. In ([7]) Kunen has proved that, assuming a small amount of reection at the level of \( \mathcal{P}(\mathbb{N}) \), \( (\omega_1, \omega_1) \) gaps are the only linear Hausdor gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \). Since this reection principle (later incorporated into Todorcevic's Open Coloring Axiom, see [16, §8]) is compatible with Martin's Axiom, \( \omega_1 \) is the only standard

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cardinal invariant of the continuum such that \( \mathcal{P}(\mathbb{N})/\text{Fin} \) always has \( (\omega,\omega) \)- or \( (\omega_1,\omega_1) \)-gaps. An important consequence of Kunen’s proof was that every Hausdorff gap can be made \emph{indestructible} under any further \( \aleph_1 \)-preserving forcing; this fact has played a key role in Woodin’s consistency proof of ‘all homomorphisms of Banach algebras are automatically continuous’ (see [3]).

Inspired by these results, Todorcevic proved a separation principle implying that there are no analytic Hausdorff gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) ([17, Corollary 1]; see also [5, §5.5]). (Recall that a subset of a compact metric space is \emph{analytic} if it is a continuous image of the irrationals.) His proof shows that in Solovay’s model ([15]) there are no Hausdorff gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) at all, thus indicating that some form of Axiom of Choice is necessary to construct a Hausdorff gap in \( \mathcal{P}(\mathbb{N})/\text{Fin} \). This may be the right place to point out to a rather optimal construction of a Hausdorff gap—also due to Todorcevic—given in [2, pages 96–98]. The gap is dened by a simple formula whose parameters are an \( \omega_1 \)-chain in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) and a ladder system on \( \omega_1 \).

If \( I \) is an ideal on \( \mathbb{N} \) other than \( \text{Fin} \), we say that two families \( A, B \) are \emph{I-orthogonal} (or \emph{orthogonal over} \( I \)) if \( A \cap B \in I \) for all \( A \in A \) and all \( B \in B \). The notions ‘separated over \( I \)’ and ‘(Hausdorff) gap in \( \mathcal{P}(\mathbb{N})/I \)’ are dened in a natural way. The question of computing the gap-spectra of quotients other than \( \mathcal{P}(\mathbb{N})/\text{Fin} \) was rst raised by W. Just (see [11]), and some partial answers were obtained by K. Mazur in [9].

The interest in the structure of analytic ideals and their quotients was intensiﬁed by [17] and its sequels [19, 18]. In [19] it was proved that every embedding of \( \mathcal{P}(\mathbb{N})/\text{Fin} \) into a quotient over an analytic ideal that has a continuous (equivalently, Baire-measurable) lifting (see [5, §1.5]) preserves all Hausdorff gaps. Since by a result of Mathias ([8]) there is always such an embedding, this implies that the Hausdorff-gap spectrum of any analytic quotient \( \mathcal{P}(\mathbb{N})/I \) is at least as rich as the Hausdorff-gap spectrum of \( \mathcal{P}(\mathbb{N})/\text{Fin} \). In particular, this implies the existence of \( (\omega_1,\omega_1) \)-gaps in every analytic quotient, answering the original Just’s question. (For more on preservation of gaps see [19] and [5, §5.9].)

The analysis of gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) has played an important role in Shelah’s consistency proof of the statement that all automorphisms of the Boolean algebra \( \mathcal{P}(\mathbb{N})/\text{Fin} \) are trivial ([12, §IV]). Later on, when obtaining this conclusion from PFA ([13]) one of the key parts of the proof consisted in making a given gap in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) indestructible. Although the gaps occurring in this proof were nonlinear, the connection with Kunen’s result is evident (it is worth noting that ‘freezing’ the gap in both cases depends on the same technique; see also [1]).

After the author has extended Shelah’s result to all quotients over analytic \( P \)-ideals ([5, §3.3]) it was natural to expect that Kunen’s and Todorcevic’s results will soon be extended to all quotients over analytic \( P \)-ideals. It therefore came as a surprise when in January 1998 (see [5, Theorem 5.10.2]) the author has constructed an analytic Hausdorff gap \( A, B \) in a quotient over a certain \( F \)-\( P \)-ideal \( I \). (Shortly after, this object was used to construct a topological space with interesting cellularity properties, see [10].) Such a gap cannot be made indestructible, since by a result of Todorcevic increasing the additivity of Lebesgue measure separates such gaps (see the proof of Lemma 7).

