ABSOLUTENESS FOR UNIVERSALLY BAIRE SETS AND THE UNCOUNTABLE II

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Abstract. Using $\diamondsuit$ and large cardinals we extend results of Magidor–Malitz and Farah–Larson to obtain models correct for the existence of uncountable homogeneous sets for finite-dimensional partitions and universally Baire sets. Furthermore, we show that the constructions in this paper and its predecessor can be modified to produce a family of 2^{ω_1}-many such models so that no two have a stationary, costationary subset of ω_1 in common. Finally, we extend a result of Steel to show that trees on reals of height ω_1 which are coded by universally Baire sets have either an uncountable path or an absolute impediment preventing one.

In [4] it was shown (using large cardinals) that if a model of a theory T satisfying a certain second-order property P can be forced to exist, then a model of T satisfying P exists already. The properties P considered in [4] included the following.

(1) Containing any specified set of $\aleph_1$-many reals.
(2) Correctness about NS_{ω_1}.
(3) Correctness about any given universally Baire set of reals (with a predicate for this set added to the language).

In this paper we add the following properties, all proved under the assumption of Jensen's $\diamondsuit$ principle.

(4) Correctness about Magidor–Malitz quantifiers (and even about the existence of uncountable homogeneous sets for subsets of $[\omega_1]^{<\omega}$ and any $[\kappa]^{<\omega}$).
(5) Correctness about the countable chain condition for partial orders.
(6) Correctness about uncountable chains through (some) trees of height and cardinality ω_1.
(7) Containing a function on ω_1 dominating any such given function on a club.

These results are obtained using two main tools (both due to Woodin):

(a) iterable models (also called $\mathbb{P}_{max}$-preconditions), introduced in [22],
(b) stationary-tower forcing ([11]), or more specifically, Woodin's proof of Σ^2_1-absoluteness ([21]) .

While (b) requires higher large cardinal strength than (a), it allows one to assure (1). Aside from (1) and (7), we can obtain all of these properties simultaneously using the method (a) (with "some" being "all" for (6)). Aside from (1) and (4) we can prove all of these properties simultaneously using the method (b). Property (4)

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subsumes the next two properties in the list, but we do not see how to obtain it by stationary tower constructions. As a matter of fact, simultaneously obtaining (1) and (4), and even (1) and (6), would imply Σ^2_2-absoluteness conditioned on $\Diamond$ (see Conjecture 3.1 and Theorem 3.8). That $\Diamond$ implies that one can iterate a $\mathbb{P}_{max}$ pre-condition to be correct about the countable chain condition for partial orders on ω_1 is due to Larson and Yorioka [13].

In the fourth section we show abstractly that these arguments can be modified to build a family of 2^{ω_1} many models, no two having a stationary-costationary subset of ω_1 in common. In the final section we prove a generalized version of a theorem of Steel which can be used to show that the existence of a model of a given sentence which is correct about a given universally Baire set is absolute, given a proper class of Woodin cardinals.

1. Magidor–Malitz logic

The language $L(Q^{<\omega})$ is formed by adding to the language of set theory quantifiers Q^n for each n in ω. In this paper we restrict our attention to the so-called ω_1-interpretation of this language. That is, a formula of the form

$$Q^n x_1, x_2, \dots, x_n \phi(x_1, \dots, x_n)$$

is interpreted as saying that there is an uncountable subset of ω_1 such that every n-tuple from this set satisfies ϕ. The expressive power of this language is not diminished by requiring ϕ to be *symmetric*, i.e., invariant under permuting its free variables $x_1, \dots, x_n$. Recall that $[Z]^n$ is the set of all n-element subsets of Z and $[Z]^{<\omega}$ is the set of all finite subsets of Z. Given $K \subseteq [\omega_1]^n$, we say (following [5]) that $X \subseteq \omega_1$ is a *K-cube* if and only if $[X]^n \subseteq K$ (or just a *cube* if the corresponding partition is clear). Since an interpretation of a symmetric formula is a subset of $[\omega_1]^n$, correctness for Magidor–Malitz logic is equivalent to correctness for the existence of uncountable cubes (note that the existence of countable cubes of any given order type is absolute between transitive models.) We therefore say that a model M is *correct* for Magidor–Malitz logic (or for Ramsey quantifiers) if ω_1^M is uncountable and, for every $n \in \omega$ and every $K \subset ([\omega_1^M]^n)^M$ definable in M from parameters in M (note that we do not assume here that M satisfies ZFC), there is an uncountable K-cube in V if and only if one exists in M. The following theorem was proved in [14].

Theorem 1.1 ($\Diamond$)**.** *If T is a theory in the language $L(Q^{<\omega})$ and it is consistent with ZFC that T has a model which is correct for Magidor–Malitz logic, then T has such a model.* □

Requiring T from Theorem 1.1 to contain a large enough fragment of ZFC is not a loss of generality. Here (and throughout this paper) "large enough fragment of ZFC" means large enough to make ultrapower embeddings for generic ultrafilters on ω_1 elementary. This requires some form of the Axiom of Choice, but the theory ZFC$^\circ$ from [12] suffices. In this case Theorem 1.1 can be equivalently reformulated as follows:

Theorem 1.2 ($\Diamond$)**.** *If a theory T extends a large enough fragment of ZFC and it is consistent, then there exists a model M for T such that ω_1^M is uncountable and M is correct about the existence of uncountable cubes for partitions of $[\omega_1^M]^n$ for all $n \in \mathbb{N}$.* □

The model M guaranteed by this result is an ω-model but it is not necessarily well-founded. As a matter of fact, asserting well-foundedness of M requires some large cardinal strength (see [4, Proposition 8.9]). A model M of a large enough fragment of ZFC is *correct for partitions of* $[\omega_1^M]^{<\omega}$ if it is correct about the existence of uncountable cubes for partitions $K \subseteq [\omega_1^M]^{<\omega}$ in M. This assertion implies ω_1^M is uncountable, but note that we do not require it to be well-founded. Our first result, proved at the end of this section as Theorem 1.12, is a strengthening of Theorem 1.2.

Theorem 1.3 ($\diamondsuit$)**.** *If a theory T extends a large enough fragment of ZFC and it is ω-consistent, then there exists a model M for T such that ω_1^M is uncountable and M is correct about the existence of uncountable cubes for partitions of $[\omega_1^M]^{<\omega}$ that belong to M, for each $n \in \omega$.*

The difference between Theorem 1.2 and Theorem 1.3 is that in the latter the dimension of K is not bounded. In Proposition 2.8 we show that the conclusion of Theorem 1.3 is stronger than the conclusion of Theorem 1.1.

Continuing along the lines of [4], we also show that in the presence of large cardinals correctness for partitions of $[\omega_1]^{<\omega}$ can be combined with correctness for any given universally Baire set of reals, with respect to the logic of forceability. Analogously to [4, §5], given a set of reals A let $L(A)$ be the language of set theory with an additional unary predicate for A. We say that a model M is *correct for A and partitions of* $[\omega_1^M]^{<\omega}$ (in short, *$L(Q^{\leq\omega}, A)$-correct*) if $\omega_1^M = \omega_1$, M interprets the additional unary symbol as $A \cap M$ and it is correct for partitions of $[\omega_1^M]^{<\omega}$. Since correctness for Π^1_1-sets already implies well-foundedness of ω_1^M, assuming $\omega_1^M = \omega_1$ is not a loss of generality in this context. The symbol $\leq \omega$ in the term '$L(Q^{\leq\omega}, A)$-correct' is perhaps misleading, but it was chosen in order to emphasize the difference between partitions of $[\omega_1]^{<\omega}$ and of $[\omega_1]^n$ for a fixed n, since $L(Q^{<\omega})$ is an established notation for Magidor–Malitz logic. The reader may wish to compare the following theorem with the results in [1].

Theorem 1.4. *Suppose that there exist proper class many Woodin cardinals, let A be a universally Baire set of reals, and let T be a set of sentences in $L(A)$. Suppose that there exists an $L(Q^{\leq\omega}, A)$-correct model of T in some set forcing extension. Then there exists an $L(Q^{\leq\omega}, A)$-correct model of T in every set forcing extension satisfying $\diamondsuit$.*

Proof. Immediate from Lemma 1.5, Lemma 1.6, and Theorem 1.7 below. □

The logic $L_{\omega_1\omega}(Q^{<\omega})$ allows countable disjunctions in addition to quantifiers Q^n ($n \in \mathbb{N}$). It is well-known that an analogue of Theorem 1.1 for this logic can be proved using the methods of [14]; for a proof see e.g., [5]. By standard methods (see e.g., [3] for the case of $L_{\omega_1\omega}(Q)$), the case of Theorem 1.4 when A is a Borel set follows. This cannot be extended even to analytic sets unless large cardinals are assumed ([4, Proposition 8.7]). An alternative way for proving these results using iterated generic ultrapowers is outlined in our proofs of Theorem 1.3 and Theorem 1.12. Note that this semantical result does not recover the full strength of Keisler or Magidor–Malitz theorems. This is because these results provide completeness theorems for logics $L_{\omega_1\omega}(Q)$ and $L_{\omega_1\omega}(Q^{<\omega})$. We do not know whether this can be achieved for the logic with the quantifier corresponding to the existence of uncountable cubes of $[\omega_1]^{<\omega}$.

1.1. **Proofs.** Continuing in the vein of [4], the proofs of this section employ $\mathbb{P}_{max}$ preconditions, also known as *iterable pairs.* ZFC$^\circ$ is a fragment of ZFC that holds in $H(\theta)$ for a regular $\theta \geq \aleph_2$ (the precise definition will not be needed here; see [12, §1]). If N_0 is a transitive model of ZFC$^\circ$ and I_0 is a normal ideal on $\omega_1^{N_0}$ in N_0, then an *iteration of* (N_0, I_0) *of length* γ is $\langle (N_\eta, I_\eta), j_{\xi\eta}, G_\eta, \xi < \eta \leq \gamma\rangle$, where $j_{\xi\eta}\colon (N_\xi, I_\xi) \to (N_\eta, I_\eta)$ is a commuting family of elementary embeddings, $G_\eta \subseteq (\mathcal{P}(\omega_1)/I_\eta)^{N_\eta}$ is a generic filter, $j_{\eta\eta+1}$ is the corresponding generic ultrapower embedding, and for a limit η and $\xi < \eta$, $j_{\xi\eta}$ and N_η are the direct limit of $j_{\xi\zeta}$ and N_ζ for $\xi < \zeta < \eta$. An iteration is *well-founded* if all the models occurring in it are well-founded. A pair is *iterable* if all of its iterations are well-founded. If $A \in N_0$ is a universally Baire set then a pair (N_0, I_0) is A-*iterable* if it is iterable and its iterations compute A correctly. We shall follow the standard convention and identify an *iteration* of length γ with the final model together with the embedding $j_{0\gamma}\colon (N_0, I_0) \to (N_\gamma, I_\gamma)$. For more information we refer the reader to [22, 12].

