Borel subgroups of Polish groups

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Abstract

We study three classes of subgroups of Polish groups: Borel subgroups, Polishable subgroups, and maximal divisible subgroups. The membership of a subgroup in each of these classes allows one to assign to it a rank, that is, a countable ordinal, measuring in a natural way complexity of the subgroup. We prove theorems comparing these three ranks and construct subgroups with prescribed ranks. In particular, answering a question of Mauldin, we establish the existence of Borel subgroups which are $\mathbf{\Pi}^0_\alpha$-complete, $\alpha \geq 3$, and $\mathbf{\Sigma}^0_\alpha$-complete, $\alpha \geq 2$, in each uncountable Polish group. Also, for every $\alpha < \omega_1$ we construct an Abelian, locally compact, second countable group which is densely divisible and of Ulm length $\alpha + 1$. All previously known such groups had Ulm length 0 or 1.

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1 Introduction

By a *Polish group* we mean a topological group with a Polish (that is, completely metrizable, second countable) group topology. All locally compact, second countable groups are Polish.

We investigate subgroups of Polish groups belonging to three classes: Borel subgroups, Polishable subgroups, and maximal divisible subgroups. Polishable subgroups form an important subclass of Borel subgroups, and maximal divisible subgroups are natural examples of Polishable subgroups. It is perhaps surprising to realize that the membership of a subgroup in each of these classes is witnessed by a transfinite procedure which reflects the nature of the class and the process by which the subgroup is built. An ordinal equal to the length of this transfinite procedure gives a natural measure of how difficult it is to decide membership of the subgroup in the class in question. Of course, for different classes the procedures have very different description and, therefore, the ordinals associated with them may be different. For example: in the two classically studied cases, the class of Borel subgroups and the class of maximal divisible subgroups, these ordinals correspond to the smallest Borel pointclass to which the subgroup belongs and to the Ulm length of the group, respectively.

Our aim is twofold. First, we show that the three measures of complexity of subgroups are related to each other. Second, we construct examples of subgroups with given lengths of the transfinite procedures. These examples are interesting in the context of earlier investigations of Borel and maximal divisible subgroups.

We start with recalling the relevant definitions.

Borel subgroups

Let H be a Borel subgroup of a Polish group G. The complexity of such a subgroup can be measured by the Borel rank of H in G defined by letting

$$\mathrm{bor}(H,G) = \min\{\alpha < \omega_1 : H \text{ is a } \mathbf{\Pi}^0_\alpha \text{ subset of } G\}.$$

The study of the Borel rank for linear subspaces of Banach spaces dates back to Mazur–Steinbach and Banach–Kuratowski (see Klee, [1]). Mauldin [2] studied Borel subgroups and their ranks for Abelian connected groups.

Note that we adhere to the tradition of enumerating the classes $\mathbf{\Pi}^0_\alpha$ starting with $\alpha = 1$. Thus, the rank bor begins at 1. The other two ranks considered in this paper start at 0.

Polishable subgroups

A subgroup H of a Polish group G is called *Polishable* if there exists a Polish group topology on H whose Borel sets are precisely the intersections of Borel subsets of G with H. Such a topology, if it exists, is unique. (To see it, notice that two such topologies have the same Borel sets, hence the identity map between them is a Borel group isomorphism and, therefore, it is a homeomorphism, see [3, 9.10].) Polishability of H is easily seen equivalent to the condition that there exists a continuous isomorphism from a Polish group onto H or the condition that H is the image of a continuous homomorphism from a Polish group. All Polishable subgroups are Borel since Borel injective images of Polish spaces are Borel, see [3, 15.1].

The notion of Polishability was introduced in full generality by Kechris and Louveau in [4] but was already implicitly studied, for linear subspaces of Fréchet linear spaces, by Saint-Raymond in [5]. It has applications in certain situations in analysis, see for example [6], [5] and [7]. It is also crucial in the study of ideals of subsets of $\mathbb{N}$: analytic P-ideals are precisely the Polishable ideals considered as groups with symmetric difference as the group operation [8]. Polishable subgroups are important for (a particular case of) a conjectured dichotomy of Kechris and Louveau [4] on complexity of Borel equivalence relations.

One defines a rank measuring the complexity of a Polishable subgroup using a transfinite process of recovering the Polish group topology on H described in [7, Theorem 2.1]. (A different but analogous procedure in the context of Fréchet linear spaces was found in [5].) There exist a countable ordinal α_0 and a sequence G^p_α, $\alpha \leq \alpha_0$, of Polishable subgroups of G with Polish group topologies τ_α such that

(i) $G^p_0 = \overline{H}$, $G^p_\lambda = \bigcap_{\alpha<\lambda} G^p_\alpha$ if λ is limit;
(ii) $G^p_{\alpha_0} = H$ and $G^p_\alpha \neq H$ for $\alpha < \alpha_0$;
(iii) $G^p_{\alpha+1}$ is a $\mathbf{\Pi}^0_3$ subgroup of G^p_α taken with τ_α;
(iv) if A is a subset of G^p_α containing H and $\mathbf{\Pi}^0_3$ in τ_α, then $A \cap G^p_{\alpha+1}$ is comeager in $\tau_{\alpha+1}$.