The proof of [5, Theorem 5.10.2] proceeded by constructing suitable families \( A, B \) and then dening \( I = A \cap B \). Therefore we had no control over the choice of \( I \),
and it was rather unclear whether such a gap can be constructed in a quotient over some more natural analytic P-ideal. In [5, Question 5.13.9] and [5, Question 5.13.7] it was asked whether the quotient over the sumable ideal $I_{1/n} = \{ A : \sum_{n \in A} 1/n < \infty \}$ satisfies some natural separation principle or has analytic Hausdorff gaps. The following implies that there are analytic Hausdorff gaps in the quotient over $I_{1/n}$.

**Theorem 1.** Let $I$ be any $F$ P-ideal. Then there is an analytic Hausdorff gap in its quotient if and only if $I$ is not generated by a single set over $\text{Fin}$.

However, it turns out that quotients over all analytic P-ideals satisfy a separation principle that implies Todorcevic's separation principle in the case of $\text{Fin}$. This result will appear in [4].

Theorem 1 completely describes the class of all $F$ P-ideals that allow an analytic Hausdorff gap in their quotient, and hence gives a partial solution to [5, Problem 5.13.5], that asked for a complete description of all analytic ideals that allow an analytic Hausdorff gap in their quotient.

The rest of this note is devoted to the proof of Theorem 1. Although this paper is self-contained, familiarity with parts of [5, Chapter 1 and Chapter 5] would be useful, and we will refer to [5] for proofs of some of the results used here.

If $A$ and $B$ form an analytic Hausdorff gap in $\mathcal{P}(\mathbb{N})/I$, and $I$ is an analytic P-ideal, then $A$ and $B$ are analytic P-ideals as well, and $A \cap B = I$. By a result of Solecki (see [14]) every analytic P-ideal $I$ is of the form

$$I = \text{Exh}(\cdot) = \left\{ A : \mathbb{N} : \lim_{n \to \infty} (A \setminus [1, n)) = 0 \right\}$$

for some lower semicontinuous submeasure on $\mathbb{N}$. In particular, every such ideal is $F$. Moreover, $I$ is $F$ if and only if $\mathcal{F}$ can be chosen to be exhaustive, i.e., so that the ideal $\text{Fin}(\cdot) = \{ A : (A) < \infty \}$ coincides with $\text{Exh}(\cdot)$. Hence the existence of an analytic Hausdorff gap in a quotient over an analytic P-ideal is equivalent to the existence of a triple of lower semicontinuous submeasures with certain properties (see [5, Proposition 5.10.1]).

The following lemma gives relatively general conditions on a lower semicontinuous submeasure implying the existence of an analytic Hausdorff gap in the quotient over $\text{Exh}(\cdot)$.

**Lemma 2.** Assume is a lower semicontinuous submeasure and there are $J_n$, $k_n \in \mathbb{N}$, and $I_{ni}$ (i.e., $k_n$) for $n \in \mathbb{N}$ such that

1. $J_n \cap \mathbb{N}$ are pairwise disjoint,
2. $\max_{i,k} \{ (k) \} < 2^n$,
3. $I_{ni} (i.e., k_n)$ are pairwise disjoint subsets of $J_n$,
4. $(I_{ni})$ for all $i, k_n$, and
5. $(\forall A) (\forall n) (\forall k_n) (A \cap J_n) > 1/(n + 1)^2 \rightarrow (A \cap J_n) 1).

Then there is an analytic Hausdorff gap in the quotient over $\text{Exh}(\cdot)$.