Lemma 1.5 below is proved in [4, Lemma 3.3]. In the presence of a proper class of Woodin cardinals, universally Baire sets of reals are δ^+-weakly homogeneously Suslin for all δ.

Lemma 1.5. *Assume that $\delta < \lambda$ are a Woodin and a measurable cardinal respectively, A and $\omega^\omega \setminus A$ are δ^+-weakly homogeneously Suslin sets of reals, and ϕ is a sentence whose truth is preserved by σ-closed forcing. If there exists a partial order in V_δ that forces that ϕ holds in $H(\theta)$ for some $\theta \geq (2^\lambda)^+$, then there exists an A-iterable model $(N, {\mathrm{NS}_{\omega_1}}^N)$ that satisfies ϕ.* □

A forcing $\mathbb{P}$ has *property K_n* if for each family p_α ($\alpha < \omega_1$) of conditions there is an uncountable $I \subseteq \omega_1$ such that $p_{\alpha(1)}, \dots, p_{\alpha(n)}$ has a lower bound for all $\alpha(1), \dots, \alpha(n)$ in I. It has *precaliber* $\aleph_1$ if for each family p_α ($\alpha < \omega_1$) of conditions there is an uncountable $I \subseteq \omega_1$ such that every finite subset of p_α ($\alpha \in I$) has a lower bound. The following is well-known.

Lemma 1.6. *Assume $K \subseteq [Z]^{<\omega}$. The statement 'there are no uncountable K-cubes' is absolute for σ-closed forcing extensions and for precaliber $\aleph_1$ forcing extensions. If furthermore $K \subseteq [Z]^n$ then the statement 'there are no uncountable K-cubes' is absolute for property K_n forcing extensions.*

Proof. In all three cases the forcing preserves $\aleph_1$, and therefore we only need to check that it does not add an uncountable cube. Assume $\mathbb{P}$ is σ-closed and it forces the existence of an uncountable cube, and let $\dot{H}$ be its name. Pick a decreasing ω_1-sequence of conditions p_α such that p_α decides the first α elements of $\dot{H}$. Then the decided set is uncountable and a K-cube. If $\mathbb{P}$ has precaliber $\aleph_1$ and it forces an existence of an uncountable cube $\dot{H}$, pick p_α that decides αth element of $\dot{H}$ is ξ_α. If every finite subset of $\{p_\alpha \mid \alpha \in I\}$ has a lower bound, then $\{\xi_\alpha \mid \alpha \in I\}$ is a cube. The proof of the case when $\mathbb{P}$ has the property K_n is very similar. □

Theorem 1.7 ($\diamondsuit$)**.** *If (M, I) is an iterable pair then there is an iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1 such that M^* is correct for partitions of $[\omega_1]^{<\omega}$.*

We give two proofs of Theorem 1.7. The first uses the presentation of Magidor–Malitz logic given in [5] and its modularity makes it more susceptible to generalizations. The second is shorter and more straightforward.

1.2. **First proof of Theorem 1.7.** We use standard model-theoretic terminology as in [5] or any standard model theory text. For a transitive model N of ZFC° let L_N be the language of set theory extended by adding the constants for elements of N (and all universally Baire sets and NS_{ω_1}). Let $(\forall^{\aleph_0} x \in z)\phi(x)$ be the shortcut for 'z is uncountable and $\phi(x)$ holds for all but countably many $x \in z$.' For a 1-type Φ in L_N let

$$\partial\Phi(x) = \{(\forall^{\aleph_0} z \in x)\phi(z) \mid \phi(z) \in \Phi(z)\}.$$

Also let $\partial^\circ\Phi = \Phi$ and $\partial^{n+1}\Phi = \partial(\partial^n\Phi)$. A type Φ is *totally unsupported* in N if $\partial^n\Phi$ is not realized in N for all $n \geq 0$.

If $j\colon N \to N^*$ is an elementary embedding and Φ is an N-type then $j\Phi$ is a type defined in the natural way:

$$j\Phi(x) = \{\phi(x, j(\vec{a})) \mid \phi(x, \vec{a}) \in \Phi(x)\}$$

(here $\vec{a}$ stands for an arbitrary n-tuple of parameters and $j(\vec{a})$ has the natural interpretation). We emphasize that in the following lemma the types Φ_i are not required to belong to N.

Lemma 1.8. *Assume (N, I) is an iterable pair and types Φ_i ($i < \omega$) are totally unsupported in N. Then there is N-generic $G \subseteq I^+$ such that each $j\Phi_i$ is totally unsupported in N^*, where $j\colon N \to N^*$ is the corresponding generic embedding.*

Proof. Let $\nu = \omega_1^N$. Enumerate all pairs $(f, \partial^n\Phi_i)$ for $f\colon \nu \to \nu$ in N. Pick G recursively, by finding a decreasing sequence A_k ($k \geq 0$) in I^+. Assure that A_{2k} is in k-th dense subset of I^+ in N. To find A_{2k+1}, consider the k-th pair $(f, \partial^n\Phi_i)$. If there is $\phi \in \partial^n\Phi_i$ such that the set

$$B_\phi = \{\alpha \in A_{2k} \mid N \models \neg\phi(f(\alpha))\}$$

is I-positive then let $A_{2k+1} = B_\phi$.

We claim that such a ϕ has to exist. Otherwise let $D = \nabla\{B_\phi \mid B_\phi \in \mathrm{NS}_{\omega_1}\}$. Then $D \in N$ and $C = A_{2k} \setminus D$ is equal to A_{2k} modulo I. Also, for every $\phi \in \partial^n\Phi_i$ we have that $N \models (\forall^{\aleph_0}\alpha \in C)\phi(f(\alpha))$. We consider two possibilities. First, if $C' = f[C]$ is uncountable, then C' realizes $\partial^{n+1}\Phi_i$, contradicting our assumption that this type is totally unsupported in N. Otherwise there is $\alpha \in N$ such that $f^{-1}(\{\alpha\}) \cap C$ is uncountable. Therefore $N \models \phi(\alpha)$ for all $\phi \in \partial^n\Phi_i$, contradicting the assumption that $\partial^n\Phi_i$ is totally unsupported in N.

The construction clearly satisfies the requirements. We need to check that N^* does not realize any one of $j\partial^n\Phi_i$. Fix a name $\dot{x}$ for an element of N^*, $n \in \mathbb{N}$, and $i \in \mathbb{N}$. Then $\mathrm{Int}_G(\dot{x}) = [f]_G$ for some $f \in N$. Let k be such that $(f, \partial^n\Phi_i)$ appears as the kth pair. Then $A_{2k+1} \subseteq \{\alpha \in \omega_1 \mid N \models \neg\phi(f(\alpha), \vec{a})\}$ for some $\phi \in \partial^n\Phi_i$, hence $A_{2k+1} \Vdash \neg\phi(\dot{x}, j(\vec{a}))$ and therefore $\dot{x}$ does not realize $\partial^n\Phi_i$. □

Let N be a model of ZFC°, let $X \subseteq \omega_1^N$ (not necessarily in N), and let $\psi(x)$ be a formula. We write

$(\mathrm{aa}\, x \in X)\psi(x)$ for 'the set of $x \in X$ such that $\neg\psi(x)$ holds (in V) is bounded in ω_1^N,'

$(\mathrm{aa}\, \vec{x} \in X^n)\psi(\vec{x})$ for $(\mathrm{aa}\, x_1 \in X)(\mathrm{aa}\, x_2 \in X)\dots(\mathrm{aa}\, x_n \in X)\psi(\vec{x})$, where $\vec{x}$ is an n-tuple of variables.

For a type $\Phi = \Phi(x_0, x_1, \dots)$ in N and $X \subseteq \omega_1^N$ write ($\vec{x}$ is assumed to be of appropriate length, this length being ω in the definition of $\Phi_X^{<\omega}$)

$$(*) \qquad \begin{aligned} \Phi_X(x) &= \{\phi(x, \vec{a}) \in \Phi \mid \vec{a} \in N, (\mathrm{aa}\, x \in X) N \models \phi(x, \vec{a})\}. \\ \Phi_X^n(\vec{x}) &= \{\phi(\vec{x}, \vec{a}) \in \Phi \mid \vec{a} \in N, (\mathrm{aa}\, \vec{x} \in X^n) N \models \phi(\vec{x}, \vec{a})\}. \\ \Phi_X^{<\omega}(\vec{x}) &= \bigcup_n \Phi_X^n(\vec{x} \restriction n). \end{aligned}$$

Assume $K \subseteq [\omega_1^N]^{<\omega}$ is in N and X is a K-cube. Then every finite set realizing $\Phi_X^{<\omega}$ is in K. Also, since we are allowing parameters from N in the definitions of Φ_X, the set $\{a \in \omega_1^N \mid a \text{ realizes } \Phi_X\}$ is in this situation automatically a K-cube. Finally, note that if $Z \in N$ realizes $\partial^1 \Phi_X$ then 'almost all' finite subsets of Z are in K.

We suppress writing parameters $\vec{a} \in N$ from now on, with the understanding that ϕ is a formula in the language extended by adding constants for all elements of the model N. The proof of the following lemma is modeled on [5, Lemma 7.3.4].

Lemma 1.9. *Assume that N is a model of ZFC° and $X \subseteq \omega_1^N$. If $\partial^d \Phi_X$ is realized in N for some $d \geq 1$ then there is an uncountable $Y \in N$ such that $\Phi_Y^{<\omega} \supseteq \Phi_X^{<\omega}$.*

In particular, if X is an uncountable K-cube and $\partial^d \Phi_X$ is realized in N for some $d \geq 1$, then in N there exists an uncountable K-cube.