It is easy to check that such a sequence is canonical. That is, if $\hat{G}^p_\alpha$, $\alpha \leq \alpha_1$, is another sequence of Polishable groups fulfilling (i)–(iv), then $\alpha_1 = \alpha_0$ and, for each $\alpha \leq \alpha_0$, $\hat{G}^p_\alpha = G^p_\alpha$. Therefore, we can define a rank value for H as a Polishable subgroup of G by letting

$$\mathrm{pol}(H, G) = \alpha_0.$$

Maximal divisible subgroups

For each Abelian group G there exists a largest divisible subgroup of G. This *maximal divisible subgroup* of G is denoted by $d(G)$. This is a classically studied object in Abelian group theory, see for example [9] and [10]. From our point of view, these groups are important since, as we will show, they provide natural examples of Polishable subgroups.

There exists a transfinite process by which one obtains $d(G)$. Its definition (and properties) can be found, for example, in [10, Sections I.6, VI.37]. We recall it here.

(i) $G_0^u = G$;
(ii) $G_{\alpha+1}^u = \bigcap_{n\in\mathbb{N}} nG_\alpha^u$;
(iii) $G_\lambda^u = \bigcap_{\alpha<\lambda} G_\alpha^u$ if λ is limit.

The group G_α^u is called the αth Ulm group of G. There exists an ordinal α for which $G_\alpha^u = G_{\alpha+1}^u$. In fact, by obvious cardinality considerations, there is such an $\alpha < |G|^+$, where $|G|^+$ stands for the successor cardinal of the cardinality of G. It is easy to see that $d(G) = G_\alpha^u$ for such an α. Again this allows us to define a rank value for $d(G)$ by letting

$$\mathrm{ulm}(G) = \min\{\alpha :\, G_\alpha^u = d(G)\}.$$

We will call $\mathrm{ulm}(G)$ the Ulm rank. (In [10], $\mathrm{ulm}(G)$ is called the Ulm length of G.)

An outline of results

We consider two types of questions: the question of existence of groups with a given value of a rank and the question of the relationship between different ranks.

Regarding comparison of the various ranks defined above we obtain the following.

1. We show that the Borel and the Polishable ranks essentially coincide on Polishable subgroups of Polish groups. They differ by at most two. A precise formula relating them is given in Theorem 3.1.

2. Zippin's examples of discrete countable groups with arbitrary Ulm rank and with the maximal divisible subgroup equal to $\{0\}$ (see [10, Corollary 76.2]) show that it is possible to have a Polish Abelian group whose Ulm rank is an arbitrary countable ordinal while the Polishable rank of its maximal divisible

subgroup is 0. We show, however, that if a Polish Abelian group is densely divisible, then its Ulm rank is less than or equal to the Polishable rank of its maximal divisible subgroup (Theorem 4.2). (An Abelian topological group is called *densely divisible* if its maximal divisible subgroup is dense in it. Dense divisibility is a notion dual to torsion freeness. A theorem of Robertson, see [9, Theorem 4.15], says that a locally compact Abelian group G is densely divisible precisely when its dual group is torsion free.) This, in view of point 1 above, gives a lower estimate on the Borel class of the maximal divisible group in terms of its Ulm rank (Corollary 4.4).

3. The remaining question, to which we do not know the answer, is whether it is possible to have a densely divisible Polish Abelian group with low Ulm rank and whose maximal divisible subgroup has high Polishable rank (see Question 6.2).

We have the following existence results.

4. We prove that any Polish uncountable group contains a subgroup with any given value α of the Borel rank with $\alpha \geq 3$ (Theorem 2.1). In fact, we show that it contains subgroups which are $\mathbf{\Pi}^0_\alpha$-complete for $\alpha \neq 2$ and $\mathbf{\Sigma}^0_\beta$-complete for $\beta \geq 2$. (The cases $\alpha = 2$ and $\beta = 1$ are easily excluded.) This answers a question of Mauldin from [11].

5. Regarding the existence of Polishable groups with a prescribed value of Polishable rank, we point out that examples of such groups can be found in [5, Theoreme 21], [12, Lemmas 5.4, 5.5, 5.6], and Section 5 of the present paper. Hjorth proved in [13] that every uncountable Abelian Polish group has Polishable subgroups of arbitrarily high Polishable rank. However, a general theorem of the kind mentioned in point 4 is still not known to be true (see Question 6.1).