**Proof.** For $A \in \mathbb{N}$ and $n \in \mathbb{N}$ define submeasures

$$n(A) = \{| j \in \mathbb{N} : A \cap I_{nj} \neq \emptyset | \}
$$

$$n(A) = \sup_{j,k_n} (A \cap I_{nj}).$$


Since both values $n(A)$ and $n(A)$ depend only on a nite set $A \cap J_n$, both sub-measures

$$(A) = \sum_{n=1}^{\infty} \frac{n(A)}{n},$$

$$(A) = \operatorname{sup}_{n \in \mathbb{N}} n(A)$$

are lower semicontinuous. Let $J = \bigcup_{n=1}^{\infty} J_n$; then

$$A = \operatorname{Exh}(\cdot) \cap \mathcal{P}(J),$$

$$B = \operatorname{Exh}(\cdot) \cap \mathcal{P}(J)$$

are analytic $\mathbb{P}$-ideals. In particular, $\mathcal{A}/\operatorname{Exh}(\cdot)$ and $\mathcal{B}/\operatorname{Exh}(\cdot)$ are $\mathbb{P}$-directed. To prove that $\mathcal{A}, \mathcal{B}$ form a Hausdorff gap in $\mathcal{P}(\mathbb{N})/\operatorname{Exh}(\cdot)$, we need only prove that $A$ and $B$ are $\operatorname{Exh}(\cdot)$-orthogonal and that they are not separated by a single set over $\operatorname{Exh}(\cdot)$.

**Claim 1.** $A$ and $B$ are $\operatorname{Exh}(\cdot)$-orthogonal.

**Proof.** Pick $A, B, J$. Then $(A \cap B \cap J_n) n(A) n(B)$, thus

$$(A \cap B) = \sum_{n=1}^{\infty} \frac{(A \cap B \cap J_n)}{n} n(A) n(B)) (A) (B).$$

If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $(A) < \infty$ and $\lim_{m \to \infty} B \setminus \bigcup_{n=1}^{m} J_n = 0$, thus by the above, $\lim_{m \to \infty} ((A \cap B) \setminus [1, J]) = 0$, and $A \cap B \in \operatorname{Exh}(\cdot)$, as required. 

**Claim 2.** $A$ and $B$ are not separated over $\operatorname{Exh}(\cdot)$.

**Proof.** Assume the contrary, that for some $C \subseteq \mathbb{N}$ we have $A \setminus C \in \operatorname{Exh}(\cdot)$ and $B \cap C \in \operatorname{Exh}(\cdot)$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. We claim that

$$(6) \lim_{m \to \infty} \operatorname{sup}_{n, j} \mathbb{k}_n (I_{n,f(m)} \setminus C) = 0.$$ 

Otherwise, we may nd an nite $X \subseteq \mathbb{N}, \varepsilon > 0$, and a ‘choice function’ $f$ in $\prod_{n \in X} [1, \mathbb{k}_n]$ such that

$$(I_{n,f(n)} \setminus C) > \varepsilon$$

for all $n \in X$. We may furthermore shrink $X$ so that $\sum_{n \in X} 1/n < \infty$. Let $A = \bigcup_{n \in X} I_{n,f(n)}$; then $(A) \sum_{n \in X} \varepsilon/n < \infty$, thus $A \in \mathcal{A}$. Note that $A \cap C = \emptyset$. However, for $n \in X$ we have $(A \cap J_n) (A \cap I_{n,f(n)}) \varepsilon$, therefore $A \notin \operatorname{Exh}(\cdot)$, contradicting the assumption on $C$.

By (6) for all but nitely many $n$ we have $\operatorname{sup}_{j} \mathbb{k}_n (I_{n,Jj} \setminus C) < 1/2$, and therefore

$$(7) (I_{n,\cdot} \cap C) > 1/2$$

for all but nitely many $n$ and all $j \mathbb{k}_n$.

Let $Y$ be the set of such $n$. We shall now nd $B_n J_n \cap C (n \in Y)$ so that for all $n \in Y$ we have

$$(8) (B_n) 1,$$

$$(9) n(B_n) 1/n^2.$$ 

Once these sets are constructed, let $B = \bigcup_{n \in Y} B_n$. Then $B \cap C$ and $n(B)$ $1/n$, thus $B \in \mathcal{B}$, yet $B \notin \operatorname{Exh}(\cdot)$ by (8). Since this contradicts the assumption that $C$ separates $A$ and $B$, it will suce to describe the construction of $B_n (n \in Y)$.