Proof. For $n \in \mathbb{N}$ and $d \in \mathbb{N}$ write

$(\forall^{\aleph_0} x \in^d Z)\phi(x)$ for $(\forall^{\aleph_0} x_1 \in Z)(\forall^{\aleph_0} x_2 \in x_1) \dots (\forall^{\aleph_0} x_d \in x_{d-1})\phi(x)$,
$(\forall^{\aleph_0} \vec{x} \in Z^n)\phi(x)$ for $(\forall^{\aleph_0} x_1 \in Z)(\forall^{\aleph_0} x_2 \in Z) \dots (\forall^{\aleph_0} x_n \in Z)\phi(\vec{x})$,
$(\forall^{\aleph_0} \vec{x} \in^d Z^n)\phi(x)$ for $(\forall^{\aleph_0} x_1 \in^d Z)(\forall^{\aleph_0} x_2 \in^d Z) \dots (\forall^{\aleph_0} x_n \in^d Z)\phi(\vec{x})$.

Hence Z realizes $\partial^d \Phi$ in N if and only if $N \models (\forall^{\aleph_0} \vec{x} \in^d Z)\phi(\vec{x})$ for each $\phi \in \Phi$.

Claim. *Assume H realizes $\partial^d \Phi_X$ for some $d \geq 1$. Then for all $m \geq 0$ and $n \geq 1$ we have $(\forall \phi \in \Phi_X^{m+n})(\mathrm{aa}\, \vec{a} \in X^m) N \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{a}, \vec{x})$.*

Proof. Induction on n, for all m simultaneously. Assume $n = 1$ and pick $\phi \in \Phi_X^{m+1}$ (with parameters suppressed) so that $(\mathrm{aa}\, \vec{x} \in X^{m+n}) N \models \phi(\vec{x})$. For $\vec{y} \in X^m$ let $\psi_{\vec{y}}(x)$ be $\phi(\vec{y}, x)$. Since $(\mathrm{aa}\, \vec{y} \in X^m)\psi_{\vec{y}}(x) \in \Phi_X(x)$, by the assumption on H we have

$$(\mathrm{aa}\, \vec{y} \in X^m) N \models (\forall^{\aleph_0} x \in^d H)\psi_{\vec{y}}(x).$$

Now assume the assertion holds for n and fix $\phi \in \Phi_X^{m+n+1}$. By the inductive assumption, $(\mathrm{aa}\, \vec{w} \in X^m)(\mathrm{aa}\, z \in X) N \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{w}, z, \vec{x})$. Fix $\vec{w} \in X^m$ such that

$$(\mathrm{aa}\, z \in X) N \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{w}, z, \vec{x}).$$

If we let $\psi_{\vec{w}}(y)$ be $(\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{w}, y, \vec{x})$, then $\psi_{\vec{w}}(y) \in \Phi_X(y)$ and therefore

$$N \models (\forall^{\aleph_0} y \in^d H)(\forall^{\aleph_0} \vec{x} \in^d H)\phi(\vec{w}, y, \vec{x}),$$

equivalently $N \models (\forall^{\aleph_0} \vec{x} \in^d H^{n+1})\phi(\vec{w}, \vec{x})$. □

Assume $\partial^d \Phi_X$ is realized in N for some d. By the claim, for every n and $\phi \in \Phi_X^n$ we have $M \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{x})$. Write $x \in^d Z$ for $x \in \bigcup \cdots \bigcup Z$, where $\bigcup$ occurs $d-1$ times, and $A \subseteq^d B$ if $A \subseteq \bigcup \cdots \bigcup B$, where $\bigcup$ occurs $d-1$ times. Note that the quantifier $(\forall^{\aleph_0} x \in^d z)$ introduced earlier agrees with these conventions.

A set $E \subseteq^d H$ is *solid* if for all $m, n \in \mathbb{N}$, every $\vec{e} \in E^m$, and every $\phi \in \Phi_X^{m+n}$ we have $M \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{e}, \vec{x})$. Since E is solid if and only if each of its finite subsets is solid, by Zorn's Lemma we can find a maximal solid $E \subseteq X$. We claim E is uncountable. Assume otherwise. For all $m, n \in \mathbb{N}$, $\vec{e} \in E^n$ and $\phi \in \Phi_X^{m+n}$ we have $N \models (\forall^{\aleph_0} \vec{x} \in^d H^n)\phi(\vec{e}, \vec{x})$. Since there are only countably many such quadruples $(m, n, \vec{e}, \phi)$, we can find $a \in^d H$ such that $a \notin E$ and $E \cup \{a\}$ is still solid, contradicting the maximality of E.

Let $Y \subseteq^d H$ be uncountable and solid. Then for every $n \geq 1$ and $\phi \in \Phi_X^n$ we have $N \models \phi(\vec{b})$ for all $\vec{b} \in Y^n$, therefore Y is as required. □

The following consequence of Lemma 1.9 is an extension of [5, Lemma 7.3.4].

Lemma 1.10. *Assume N is a model of ZFC$^\circ$ and $K \in N$ is such that N models '$K \subseteq [\omega_1]^{<\omega}$ and there are no uncountable K-cubes.' If $X \subseteq \omega_1^N$ is a maximal K-cube, then Φ_X is totally unsupported in N.*

Proof. If Φ_X is realized by some $b \in N$, then $b \neq a$ for all $a \in X$ and $X \cup \{b\}$ is still a K-cube, contradicting the maximality of X. Now assume $\partial^n \Phi_X$ for some $n \geq 1$ is realized by some H. By Lemma 1.9 there is an uncountable $Y \in N$ such that every $\vec{a} \in Y^{<\omega}$ satisfies $\vec{a} \in K$, contradicting our assumption. □

First Proof of Theorem 1.7. It will suffice to construct $M^* = M_{\omega_1}$ with correct ω_1 and such that for every $K \subseteq [\omega_1]^{<\omega}$ in M, if there are no uncountable K-cubes in M then there are no uncountable K-cubes in V.

Let $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond$-sequence. We recursively build an iteration

$$\langle (M_\alpha, I_\alpha), G_\beta, j_{\alpha\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

of (M, I) and a set $U \subset \omega_1$ as follows. For each $\alpha < \omega_1$, let Φ^α be Φ_{σ_α} as defined in (*) using M_α for N, and put $\alpha \in U$ if and only if Φ^α is totally unsupported in M_α. When constructing G_α, we apply Lemma 1.8 to ensure that each $j_{\beta(\alpha+1)}\Phi^\beta$ $(\beta \in U \cap (\alpha+1))$ is totally unsupported in $M_{\alpha+1}$.

Having completed the construction of the iteration, fix $K \subset [\omega_1]^{<\omega}$ in M_{ω_1} such that in M_{ω_1} there exists no uncountable K-cube $Y \subset \omega_1$. Let X be a maximal K-cube of ω_1, i.e., such that $[X]^{<\omega} \subset K$ but $[X \cup \{\xi\}]^{<\omega} \not\subset K$ for any $\xi \in \omega_1 \setminus X$. Let $\alpha < \omega_1$ and $k \in M_\alpha$ be such that $K = j_{\alpha\omega_1}(k)$, and let $\beta \in [\alpha, \omega_1)$ be such that

1. $\omega_1^{M_\beta} = \beta$;
2. $\sigma_\beta = X \cap \beta$;
3. $j_{\beta\omega_1}\Phi^\beta$ is contained in Φ_X as computed over M_{ω_1}; $(M_\beta, X \cap \beta)$ is an elementary submodel of (M_{ω_1}, X), in particular σ_β is $j_{\alpha\beta}(k)$-maximal over M_β.

Then Lemma 1.10 implies Φ_β is totally unsupported in M_β, hence $\beta \in U$. Then $j_{\beta\omega_1}\Phi^\beta$ is totally unsupported in M_{ω_1}. If $\xi \in X \setminus \beta$, then (3) implies $j_{\beta\omega_1}\Phi_{X\cap\beta}$ is realized by ξ in M_{ω_1}, a contradiction. Therefore $X \subseteq \beta$, and we conclude that there are no uncountable K-cubes in M_{ω_1}. □

1.3. **Second proof of Theorem 1.7.** This proof is similar and uses the following notion from [9]: a subset of $[\omega_1]^{<\omega}$ is *stationary* if it contains a subset of every club subset of ω_1. More generally, given a normal uniform ideal I on ω_1 we say that a subset of $[\omega_1]^{<\omega}$ is I-positive if it contains a subset disjoint from each member of I. We also let $a < b$ mean $\sup(a) < \inf(b)$, when a and b are sets of ordinals.

Let $\langle \sigma_\delta : \delta < \omega_1 \rangle$ be a $\Diamond$-sequence. We construct an iteration

$$\langle M_\alpha, I_\alpha, G_\beta, j_{\alpha,\gamma} : \beta < \omega_1, \alpha \leq \gamma \leq \omega_1 \rangle$$

in the usual way, with the following modifications. We allow the ordinary construction to determine cofinally many members of each G_β, including the first one, and fill in the intervening steps ourselves. For each $\beta < \omega_1$, let Φ_β be the set of unary formulas with constants in M_β satisfied by every member of σ_β, and for $\gamma \in [\beta, \omega_1]$, let Φ^γ_β be $j_{\beta\gamma}\Phi_\beta$, the set of ϕ such that for some $\phi' \in \Phi_\beta$, ϕ is ϕ' with its constants replaced by their $j_{\beta\gamma}$-images.

While constructing G_β, we include a stage for each tuple (B, f, ξ) of the following type:

- B is a stationary subset of $[\omega_1^{M_\beta}]^p$ in M_β for some nonzero $p \in \omega$;
- $f\colon B \to \omega_1^{M_\beta}$ is a function in M_β with $f(b) \geq \max(b)$ for all $b \in dom(f)$;
- $\xi \leq \beta$.

When we come to the stage for a given (B, f, ξ), we have some $A \in I^+_\beta$ which we have decided to put into G_β. If A has stationary intersection with the complement of the first-coordinate projection of B, then we put this intersection in G_β. Otherwise, if possible, we find some $\phi \in \Phi^\beta_\xi$ such that the following set A' is in I^+_β.

- if $p = 1$, $A' = \{\alpha \in A \mid M_\beta \models \neg\phi(f(\alpha))\}$;
- if $p = m + 1$, A' is the set of $\alpha \in A$ for which for I^+_β-many $a \in [\omega_1^{M_\beta}]^m$, $(\alpha, a) \in B$ and $M_\beta \models \neg\phi(f(\alpha, a))$.

Then we put A' in G_β. If there is no such ϕ, we do nothing at this stage.

Now, at the end of our construction, consider some $K \subset [\omega_1]^{<\omega}$ in M^* and suppose that $X \subset \omega_1$ is an uncountable K-cube. Fix α and K' such that $K = j_{\alpha\omega_1}(K')$. We will derive a contradiction from the assumption that for no $\beta \in (\alpha, \omega_1)$ is there an uncountable $j_{\alpha\beta}(K')$-cube in M_β.

Let Φ be the set of unary formulas with constants from M that are satisfied in M by every member of X. Note then that the set of countable ordinals satisfying all the members of Φ in M is uncountable. Fix $\xi \in [\alpha, \omega_1)$ such that $\omega_1^{M_\xi} = \xi$, $\sigma_\xi = X \cap \xi$ and $\Phi^{\omega_1}_\xi$ is the set of formulas in Φ with constants in the $j_{\xi\omega_1}$-image of M_ξ.

Now suppose that $\beta \geq \xi$, $p \in \omega \setminus \{0\}$, $A \in G_\beta$, $B \in M_\beta$ is a stationary subset of $[\omega_1^{M_\beta}]^p$ with first-coordinate projection containing A modulo I_β, and $f\colon B \to \omega_1^{M_\beta}$ is a function in M_β. Then there exist a (possibly 0) $k \in \omega$, a k-tuple b contained in the critical sequence of $j_{\xi,\beta}$, an I_ξ-positive $B' \subset [\omega_1^{M_\xi}]^{k+p}$ in M_ξ and a function $f'\colon B' \to \omega_1^{M_\xi}$ in M_ξ such that $B = \{a \mid (b \cup a) \in j_{\xi\beta}(B') \wedge \max(b) < \min(a \setminus b)\}$ and $f(a) = f'(b \cup a)$ for all $a \in B$. By induction on k, we show, under the assumption that $f(b) \geq \max(b)$ for all $b \in dom(f)$, that there exists a $\phi \in \Phi^\beta_\xi$ such that the following set A'_ϕ is in I^+_β.

- if $p = 1$, $A'_\phi = \{\alpha \in A \mid M_\beta \models \neg\phi(f(\alpha))\}$;
- if $p = m + 1$, A'_ϕ is the set of $\alpha \in A$ for which for I^+_β-many $a \in [\omega_1^{M_\beta}]^m$, $(\alpha, a) \in B$ and $M_\beta \models \neg\phi(f(\alpha, a))$.

By our construction, this shows that $X \subset \xi$.

In the case where $k = 0$ and $p = 1$, if there is no ϕ as desired then $A \in \mathcal{P}(\omega_1)^{M_\beta} \setminus I_\beta$ and $f \in (\omega_1^{\omega_1})^{M_\beta}$ are such that A forces $[f]_{G_\beta}$ to satisfy each member of $\Phi^{\beta+1}_\xi$. For each $n \in \omega$, we show that there is a club $E_n \subset \omega_1^{M_\beta}$ in M_β such that for all increasing n-tuples $\langle \nu_0, \ldots, \nu_{n-1}\rangle$ from $A \cap C$, $\langle f(\nu_0), \ldots, f(\nu_{n-1})\rangle$ is an increasing sequence and $\{f(\nu_0),\ldots,f(\nu_{n-1})\}$ is in $j_{\alpha\beta}(K')$." Then in M_β there exists a sequence of clubs $\langle E'_n : n<\omega\rangle$ such that each E'_n satisfies this statement for n, and their intersection is the desired set.

Note first of all that since $f(\alpha)\geq\alpha$ for all $\alpha\in dom(f)$, we may assume by shrinking if necessary that for all finite sequences $\langle\nu_0,\ldots,\nu_n\rangle$ from $A\cap C$, $\langle f(\nu_0),\ldots,f(\nu_n)\rangle$ is an increasing sequence.

For each n, by reverse (finite) induction starting at $i=n-1$ and ending at $i=0$ we show that the following holds for each i: for each i-tuple a from σ_ξ, M_β satisfies the sentence "there is a club subset $C\subset\omega_1$ such that for all $(n-i)$-tuples $\langle\nu_0,\ldots,\nu_{n-i-1}\rangle$ from $A\cap C$, if $\langle f(\nu_0),\ldots,f(\nu_{n-i-1})\rangle$ is an increasing sequence above $\sup(a)$, then $a\cup\{f(\nu_0),\ldots,f(\nu_{n-i-1})\}$ is in $j_{\alpha\beta}(K')$." Since σ_ξ is a $j_{\alpha\xi}(K')$-cube, this holds for $i=n-1$. It if holds for $i=j+1$, then for each j-tuple a from σ_ξ there is a club set $D_a\in\mathcal{P}(\omega_1)^{M_\beta}$ such that in M_β, for each $\chi\in D_a\cap A$ there is a club $C_{a,\xi}$ such that for all $(n-i)$-tuples $\langle\nu_0,\ldots,\nu_{n-i-1}\rangle$ from $A\cap C_{a,\xi}$, if $\langle f(\chi),f(\nu_0),\ldots,f(\nu_{n-i-1})\rangle$ is an increasing sequence above $\sup(a)$, then $a\cup\{f(\chi),f(\nu_0),\ldots,f(\nu_{n-i-1})\}$ is in $j_{\alpha\beta}(K')$. Then letting $E^{i,a}_n=D_a\cap\triangle\{C_{a,\chi}:\chi\in D\cap A\}$, we have the desired statement for i and a, and $E^{0,\emptyset}_n$ is the desired club E_n.

The case where $k=0$ and $p=m+1$ is similar. Suppose that for every $\phi\in\Phi_\beta$ the set A'_ϕ is nonstationary. Let B' be the set of members of B whose least members are in A. For each $n\in\omega$ we find a club $E_n\subset\omega_1^{M_\beta}$ in M_β such that for all increasing n-tuples $\langle b_0,\ldots,b_{n-1}\rangle$ from $B'\cap[E_n]^n$, $\langle f(b),f(b_0),\ldots,f(b_{n-1})\rangle$ is an increasing sequence and $\{f(b),f(b_0),\ldots,f(b_{n-1})\}$ is in $j_{\alpha\beta}(K')$. Then M_β has an intersection of such clubs as above. Again, we may assume by shrinking if necessary that $f(b)<\alpha$ for all $\alpha\in A$ and $b\in B'\cap[\alpha]^{<\omega}$.

Again, by reverse finite induction starting at $i=n-1$ and ending at $i=0$ we show that the following holds for each i: for each i-tuple a from σ_ξ, M_β satisfies the sentence "there is a club subset $C\subset\omega_1$ such that for all $(n-i)$-tuples $\langle b_0,\ldots,b_{n-i-1}\rangle$ from $B'\cap\mathcal{P}(C)$, if $\langle f(b_0),\ldots,f(b_{n-i-1})\rangle$ is an increasing sequence above $\sup(a)$, then $a\cup\{f(b_0),\ldots,f(b_{n-i-1})\}$ is in $j_{\alpha\beta}(K')$." Since σ_ξ is a $j_{\alpha\xi}(K')$-cube, this holds for $i=n-1$. It if holds for $i=j+1$, then for each j-tuple a from σ_ξ there is a club set $D_a\in\mathcal{P}(\omega_1)^{M_\beta}$ such that in M_β, for each $b\in B'\cap\mathcal{P}(D)$ there is a club $C_{a,b}$ such that for all $(n-i)$-tuples $\langle b_0,\ldots,b_{n-i-1}\rangle$ from $B'\cap\mathcal{P}(C_{a,b})$, if $\langle f(b),f(b_0),\ldots,f(b_{n-i-1})\rangle$ is an increasing sequence above $\sup(a)$, then $a\cup\{f(b),f(b_0),\ldots,f(b_{n-i-1})\}$ is in $j_{\alpha\beta}(K')$. Then letting

$$E^{i,a}_n=D_a\cap\triangle\{C_{a,b}:b\in B'\cap\mathcal{P}(D)\},$$

we have the desired statement for i and a, and $E^{0,\emptyset}_n$ is the desired club E_n.

If $k=j+1$, let $\eta=\max(b)$ and let $b^-=b\setminus\{\eta\}$. Then by our induction hypothesis there is a $\phi\in\Phi_\xi$ such that the set of $\alpha<\omega_1^{M_\eta}$ for which for stationarily many $a\in[\omega_1^{M_\eta}]^n$, $b^-\cup\{\alpha\}\cup a\in j_{\xi\eta}(B')$ and $M_\eta\models\neg\phi(j_{\xi\eta}(f)(b^-\cup\{\alpha\}\cup a))$ is in G_η. Then ϕ is as desired.

Remark. If M is a model of a sufficient fragment of ZFC which is correct about ω_1 we say that M is *correct about* NS_{ω_1} if $\mathrm{NS}_{\omega_1}\cap M={\mathrm{NS}_{\omega_1}}^M$. We note that either of the above proofs of Theorem 1.3 allows one to easily add correctness about NS_{ω_1} to M^* to the conclusion. To see this, note the the proofs of Theorem 1.7 do not require putting any specific set into the generic filter at a given stage. The standard $\mathbb{P}_{max}$ bookkeeping argument then allows putting the images of each stationary subset of ω_1 in each model of the iteration into the generic filter stationarily often, thus assuring NS_{ω_1}-correctness (see the game-theoretic formulation of the basic iteration lemma for $\mathbb{P}_{max}$ in [12]).

In [19], to $S \subseteq \omega_1$ Todorcevic associates $K_S \subseteq [\omega_1]^2$ such that if there is an uncountable K-cube then S contains a club and if S contains a club then a proper forcing notion adds an uncountable K-cube. Hence one may ask whether correctness about partitions of $[\omega_1]^2$, or for partitions of $[\omega_1]^{<\omega}$, implies correctness about NS_{ω_1}. However, the above proof can easily be adapted to make M^* *in*correct about NS_{ω_1}, showing that correctness about partitions of $[\omega_1]^{<\omega}$ does not imply correctness about NS_{ω_1}. To see this, take some costationary subset of ω_1 in some model and keep it and its images out of all the generic filters, thus assuring that the image of this set will be nonstationary in V even though it is stationary in the final model. With these observations one gets the following strengthening of Theorem 1.4.

Theorem 1.11. *Suppose that there exist proper class many Woodin cardinals, let A be a universally Baire set of reals, and let T be a set of sentences in $L(A)$. Suppose that there exists an $L(Q^{\leq\omega}, A)$-correct model of T in some set forcing extension. Then in every set forcing extension satisfying $\diamondsuit$ there exist $L(Q^{\leq\omega}, A)$-correct models M, M' of T such that M is correct about NS_{ω_1} and M' is not.* □

1.4. **Proof of Theorem 1.3.** Theorem 1.3 follows from the proof of Theorem 1.7 once we notice that that proof of correctness for partitions of $[\omega_1]^{<\omega}$ did not require iterability (i.e., we did not use the fact that the models produced were wellfounded). One could rephrase Theorem 1.7 as follows.