Fix $n \in Y$. By (7) and (2) we may nd $B_n J_n \cap C$ such that

$$(10) 1/n^2 2^n (B_n \cap I_{n,Jj}) 1/n^2.$$
for all \( j \) \( k_n \), thus (9) holds. By (5), \((B_n \cap J_n) = 1\), thus (8) holds as well. This completes the proof.

Claim 1 and Claim 2 complete the proof of lemma.

**Lemma 3.** Assume that is an exhaustive lower semicontinuous submeasure such that \( \lim_{n \to \infty} (\{ n \}) = 0 \) and \( \text{Exh}( ) \) is a proper ideal. Then satisfies the conditions of Lemma 2.

**Proof.** Since \( \text{Fin}( ) = \text{Exh}( ) \) is a proper ideal, we have \((N) = \infty\). We describe the recursive construction of \( J_n (n \in \mathbb{N}) \). Suppose \( J_i \) \((i < n)\) have already been constructed. Since \( \lim_{k \to \infty} (\{ k \}) = 0 \) and \( \text{Exh}( ) \) is lower semicontinuous, we may nd \( m \) such that

\[
\bigcup_{i=1}^{n} J_i \cap [1,m) \text{ and } \sup_{j \in m} (\{ j \}) < 2^{-n}.
\]

We will have \( \min(J_n) = m \), assuring (1) and (2). Since \((N) = \infty\) and \( \text{Exh}( ) \) is lower semicontinuous, we can nd \( m = m_0 < m_1 < m_2 < \ldots \) such that \( 1 - (m_i, m_{i+1}) \) for all \( i \). Let \( T \) be the family of all \( B \in [m, \infty) \) such that for some \( j \in \mathbb{N} \) and all \( i \) \( j \), we have

\[
(11) \quad 1/n^2 < (B \cap [m_i, m_{i+1}]), \text{ and}
\]

\[
(12) \quad (B) = 1.
\]

We claim that \( T \not\subseteq \{ (m_0, m_k) \} \) for some \( k \in \mathbb{N} \). Otherwise \( T \) is nite, and by Knig's lemma there is an nite sequence of sets \( B_i \subseteq [m_i, m_{i+1}) \) whose union satises (11) for all \( i \) yet has submeasure at most 1. But this contradicts the exhaustivity of \( \text{Exh}( ) \). Thus \( T \not\subseteq \{ (m_0, m_k) \} \) for some \( k \). Let \( k_n = m_{k+1} \), and \( I_{ni} = [m_i, m_i] \) for \( i \neq k_n \). Then clearly (3) and (4) hold. Let us check (5). If \( A \cap J_n \) is such that \( (A \cap I_{ni}) > 1/n^2 \) for all \( i \neq k_n \), then since \( A \cap [m_0, m_k) \) we have \( A \in T \), and therefore \( (A) > 1 \), as required.

This describes the recursive construction of sets satisfying the assumptions of Lemma 2.

I do not know whether the assumption that \((N) = \infty\) is necessary in Lemma 3, but a satisfying all assumptions of Lemma 3 with exhaustivity replaced by \((N) = \infty\) and failing the conclusion of Lemma 3 would have to be rather peculiar.

An ideal \( I \) is dense if every nite set has an nite subset in \( I \). It is dense below \( A \) (for some \( A \subseteq \mathbb{N} \)) if every nite subset of \( A \) has an nite subset in \( I \). Note that \( \text{Exh}( ) \) is dense if and only if \( \lim_{n \to \infty} (\{ n \}) = 0 \). An ideal has the Fachet property if it is not dense below any positive set. The following special case of a result of Todorcevic shows that it suces to prove Theorem 1 for dense ideals.

**Lemma 4.** An \( F \) \( P \)-ideal has the Fachet property if and only if it is generated by a single set over \( \text{Fin} \).

**Proof.** This follows immediately from [18, Theorem 2]; for a proof see [5, Corollary 1.2.11].

**Proof of Theorem 1.** The direct implication follows from [17, Corollary 1], since an ideal generated by a single set over \( \text{Fin} \) is isomorphic to \( \text{Fin} \).