Theorem 1.12 ($\diamondsuit$). *Assume M is a countable model of a large enough fragment of ZFC. Then M has an elementary extension M^* whose ω_1 is uncountable and which is correct about partitions of $[\omega_1^{M^*}]^n$ for each $n \in \omega$. If M is an ω-model, then it has an elementary extension M^* whose ω_1 is uncountable and which is correct about partitions of $[\omega_1^{M^*}]^{<\omega}$. Moreover, M^* is correct about all Borel sets with codes in the well-founded part of M.*

Proof. The proof of this is largely the same as the proofs of Theorem 1.7. Let $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. We recursively build a sequence

$$\langle (M_\alpha, I_\alpha), G_\beta, j_{\alpha\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1 \rangle$$

such that each G_β an M_β-normal ultrafilter of $\mathcal{P}(\omega_1^{M_\beta})^{M_\beta}$ and $M_{\beta+1}$ is the G_β-ultrapower of M_β. We don't require the ultrafilters G_β to be generic over the models M_β.

Note that if $X \in M_\beta$ is countable in M_β and $f\colon \omega_1^{M_\beta} \to X$ is a function in M_β, then the M_β-normality of G_β implies that f is constant on a set in G_β. Conversely, if X is uncountable and f is injective, then f represents a new element of $j(X)$ in the ultrapower (these facts are standard; the point is just that they don't depend on wellfoundedness). It follows that elements of M_{ω_1} will have uncountable extent if and only if they are uncountable in M_{ω_1}.

One can likewise construct the iteration following the construction in the proof of Theorem 1.7. The argument goes through without change except for one point. If n is a nonstandard integer of M_β, then clearly we cannot argue by finite reverse induction on n. If the integers of M are nonstandard, then we have to settle for correctness about partitions of $[\omega_1^{M^*}]^n$ for each standard integer n. □

While M^* constructed in the above proof of Theorem 1.12 need not be well-founded, the wellfounded part of its ω_1 contains the well-founded part of ω_1^M. Therefore, assuming M is well-founded, M^* is correct about $L_{\omega_1\omega}$ sentences belonging to M. Note that the method of the proof gives proof of the following consequence of Keisler's completeness theorem for $L_{\omega_1\omega}(Q)$: For any $L_{\omega_1\omega}(Q)$ sentence ϕ the statement 'ϕ has a correct model' is forcing-absolute.

In [14] Magidor and Malitz provide an axiomatization for $L(Q^{<\omega})$ and, using $\diamondsuit$, prove the corresponding completeness theorem. Their axiomatization involves schemata of arbitrarily high complexity (and necessarily so; see [15]), Our result is purely semantic and we do not know whether there is a reasonable axiomatization for the logic of 'correctness for partitions of $[\omega_1]^{<\omega}$.' Note that we have completely avoided the problem of defining the syntax for this logic by embedding T into ZFC.

2. More on correctness for partitions of finite sets

2.1. **Partitions of $[\kappa]^{<\omega}$ for $\kappa > \omega_1$.** If (M, I) is an iterable pair and $j\colon (M, I) \to (M^*, I^*)$ is an iteration, then M^* is equal to the collection of all sets of the form $j(f)(a)$, where f is a function in M and a is a finite subset of the critical sequence corresponding to j. It follows that if M is countable and j is an iteration of length ω_1, then M^* is the union of countably many sets each having cardinality $\aleph_1$ in M^*. The results of the previous section then give the following.

Theorem 2.1 ($\diamondsuit$). *If (M, I) is an iterable pair, then there is an iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1 such that M^* is correct about the existence of uncountable cubes for partitions of $[\kappa]^{<\omega}$ for every $\kappa \in M$.* □

Theorem 2.2. *Suppose that there exist proper class many Woodin cardinals, let A be a universally Baire set of reals, and let T be a set of sentences in $L(A)$. Suppose that there exists an A-correct model of T that is correct about the existence of uncountable cubes for partitions of any $[\kappa]^{<\omega}$ in some set forcing extension. Then there exists such model in every set forcing extension satisfying $\diamondsuit$.* □

Correctness about the existence of uncountable cubes for partitions of pairs implies the following.

Corollary 2.3 ($\diamondsuit$). *Suppose that there exist proper class many Woodin cardinals. Let T be a large enough fragment of ZFC that holds in some forcing extension. Then there is an uncountable transitive model M of T that is correct about the countable chain condition of all partial orders in M. We can also assure M is A-correct for any given universally Baire set A.* □

Analogously to Theorem 1.3 we obtain the following.

Theorem 2.4 ($\diamondsuit$). *If a theory T extends a large enough fragment of ZFC and it is consistent, then there exists a model M for T such that ω_1^M is uncountable and M is correct about the existence of uncountable cubes for partitions of $[\kappa]^{<\omega}$ that belong to M for every $\kappa \in M$.* □

2.2. **$[\omega_1]^n$ vs. $[\omega_1]^{<\omega}$.** By the results of §1, the existence of class many Woodin cardinals and $\diamondsuit$ imply the following.

$R^{<\omega}$ If A is universally Baire and ϕ is a sentence of $L(Q^{<\omega}, A)$ that has a correct model in some forcing extension, then ϕ has a correct model.

$R^{\leq\omega}$ If A is universally Baire and ϕ is a sentence of $L(Q^{\leq\omega}, A)$ that has a correct model in some forcing extension, then ϕ has a correct model.

The case of $R^{<\omega}$ ($R^{\leq\omega}$, respectively) when A is a Borel set easily follows from the method of [14] (Theorem 1.12, respectively) and it does not require large cardinals. The general case of $R^{<\omega}$ (and therefore of $R^{\leq\omega}$) requires large cardinals; this follows from [4, §8.2], where the weaker logic $L(Q, A)$ was considered.

In this section we shall show that $R^{\leq\omega}$ is a genuine strengthening of $R^{<\omega}$ already in the case when A is Borel. For a universally Baire set A consider the following two assertions.

$R^{<\omega}(A)$ If ϕ is a sentence of $L(Q^{<\omega}, A)$ that has a correct model in some forcing extension, then ϕ has a correct model.

$R^{\leq\omega}(A)$ If ϕ is a sentence of $L(Q^{\leq\omega}, A)$ that has a correct model in some forcing extension, then ϕ has a correct model.

For a collection Γ consisting of universally Baire sets of reals, $R^{<\omega}(\Gamma)$ and $R^{\leq\omega}(\Gamma)$ assert respectively that $R^{<\omega}(A)$ and $R^{\leq\omega}(A)$ hold for each $A \in \Gamma$. Along with proving that $\diamondsuit$ implies $R^{<\omega}$(Borel), Magidor and Malitz showed that forcing with a Cohen algebra preserves $R^{<\omega}$(Borel) ([14, p. 257]). This shows that $R^{<\omega}$(Borel) does not imply CH. Below we dwell on their ideas and further investigate in which models $R^{<\omega}$(A) and $R^{\leq\omega}(A)$ hold.

Lemma 2.5. *Assume $R^{<\omega}$(Borel). Then there exists a Suslin tree, a ccc-destructible (ω_1, ω_1)-gap in $P(\omega)/Fin$ and an entangled set of reals.*

Proof. This is immediate; for the definitions see e.g., [18]. □

Before stating a less trivial consequence of $R^{\leq\omega}$, let us record an immediate consequence of Lemma 1.6.

Lemma 2.6. *Assume A is universally Baire.*

(1) If $R^{<\omega}(A)$ holds then it holds in every forcing extension by a forcing that has property K_n for all n.

(2) If $R^{\leq\omega}(A)$ holds then it holds in every forcing extension by a forcing that has precaliber $\aleph_1$. □

The content of (1) of Lemma 2.7 below is in the well-known equivalence of statements 'the real line is not covered by $\aleph_1$ many Lebesgue null sets' and 'the Lebesgue measure algebra has precaliber $\aleph_1$.' We reproduce its proof for the convenience of the reader and to ensure that the former assertion's expressibility in $L_{\omega_1\omega}(Q^{\leq\omega})$ is transparent. Clause (2) is essentially given in [14, p. 257], where it was shown that $R^{<\omega}$ is preserved by the forcing for adding any number of Cohen reals. We don't state the obvious variations for $R^{<\omega}(A)$ or $R^{\leq\omega}(A)$ of two propositions below.

Proposition 2.7. *(1) Assume $R^{\leq\omega}$(Borel). Then the real line can be covered by $\aleph_1$ Lebesgue null sets.*

(2) Every model of ZFC has a forcing extension in which $R^{\leq\omega}$(Borel) holds but the real line cannot be covered by $\aleph_1$ meager sets.

(3) Every model of ZFC has a forcing extension in which $R^{<\omega}$(Borel) holds but the real line cannot be covered by $\aleph_1$ Lebesgue null sets.

Proof. (1) We shall find a sentence ϕ of $L(Q^{\leq\omega}$,Borel) that has a correct model if and only if the real line can be covered by $\aleph_1$ null sets.

Assume for a moment there is an increasing sequence of null G_δ sets N_α ($\alpha < \omega_1$) such that $\bigcup_{\alpha<\omega_1} N_\alpha = \mathbb{R}$. Let $F_\alpha \subseteq \mathbb{R}$ be a compact set of positive measure disjoint from N_α, and define $K \subseteq [\omega_1]^{<\omega}$ by $s \in K$ if and only if $\bigcap_{\alpha\in s} F_\alpha \neq \emptyset$. An uncountable K-cube gives a family of compact sets with a finite intersection property, and the intersection of this family is disjoint from $\bigcup_\alpha N_\alpha$. Therefore a sentence ϕ asserting enough ZFC plus 'There exist compact sets of positive measure F_α ($\alpha < \omega_1$) such that the partition K defined by $s \in K$ if and only if $\bigcap_{\alpha\in s} F_\alpha \neq \emptyset$ has no uncountable cube' has a correct model in every extension in which the real line can be covered by $\aleph_1$ many null sets. (Note that we only need correctness for a rather simple Borel set.)

We claim that the converse is also true. Assume otherwise. Let M be a model correct for ϕ and assume the real line cannot be covered by $\aleph_1$ many null sets. Let F_α ($\alpha < \omega_1$) be compact positive sets witnessing ϕ in M. By downward Löwenheim–Skolem theorem we may assume M is of size $\aleph_1$. By the ccc-ness of Lebesgue measure algebra there is a compact positive set $F \in M$ forcing that the generic filter contains uncountably many of the F_α's. Let $r \in F$ be a real that avoids all null sets coded in M. Then r is a random real over M, hence $H = \{\alpha < \omega_1 \mid r \in F_\alpha\}$ is uncountable. Then H is an uncountable cube, contradicting the assumption on M.

(2) A model of $\diamondsuit$ satisfies $R^{<\omega}$(Borel) by the $L_{\omega_1\omega}$ variant of Theorem 1.1, e.g., Theorem 1.12. The standard forcing for adding $\aleph_2$ Cohen reals has precaliber $\aleph_1$ and it forces that the real line cannot be covered by fewer than $\aleph_2$ meager sets . By Lemma 2.6 the extension obtained by adding Cohen reals to a model of $\diamondsuit$ is as required.

(3) This is similar to the proof of (2), using the well-known fact that every measure algebra has property K_n for all n. □

The model of (3) of Proposition 2.7 gives the following.

Proposition 2.8. *Every model of ZFC has a forcing extension in which $R^{<\omega}$(Borel) holds, but $R^{\leq\omega}$(Borel) fails.* □

3. Extensions of the Σ^2_1-absoluteness argument

Let us recall a conjecture of John R. Steel presented in [20].

Conjecture 3.1. *Assuming sufficient large cardinals, every Σ^2_2 sentence ϕ that holds in some forcing extension satisfying $\diamondsuit$ holds in all forcing extensions satisfying $\diamondsuit$.*

Since $\neg$CH is Σ^2_2, the requirement that $\diamondsuit$ holds in the forcing extension in which ϕ holds cannot be dropped in Conjecture 3.1. Note the resemblance to the following result of Woodin ([21], [11], [2]).

Theorem 3.2. *Assume there are class many measurable Woodin cardinals. Then every Σ^2_1 sentence ϕ that holds in some forcing extension holds in all forcing extensions satisfying CH.* □

This was one of the starting points to the first part of this paper ([4]). By standard facts about Woodin cardinals ([11, Theorem 2.5.10]), Conjecture 3.1 is equivalent to its consequence stating that $\diamondsuit$ (together with appropriate large cardinals) implies every Σ^2_2 statement true in some forcing extension satisfying $\diamondsuit$. Results of §1 can be interpreted as confirmation of Conjecture 3.1 in the case when the Σ^2_2 sentence ϕ states the existence of a partition of $[\omega_1]^{<\omega}$ satisfying some first-order properties with no uncountable cubes. However, Conjecture 3.1 is not likely to be proved by iterating $\mathbb{P}_{max}$ preconditions as in §1. A major obstacle is that for each $\mathbb{P}_{max}$ precondition (N, I) there exists a real number not belonging to any of the iterates of (N, I) (take e.g., the real coding (N, I)). At this point we do not see how to prove a version of absoluteness for Magidor–Malitz logic using the stationary tower or Todorcevic's method of using a saturated ideal in a Lévy collapse of a large cardinal to $\aleph_2$ (see [2]). In this section we solve some other technical problems related to Conjecture 3.1. Assuming $\diamondsuit$ and using stationary tower, we find a model containing all reals and satisfying the following

(1) Correctness about the countable chain condition for partial orders on ω_1 (Theorem 3.6).
(2) Correctness about uncountable chains through (some) trees of height and cardinality ω_1 (Theorem 3.6).
(3) Containing a function on ω_1 dominating any such given function on a club (Proposition 3.10).

While both (1) and (2) are consequences of correctness for the existence of uncountable cubes for partitions of $[\omega_1]^2$, (3) cannot be obtained using $\mathbb{P}_{max}$ preconditions. The following fundamental fact ([22, 12]) about $\mathbb{P}_{max}$ iterations shows that for every iterable pair (M, I) there is a function $f\colon \omega_1 \to \omega_1$ such that for every iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1, f dominates every member of $\omega_1^{\omega_1} \cap M^*$ on a club: if (M, I) is an iterable pair coded by a real x such that M is countable and $x^\#$ exists, then for every countable ordinal β and every iteration $j\colon (M, I) \to (M^*, I^*)$ of length β, the ordinal height of M^* is less than the least x-indiscernible above β. This is one of the points in Woodin's proof that the saturation of NS_{ω_1} together with the existence of $H(\aleph_2)^\#$ implies CH fails ([22, §3.1]).

The version for correctness about the countable chain condition was proved in [13] before the work in this paper and its predecessor. The version for trees on ω_1 is left to the reader.

3.1. **The setup.** Definitions of the stationary towers $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$ can be found e.g., in [22] or [11]. We work with the terms from Section 4 of [4]. There, $V[h]$ is a forcing extension of V, and M is a model whose ω_1 (which we also call λ) is a Woodin cardinal in $V[h]$, which sees a club $C \subset \lambda$ contained in the Woodin cardinals of $V[h]$ whose limit points β have the property that $C \cap \beta$ is contained modulo a tail in each club subset of β in $V[h]$, and such that $V_\zeta[h] \in M$ for some strongly inaccessible cardinal $\zeta > \lambda$ of $V[h]$. Inside the model M, then, one can construct $V[h]$-generics for $\mathbb{Q}^{V[h]}_{<\lambda}$. The following theorem (due to Woodin, see [11, 4]) summarizes the situation. As discussed in [4], the assumption of a measurable Woodin cardinal can be replaced with a weaker, so-called *full*, Woodin cardinal.

Theorem 3.3. *Suppose that δ is a measurable Woodin cardinal and $\kappa > \delta$ is a Woodin cardinal. Then there is a condition $a \in \mathbb{P}_{<\kappa}$ such that if $G \subset \mathbb{P}_{<\kappa}$ is a V-generic and $a \in G$, then $G \cap V_\delta$ is a V-generic filter for $\mathbb{Q}_{<\delta}$ and, letting $j\colon V \to M$ be the generic ultrapower induced by G,*

- *$j(\omega_1^V) = \delta$;*
- *κ is a Woodin cardinal in $V[G]$;*
- *M is closed under sequences of length less than κ in $V[G]$;*

- *there exists in M a club set $C \subset \delta$ contained in the Woodin cardinals of V such that for each limit point β of C, $(C \cap \beta) \setminus D$ is a bounded subset of β for each club $D \subset \beta$ in V.* □

Note that in this context, since δ is strongly inaccessible in V and ω_1 is in M, in M there exist V-generic filters for each partial order in V_δ. In Theorem 3.6 we show that if $\Diamond$ holds in M then M can build the generic so that the final image model is correct about the countable chain condition for its partial orders on ω_1, and correct about whether its trees of height and cardinality ω_1 have uncountable paths. In each case the argument involves inserting cofinally many steps into the construction of each generic filter H_α (or just stationarily many), in order to ensure that a set given by a fixed $\Diamond$-sequence is not an initial segment of an uncountable path or antichain.

Lemma 3.4. *Suppose that $\delta < \lambda$ are Woodin cardinals, $G \subset \mathbb{Q}_{<\delta}$ is V-generic, $a \in \mathbb{Q}_{<\lambda}$ is such that $\mathbb{Q}_{<\delta}$ regularly embeds into the restriction $\mathbb{Q}_{<\lambda}(a)$ of $\mathbb{Q}_{<\lambda}$ to a. Let $j \colon V \to N$ be the embedding induced by G. Let $T \in N$ be a tree on ω_1^N of height ω_1^N with no uncountable branches in N. Let p be a cofinal branch of T in some outer model of $V[G]$, let b be a condition in $\mathbb{Q}_{<\lambda}(a)/\mathbb{Q}_{<\delta}$ and let f be a function in V from b to ω_1^V. Then there is a $b' \leq b$ forcing that $[f]_H$ does not extend p, where H is the induced $\mathbb{Q}_{<\lambda}$-generic.*

Proof. It is a standard fact that in this situation N and $V[G]$ agree about the existence or nonexistence of cofinal paths through T. More generally, they agree about Σ_1 sentences with parameters in $\mathcal{P}(\delta)^N$. This follows from the fact that there is in some outer model an elementary embedding with critical point above δ from N into a model containing $\mathcal{P}(\delta)^{V[G]}$; as an example of this, see the relationship between M_λ and M on page 95 of [11]. Consider then the set of nodes in T which b forces $[f]_H$ to extend. This set cannot be p, but it must be a pairwise compatible set, so it cannot contain p, either. So extend b to b' forcing that $[f]_H$ does not extend some fixed member of p. □

Lemma 3.5. *Suppose that $\delta < \lambda$ are Woodin cardinals, $G \subset \mathbb{Q}_{<\delta}$ is V-generic, $a \in \mathbb{Q}_{<\lambda}$ is such that $\mathbb{Q}_{<\delta}$ regularly embeds into the restriction $\mathbb{Q}_{<\lambda}(a)$ of $\mathbb{Q}_{<\lambda}$ to a. Let $j \colon V \to N$ be the embedding induced by G. Let $P \in M$ be a partial order on ω_1^N which is c.c.c. in M. Let A be a predense subset of P in some outer model of $V[G]$, let b be a condition in $\mathbb{Q}_{<\lambda}(a)/\mathbb{Q}_{<\delta}$ and let f be a function in V from b to ω_1^V. Then there is a $b' \leq b$ forcing that $[f]_H$ is compatible with some member of A, where H is the induced $\mathbb{Q}_{<\lambda}$-generic.*

Proof. By the same standard fact as in the proof of Lemma 3.4, N and $V[G]$ agree about the existence or nonexistence of uncountable antichains of P. Consider then the set X of elements of P which b forces $[f]_H$ to be incompatible with. If X does not contain A then the lemma clearly holds, so assume otherwise. In $V[G]$, and thus in M there is a countable $X' \subset X$ such that every element of P is compatible with an element of X if and only if it is compatible with an element of X'. Since A is predense and $A \subset X$, this means that X' is predense, so every element of P is compatible with some member of X'. Since X' is countable, it will continue to have this property in the $\mathbb{Q}_{<\lambda}$-ultrapower, contradicting that b forces that $[f]_H$ will be incompatible with every member of X'. □

We say that a model N is correct about the countable chain condition on partial orders on ω_1 if $\omega_1^N = \omega_1$ and for every partial order P on ω_1 in N, P has an uncountable antichain in N if and only if it has one in V. We say that a model N is correct about uncountable paths through trees of height and cardinality ω_1 if $\omega_1^N = \omega_1$ and for every tree of height and cardinality ω_1 in N, P has an uncountable branch in N if and only if it has one in V.