Let \( I \) be an \( F \) \( P \)-ideal not generated by a single set over \( \text{Fin} \). By Solecki's theorem, let \( I \) be a lower semicontinuous exhaustive submeasure such that \( I = \)
Exh( ). By Lemma 4, there is an \( I \)-positive set \( A \) such that \( I \) is dense below \( A \); we may assume that \( A = \mathbb{N} \), i.e., that \( I \) is dense. This is easily equivalent to \( \lim_{k \to \infty} \langle \{k\} \rangle = 0 \). By the exhaustivity of \( \mathcal{P} \), \( (\mathbb{N}) = \infty \), thus by Lemma 3 the assumptions of Lemma 2 are satisfied and there is an analytic Hausdorff gap in \( \mathcal{P}(\mathbb{N})/I \) as required. \( \square \)

Recall that the summable ideal is dened by

\[ I_{1/n} = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty \right\}. \]

For some time Fin and \( I_{1/n} \) (and its obvious variations) were essentially the only known examples of \( F \) \( \mathcal{P} \)-ideals, until a number of radically dierent examples was found (see [5, \S 1.11]), but \( I_{1/n} \) still stands out as the simplest nontrivial \( F \) \( \mathcal{P} \)-ideal.

**Corollary 5.** There is an analytic Hausdorff gap in \( \mathcal{P}(\mathbb{N})/I_{1/n} \). \( \square \)

Corollary 6 below gives partial answers to [19, Problem 2], [5, Problem 5.13.16], and [5, Question 5.13.17]. Together with the Kunen-Todorcvec result, it also shows that the gap-spectra of quotients over \( F \) \( \mathcal{P} \)-ideals other than \( \text{Fin} \) radically dier from the gap-spectrum of \( \mathcal{P}(\mathbb{N})/\text{Fin} \). Recall that the ideals \( I \) and \( J \) are isomorphic if there are \( A \in I \), \( B \in J \) and a bijection \( h \) between \( \mathbb{N} \setminus A \) and \( \mathbb{N} \setminus B \) such that \( C \in I \) if and only if \( h^{-1}(C) \in J \). Clearly, this implies that the quotients over \( I \) and \( J \) are isomorphic.

**Corollary 6.** Assume Martin’s Axiom. Then there is a \( (\mathfrak{c}, \mathfrak{c}) \)-gap in the quotient over every \( F \) \( \mathcal{P} \)-ideal not isomorphic to \( \text{Fin} \).

**Proof.** Since Martin’s Axiom implies that \( add(\mathcal{N}) = cof(\mathcal{N}) = \mathfrak{c} \), this follows immediately from Theorem 1 and Lemma 7 below. \( \square \)

Recall that \( add(\mathcal{N}) \) is the additivity of the Lebesgue measure and that \( cof(\mathcal{N}) \) is the corality of the ideal of Lebesgue null sets.

**Lemma 7.** Assume \( add(\mathcal{N}) = cof(\mathcal{N}) \). Then there is an \( (add(\mathcal{N}), add(\mathcal{N})) \)-gap in the quotient over every analytic \( \mathcal{P} \)-ideal whose quotient contains an analytic Hausdorff gap.

**Proof.** This proof is identical to the proof of [5, Lemma 5.6.3], but we recall it for the reader’s convenience. Let \( I \) be an analytic \( \mathcal{P} \)-ideal and let \( A_0, B_0 \) be an analytic Hausdorff gap in \( \mathcal{P}(\mathbb{N})/I \). Let \( A \) be the ideal generated by \( A_0 \) and \( I \), and let \( B \) be the ideal generated by \( B_0 \) and \( I \). Both \( A \) and \( B \) are analytic \( \mathcal{P} \)-ideals.