Theorem 3.6. *Suppose that κ is a measurable Woodin cardinal. Let A be a κ-universally Baire set of reals and let ϕ be a sentence in the language of set theory with one additional unary predicate. Then the following hold, where the models are taken to be over a language with an additional unary predicate for the interpretation of A in the corresponding model.*

1. *Suppose that some partial order $P \in V_\kappa$ forces the existence of a model N of ϕ which is correct about uncountable paths through trees of height and cardinality ω_1. Then in every set forcing extension of V by a forcing in V_κ which satisfies $\diamondsuit$ there exists a model M of ϕ which is correct about uncountable paths through trees of height and cardinality ω_1.*
2. *Suppose that some partial order $P \in V_\kappa$ forces the existence of a model N of ϕ which is correct about the ccc on partial orders on ω_1. Then in every set forcing extension of V by a forcing in V_κ which satisfies $\diamondsuit$ there exists a model M of ϕ which is correct about the ccc on partial orders on ω_1.*

Correctness about NS_{ω_1} can be added to conclusion of Theorem 3.6 and Theorem 3.7 below. The proof of each theorem involves adding a few steps to the construction of each H_α in the proof of the corresponding theorem in [4]. The point is that the model M from that proof constructs a collection of $V[h]$-generic filters H_α ($\alpha < \lambda$), and if at a given stage a $\diamondsuit$-sequence in M guesses a cofinal branch in a given tree in the current model $((V[h][H_\alpha])_\zeta)$, Lemma 3.4 says that we can extend our construction in such a way that that branch is not extended in the extension of the tree. Similarly, if at a given stage a $\diamondsuit$-sequence in M guesses a maximal antichain in a given partial order in the current model, Lemma 3.5 says that we can extend our construction in such a way that that antichain is not extended in the extension of the partial order. The new elements of the construction discussed here require only cofinally many stages of the construction of each H_α, and so do not interfere with the original argument. They do not interfere with each other, either: one can combine these two arguments to obtain both correctness properties. However, they do interfere with the argument that allows the construction in M to put any given real in the model it is constructing, as adding a given real to a model requires control over the entire construction of the generic filter at that stage. If we restrict to the set of ω_1-trees, however, then we can obtain correctness about paths while picking up all the reals. The point here is that for each level of each ω_1-tree in the construction, there is only one stage where nodes on that level are created. So once Lemma 3.4 has been applied to make sure that a given path is not extended, that path can never be extended accidentally later in the construction, while picking up a given real, say. Combining this observation with the arguments from Section 4 of [4], we have the following.

Theorem 3.7. *Suppose that κ is a measurable Woodin cardinal. Let A be a κ-universally Baire set of reals and let ϕ be a sentence in the language of set theory with one additional unary predicate. Suppose that some partial order $P \in V_\kappa$ forces* *the existence of a model N of ϕ (with the additional symbol interpreted as A^{V^P}) which is correct about uncountable paths through ω_1-trees and which contains all the reals. Then in every set forcing extension of V by a forcing in V_κ which satisfies $\Diamond$ there exists a model M of ϕ (with the additional symbol interpreted as $A \cap M$) which is correct about uncountable paths through ω_1-trees and contains any given $\aleph_1$-many reals.* □

The following well-known observation shows that a version of this construction which obtained correctness about uncountable paths through trees of height and cardinality ω_1 while picking up all the reals would show that $\Diamond$ decides all Σ^2_2 sentences with respect to models obtained by set forcing.

Theorem 3.8. *Suppose that M is a transitive model of ZFC + CH which contains the reals, and for every tree T of height and cardinality ω_1 in M, T has an uncountable path in M if and only if it has one in V. Suppose that M satisfies a sentence ϕ of the form $\exists A \subset \mathbb{R} \forall B \subset \mathbb{R} \psi(A, B)$, where the quantifiers of ψ range over the reals. Then ϕ holds in V.*

Proof. Let $A \subset \mathbb{R}$ be such that $\forall B \subset \mathbb{R}\psi(A, B)$ holds in M, let $\langle x_\alpha : \alpha < \omega_1\rangle$ be a listing of the real in M, and for each $\alpha < \omega_1$ and any set of reals X let $X \restriction \alpha$ denote $X \cap \{x_\beta : \beta < \alpha\}$. Then for any $X \subset \mathbb{R}$ and any formula θ whose quantifiers range only over reals, $\theta(A, B)$ holds if and only if there is a club $C \subset \omega_1$ such that for all $\alpha \in C$, $\theta_\alpha(X \restriction \alpha)$ holds, where θ_α is the formula θ with its quantifiers restricted to $\{x_\alpha : \alpha < \omega_1\}$. Since CH holds in M, there is a natural tree in M of height and cardinality ω_1 giving the initial segments of a supposed club $C \subset \omega_1$ and set $B \subset \mathbb{R}$ such that for all $\alpha \in C$, $\neg\psi_\alpha(A \restriction \alpha, B \restriction \alpha)$ holds. Since A witnesses ϕ in M, there is no uncountable path through this tree in M, and thus by the assumption of the theorem, there is none in V, which means that A witnesses ϕ in V. □

As we noted above, the following theorem follows easily from Theorem 1.7, though it can be proved much more easily using the approach from Lemmas 3.4 and 3.5.

Theorem 3.9 ($\Diamond$). *If (M, I) is an iterable pair, then there is an iteration $j : (M, I) \to (M^*, I^*)$ of (M, I) such that the model M^* is correct about the ccc on partial orders on ω_1 and about the existence of uncountable paths through trees of height and cardinality ω_1.* □

Proposition 3.10. *In the situation of Theorem 3.3, assuming $\Diamond$ holds in $V[h]$ then there is a function in $V[h]$ whose image under the $\mathbb{Q}^{V[h]}_{<\lambda}$-generic embedding can be made to dominate any function from λ to λ in M on a club.*

The proof given below uses the following standard fact about the stationary tower $\mathbb{Q}_{<\lambda}$ (see [11]): for any ordinal $\gamma < \lambda$, the function on $\mathcal{P}_{\aleph_1}(\gamma)$ which takes each $X \subset \gamma$ to the ordertype of X represents γ in the generic ultrapower. The stationary set defined in Lemma 3.11 below then forces that the image of g will take the value γ at δ. This contrasts with the situation when *canonical function bounding* (see [10], for instance) holds; then, no function in $\omega_1^{\omega_1}$ can represent any ordinal above the ω_2 of the ground model.

Lemma 3.11. *Let δ be a Woodin cardinal and let $\langle \sigma_\alpha : \alpha < \omega_1\rangle$ be a sequence witnessing that $\Diamond$ holds. Define $g : \omega_1 \to \omega_1$ by letting $g(\alpha)$ be the corresponding ordertype if σ_α codes a wellordering of α, and 0 otherwise. Then for any $\gamma > \delta$ the* *set of countable $X \subset V_\gamma$ such that* $\mathrm{o.t.}(X \cap \gamma) = g(\mathrm{o.t.}(X \cap \delta))$ *and X captures every predense subset of $\mathbb{Q}_{<\delta}$ in X is compatible with every condition in $\mathbb{Q}_{<\delta}$.*

Proof. Pick $a \in \mathbb{Q}_{<\delta}$ and $F\colon [V_\gamma]^{<\omega} \to V_\gamma$. By a standard argument (see Corollary 2.7.12 of [11]), there exists a continuous increasing $\subset$-chain $\langle X_\alpha : \alpha \leq \omega_1\rangle$ of countable subsets of V_γ such that

- $X_0 \cap \cup a \in a$;
- each X_α is closed under F;
- each $X_{\alpha+1}$ end-extends X_α below δ;
- each X_α captures every predense subset of $\mathbb{Q}_{<\delta}$ in X_α.

Let $f\colon \omega_1 \to (X_{\omega_1} \cap \gamma)$ be a bijection, and let S be the set of $(\alpha, \beta) \in \omega_1^2$ such that $f(\alpha) \leq f(\beta)$. For club many $\alpha < \omega_1$, the ordertype of $X_\alpha \cap \delta$ is α and $f[\alpha] = X_\alpha \cap \gamma$. For some such α, σ_α codes $S \cap \alpha^2$, and this α is as desired. □

Proof of Proposition 3.10. Suppose $H \subset \mathbb{Q}_{<\eta_\alpha}$ is a $V[h]$-generic filter as in the Σ^2_1-absoluteness proof in [4]. Suppose that γ is less than $\eta_{\alpha+1}$ (which itself can be chosen to arbitrarily large below λ). Let a be the stationary set of countable subsets of V_γ given by Lemma 3.11. Then $\mathbb{Q}_{<\eta_\alpha}$ regularly embeds into the restriction of $\mathbb{Q}_{<\eta_{\alpha+1}}$ to conditions below a, and a forces that the image of g under the induced $\mathbb{Q}_{<\eta_{\alpha+1}}$-embedding will take value γ at η_α. In this way, Lemma 3.11 can be used in M to ensure that the image of the function g dominates any given function in M on a club. □

An another warm-up problem towards proving Σ^2_2-absoluteness from $\diamondsuit$ is the question of whether the model M^* constructed in the Σ^2_1 absoluteness proof can be made to contain a sequence which is a $\diamondsuit$-sequence in M. If M had contained a canonical function which necessarily dominated every function in N on a stationary set then this would have shown that M^* could not contain a $\diamondsuit$-sequence of M. The following observation relates preserving $\diamondsuit$-sequences to the Σ^2_2 absoluteness problem. The idea is very similar to the proof of Theorem 3.8.