By a result of Todorcvec ([18, Theorem 5]), for every analytic \( \mathcal{P} \)-ideal \( A \) the partially ordered set \( (A, \subseteq) \) (i.e., \( A \) ordered by the inclusion modulo nite) is Tuloz-reducible to Lebesgue null sets ordered by the inclusion (see also [5, Theorem 5.6.2]). In particular, every subset \( X \) of \( A \) of size less than \( add(\mathcal{N}) \) is bounded in \( (A, \subseteq) \), and there is a \( Y \subseteq I \) of size \( cof(\mathcal{N}) \) such that \( Y \) is \( \mathfrak{c} \)-coinical. Since \( add(\mathcal{N}) = cof(\mathcal{N}) \), we can recursively construct two chains, \( A' \subseteq A \) and \( B' \subseteq B \), increasing modulo \( \text{Fin} \) and each of length \( add(\mathcal{N}) \) such that \( A' \) is coinal in \( A \) and \( B' \) is coinal in \( B \). Hence \( A', B' \) form an \( (add(\mathcal{N}), add(\mathcal{N})) \)-gap in \( \mathcal{P}(\mathbb{N})/I \). \( \square \)

Note that Lemma 7 shows that an analytic Hausdorff gap in a quotient over an analytic \( \mathcal{P} \)-ideal can always be lled by a ccc forcing, contrasting the Kunen’s result on the indestructibility of gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \). It should be noted that the
The only two analytic P-ideals (up to the isomorphism) that are known not to have analytic Hausdorff gaps in their quotients are Fin (by [19, Corollary 1]) and ∅ Fin (by [5, Proposition 5.3.2] and Lemma 7). By a result of Todorcevic (a special case of which is Lemma 4), these are also the only two analytic P-ideals that have the Fréchet property (see [5, Corollary 1.2.11]).

**Question 8.** (a) Assume I is an analytic P-ideal that does not have the Fréchet property. Is there an analytic Hausdorff gap in its quotient?

(b) Assume I is an F-ideal that does not have the Fréchet property. Is there an analytic Hausdorff gap in its quotient?

(Note that Fin is not the only F-ideal with the Fréchet property; e.g., the ideal I_0 of [9, §1.4] generated by the branches of the tree N^cN has the Fréchet property.) Let us explicitly state a special case of Question 8(a).

**Question 9.** Assume a dense analytic P-ideal is equal to Exh( ) for a lower semicontinuous submeasure such that \((\mathbb{N}) = \infty\). Is there an analytic Hausdorff gap in its quotient?

Lemma 2 can be used to give a positive answer to this question for many ‘natural’ submeasures. While a positive solution to Question 9 may not require much beyond what is given in the proof of Theorem 1, the following question most likely requires a different approach.

**Question 10.** Is there an analytic Hausdorff gap in the quotient over the ideal of asymptotic density zero sets, \(\mathcal{Z}_0 = \{ A : \lim sup_n |A \cap n|/n = 0 \} \)?

The existence of analytic Hausdorff gaps does not immediately give new linear gaps, unless MA or a similar axiom is assumed (see Lemma 7; to see why linear Hausdorff gaps are of some interest, see e.g., [10]). The following is a stronger version of [5, Conjecture 5.13.4].

**Conjecture 11.** There is an \(\langle \text{add}(\mathcal{N}), \text{add}(\mathcal{N}) \rangle\)-gap in the quotient over every F-ideal that is not generated by a single set over Fin.

A confirmation of Conjecture 11 would give a positive answer to [19, Problem 3], where it was asked whether some standard cardinal invariants of the continuum other than \(\omega_1, \omega_1\) and the bounding number for \(\mathbb{N}^\mathbb{N}/\text{Fin}\) naturally occur in the (linear) gap spectra of some analytic quotient.

At the end, let us point out that analytic Hausdorff gaps exist in quotients over analytic ideals that are not P-ideals. For example, the quotient over the following ideal (\(\text{otp}\) stands for the order-type of a subset of an ordinal)

\[ I_{\omega^2} = \{ A : \omega^2 : \text{otp}(A) < \omega^2 \} \]

(also known as Fin Fin) has such a gap (see [5, Proposition 5.8.3]).

**References**


DEPARTMENT OF MATHEMATICS, CUNY, COLLEGE OF STATEN ISLAND AND GRADUATE CENTER, AND MATHEMATIČKI INSTITUT, KNEZA MIHAJLA 35, BELGRADE

E-mail address: ifarah@g.c.cuny.edu

URL: http://www.math.csi.cuny.edu/~farah