Lemma 3.12. *Suppose that M is a model of ZFC + $\diamondsuit$. Then for every Σ^2_2 sentence ϕ which holds in M there is a $\diamondsuit$-sequence Σ_ϕ in M such that ϕ holds in any outer model of M in which Σ_ϕ remains a $\diamondsuit$-sequence.*

Proof. Let $\langle \sigma_\alpha : \alpha < \omega_1^M\rangle$ be a $\diamondsuit$ sequence in M. Fix a Σ^2_2 sentence

$$\phi \equiv \forall X \subset \mathbb{R} \exists Y \subset \mathbb{R}\, \psi(X, Y),$$

where all quantifiers in ψ range over reals and integers, and suppose that $A \subset \mathbb{R}^M$ witnesses ϕ in M. Let $\langle a_\alpha : \alpha < \omega_1^M\rangle$ be a wellordering of $H(\omega_1)^M$ in M, and for each $\alpha < \omega_1$, let ψ_α be the formula obtained by restricting all the real quantifiers of ψ to range only over $\mathbb{R}_\alpha = \mathbb{R} \cap \{a_\beta : \beta < \alpha\}$. For each $B \in \mathcal{P}(\mathbb{R})^M$ there is a club $C \in \mathcal{P}(\omega_1)^M$ such that for all $\alpha \in C$, the structure

$$\langle \{a_\beta : \beta < \alpha\}, A \cap \mathbb{R}_\alpha, B \cap \mathbb{R}_\alpha, \in\rangle$$

is an elementary submodel of $\langle H(\omega_1), A, B, \in\rangle$. It follows that, letting Z be the set of $\alpha < \omega_1^M$ such that $\psi_\alpha(A \cap \mathbb{R}_\alpha, \sigma_\alpha)$ holds, $\Sigma_\phi = \langle \sigma_\alpha : \alpha \in Z\rangle$ is a $\diamondsuit$-sequence in M. The same argument shows that if Σ_ϕ remains a $\diamondsuit$-sequence in some outer model of M, then A witnesses ϕ in this outer model. □

3.2. **Trees of models.** Let $S(\alpha, \beta, \gamma)$ $(\alpha, \beta, \gamma < \omega_1)$ be pairwise disjoint stationary subsets of ω_1. Let N_x $(x \in 2^{\leq \omega_1})$ be a collection of transitive models of ZFC such that $\mathcal{P}(\omega_1)^{N_x}$ is countable for each N_x, and for each pair $x \subset y$ in $2^{\leq \omega_1}$ let $j_{xy} \colon N_x \to N_y$ be an elementary embedding with critical point $\omega_1^{N_x}$. Suppose that for each $x \in 2^{\leq \omega_1}$ of limit length the model N_x is the direct limit of the models N_y $(y \subsetneq x)$ under these embeddings. For each $x \in 2^{<\omega_1}$ of limit length we let $\langle A^x_\alpha : \alpha \in dom(x) \rangle$ list the stationary, costationary subsets of $\omega_1^{N_x}$ in N_x, in such that a way that $x \subset y$ implies that $A^y_\alpha = j_{xy}(A^x_\alpha)$ for each $\alpha \in dom(x)$.

Suppose further that

- whenever $x(\gamma) = 0$ and $\omega_1^{N_x} \in S(\alpha, \beta, \gamma)$ for some β and some $\alpha < \omega_1^{N_x}$, then $\omega_1^{N_x}$ is in A^y_α for all $y \supset x$, and that
- whenever $x(\gamma) = 1$ and $\omega_1^{N_x} \in S(\alpha, \beta, \gamma)$ for some α and some $\beta < \omega_1^{N_x}$, then $\omega_1^{N_x}$ is not in A^y_β for any $y \supset x$.

Now let x, y be any two distinct elements of 2^{ω_1}, and suppose that $\gamma < \omega_1$ is such that $x(\gamma) = 0$ and $y(\gamma) = 1$. Let $C_x = \{\omega_1^{N_{x'}} : x' \subsetneq x\}$ and let $C_y = \{\omega_1^{N_{y'}} : y' \subsetneq y\}$, and let B_x and B_y be two stationary, costationary subsets of ω_1 in N_x and N_y respectively. Fix $\alpha, \beta < \omega_1$ such that $B_x = A^x_\alpha$ and $B_y = A^y_\beta$, and let η be the maximum of $\min(C_x \setminus (\alpha + 1))$ and $\min(C_y \setminus (\beta + 1))$. Then $B_x \triangle B_y$ contains $C_x \cap C_y \cap S(\alpha, \beta, \gamma) \cap (\omega_1 \setminus \eta)$, and so is stationary.

The argument just given show that the constructions given in this section can be modified to produce a $2^{<\omega_1}$-tree of models whose paths produce models with no stationary, costationary subsets of ω_1 in common. During the construction of a $\mathbb{P}_{max}$ iteration or a sequence of stationary tower generic as in the Σ^2_1 absoluteness argument, one can take any given stationary, costationary set in the current model and choose whether to put the current ω_1 in the image of this set (for $\mathbb{P}_{max}$ this is standard, for the Σ^2_1 argument this was shown in [4]). The tree-of-models construction above is an attempt to capture the idea that if CH implies some Σ^2_1 statement ϕ (which doesn't follow from ZFC), then it implies that there are 2^{ω_1} many distinct witnesses ϕ. Undoubtedly this can be made more precise.

4. Special trees on reals

In [16], Steel shows that in the presence of large cardinals, trees on reals in $L(\mathbb{R})$ without uncountable branches in V have an absolute impediment preventing such a branch from being added by forcing. In this section we generalize this result to trees coded by arbitrary universally Baire sets, using results of Woodin on the inner model HOD (the class model consisting of all hereditarily ordinal definable sets, see [6, 8]) in place of inner model theory.

Given a tree T, we let T^+ denote the set of sequences whose proper initial segments are all in T. We think of the trees on reals in this section as sets of reals.

Theorem 4.1 ([16])**.** *Assume that there exist infinitely many Woodin cardinals below a measurable cardinal. Let $T \subset \mathbb{R}^{<\omega_1}$ be a tree in $L(\mathbb{R})$. Then exactly one of the following holds.*

- *There is an uncountable branch of T in V.*
- *There is a function $f \colon T^+ \to \omega^\omega$ in $L(\mathbb{R})$ such that for each $p \in T^+$, $f(p)$ codes a wellordering of ω in ordertype $dom(p)$.*

In our generalization of this result, we can prove one of the two directions in a slightly more general context than the other.

Recall that HOD_x is the class of all sets hereditarily ordinal-definable with x as a parameter. The two following theorems are due to Woodin and appear in [7].

A cone of Turing degrees is a set of the form $\{x \subset \omega \mid y$ is Turing-reducible to $x\}$, for some $y \subset \omega$.

Theorem 4.2. *Assume ZF+AD. Suppose that Y is a set and $a \in H(\omega_1)$. Then for a Turing cone of x,*

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \text{ is a Woodin cardinal,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$. □

Given a set Y, the Y-cone of reals above a given real x is the set of all reals z such that $x \in \mathrm{HOD}_{Y,z}$.

Theorem 4.3. *Assume ZF+AD. Suppose that Y is a set, $a \in H(\omega_1)$ and $\alpha < \omega_1$. Then for a Y-cone of x,*

$$\mathcal{P}(\alpha) \cap \mathrm{HOD}_{Y,a,[x]_Y} \subset \mathcal{P}(\alpha) \cap \mathrm{HOD}_{Y,a}\,.$$

Theorem 4.4. *Let $T \subset \mathbb{R}^{<\omega_1}$ be a tree, let S be a set of ordinals coding trees on the ordinals projecting to T and its complement, and suppose that $L(S,\mathbb{R}) \models AD$. Then at least one of the following two statements is true.*

(1) There is an uncountable branch of T in V.
(2) There is a function $f\colon T^+ \to \omega^\omega$ in $L(S,\mathbb{R})$ such that for each $p \in T^+$, $f(p)$ codes a wellordering of ω in ordertype dom(p).

Furthermore, if there exists a Woodin cadinal δ and every set of reals in $L(S,\mathbb{R})$ is δ^+ weakly homogeneously Suslin in V, at least one of (1) and (2) is false.

Proof. We work in $L(S,\mathbb{R})$. First suppose that (2) fails. We show that (1) holds. Since there are wellorderings of $\mathcal{P}(\omega)^{\mathrm{HOD}_{S,p}}$ uniformly definable from p, there must be a $p \in T^+$ which is uncountable in $\mathrm{HOD}_{S,p}$. Letting Y be S, a be p and α be ω, we have from Theorems 4.3 and 4.2 that there is a real x such that p is uncountable in $M = \mathrm{HOD}_{S,p,[x]_S}$ and $\delta = \omega_2^{\mathrm{HOD}_{S,p,x}}$ is a Woodin cardinal in M.

Since δ is countable in V, we can choose an M-generic filter g for $\mathrm{Coll}(\omega_1,{<}\delta)^M$. Then the nonstationary ideal is presaturated in $M[g]$. Furthermore, since $S \in M$, there are trees in $M[g]$ projecting in V to T and its complement. This means that $M[g]$ is T-iterable [22, 12]. Stepping outside of $L(S,\mathbb{R})$ to a model of Choice and taking any iteration j of $M[g]$ of length ω_1, then, $j(p)$ is an uncountable member of T^+.

To see the last part of the Theorem, suppose that T and f are coded by δ^+-weakly homogeneously Suslin sets of reals, and suppose that p is an uncountable path through T. Then there is a countable elementary submodel X of some large enough initial segment of the universe containing δ, T, f and p whose transitive collapse M has the property that (letting $\bar{\delta}$ be the image of δ under the collapse), if $M[g]$ is a forcing extension of M by $\mathrm{Coll}(\omega_1,\bar{\delta})^M$, then $M[g]$ is (T,f)-iterable ([22, 12, 4]). Letting $\bar{p}$ be the image of p under the collapse, then, every forcing extension of $M[g]$ by $(\mathcal{P}(\omega_1)/NS_{\omega_1})^{M[g]}$ has $f(\bar{p})$ as an element, which means that $M[g]$ has $f(\bar{p})$ as an element, giving a contradiction, since $f(\bar{p})$ codes a wellordering of ω of the same length as $\bar{p}$, and this length is $\omega_1^{M[g]}$. □

The following theorems can be used to show that if there exists a proper class of Woodin cardinals and the tree T is a weakly homogeneously Suslin set of reals, then there is a model of the form $L(S, \mathbb{R})$ satisfying AD, where S is a set of ordinals coding trees projecting to T and its complement. In this context, then, exactly one of (1) and (2) above hold.

Theorem 4.5 (Steel [17, 11])**.** *Suppose that there exist proper class many Woodin cardinals. Then universally Baire sets of reals have universally Baire scales.* □

Theorem 4.6 (Woodin [17])**.** *Suppose that there exist proper class many Woodin cardinals, and let A be a weakly homogeneously Suslin set of reals. Then A is universally Baire and $L(A, \mathbb{R}) \models AD$.* □

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