6. We show that for an Abelian Polish group the Ulm rank is a countable ordinal (Theorem 4.1). Zippin gave examples of discrete countable (so Polish) Abelian groups with arbitrary Ulm rank. Since these groups are discrete, their maximal divisible subgroups are certainly not dense. However, in light of point 2 above, the right class of groups to consider in our context is that of densely divisible groups. We show that, for each countable ordinal α, there exists an Abelian Polish group which is densely divisible and whose Ulm rank is equal to α. In fact, if α is a successor ordinal, then the group can be chosen to be locally compact, second countable (Theorem 5.3). All previously known examples of Polish Abelian densely divisible groups had Ulm ranks ≤ 1.

In proving the above results, we apply a variety of methods coming from Ramsey theory, topological transformation groups, descriptive set theory, and theory of infinite Abelian groups.

2 Borel subgroups

The question of the complexity of substructures of Polishable structures was raised as early as 1933 (see [1]). In this year, Mazur and Steinbach asked whether every separable infinite-dimensional Banach space has Borel linear subspaces 'of arbitrarily high borelian type'? In the same year (and in the same issue of Studia Mathematica) Banach and Kuratowski asked whether every separable infinite-dimensional Banach space has linear subspaces that are 'analytic but not borelian'? These questions were answered by Klee. His proof relied on the fact that every separable infinite-dimensional Banach space E contains a perfect subset P consisting of linearly independent elements (see [1]).

In [2] Mauldin refined Klee's result, and used the same method to prove that each Abelian connected Polish group with an element of infinite order contains a subgroup of arbitrary Borel class. He asked in [11, Question 11, p. 482] if this is true for an arbitrary uncountable Polish group.

Theorem 2.1 *Let G be a Polish uncountable group. For any countable ordinal $\alpha \neq 2$ there exists a Borel subgroup H of G with*

$$\mathrm{bor}(H, G) = \alpha.$$

In fact, G has a subgroup H that is Γ-complete where Γ is $\mathbf{\Sigma}^0_\alpha$, $\alpha \geq 2$, or $\mathbf{\Pi}^0_\alpha$, $\alpha \neq 2$.

PROOF. A one-element subgroup is $\mathbf{\Pi}^0_1$-complete. The case $\alpha \geq 2$ is a very special case of Theorem 2.5 proved below.

The proof of Klee's theorem proceeded via obtaining a perfect set of linearly independent elements. Mauldin used this idea as well, by finding a perfect set of algebraically independent elements in the given Polish Abelian group. In order to prove Theorem 2.1 we are going to construct a perfect set of 'indiscernible' elements of the given group. The key result is Theorem 2.4 and its proof will rely on Ramsey-theoretic methods. Let us first introduce some terminology.

The lexicographical ordering on $2^{\mathbb{N}}$ will be denoted by $<_{\mathrm{Lex}}$. We will write $\vec{x} = \{x_0, \ldots, x_{n-1}\}_{<_{\mathrm{Lex}}}$ for an element of $[2^{\mathbb{N}}]^n$ such that $x_0 <_{\mathrm{Lex}} x_1 <_{\mathrm{Lex}} \ldots <_{\mathrm{Lex}} x_{n-1}$. For distinct x and y in $2^{\mathbb{N}}$, we let $\Delta(x, y) = \min\{k : x(k) \neq y(k)\}$. For

$\vec{x} \in [2^{\mathbb{N}}]^n$, $I \subseteq n$ and $J \subseteq n-1$ write

$$\vec{x}|I = \{x_i : i \in I\}$$
$$D_J(\vec{x}) = \{\Delta(x_i, x_{i-1}) : i \in J\}.$$

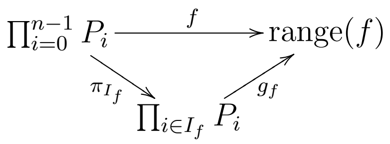
For $Q \subseteq 2^{\mathbb{N}}$ let

$$[Q]^n_{\prec} = \{\vec{x} \in [Q]^n : \Delta(x_i, x_{i+1}) < \Delta(x_{i+1}, x_{i+2}) \text{ for all } i < n-2\}.$$

This set can naturally be identified with an open subset of Q^n. In particular we consider that a function $f\colon (2^{\mathbb{N}})^n \to X$ includes $[Q]^n_{\prec}$ in its domain. A special case of Lefmann's canonical partition theorem for perfect sets [14, Theorem 1.4] is the following

> If X is a metric space and $f\colon (2^{\mathbb{N}})^n \to X$ is continuous, then there are a perfect $Q \subseteq 2^{\mathbb{N}}$ and $I \subseteq n$, $J \subseteq n-1$ such that the following are equivalent for $\vec{x}$ and $\vec{y}$ in $[Q]^n_{\prec}$:
> (1) $f(\vec{x}) = f(\vec{y})$.
> (2) $\vec{x}|I = \vec{y}|I$ and $D_J(\vec{x}) = D_J(\vec{y})$.

If $I \subseteq n$ then let $\pi_I\colon \prod_{i=0}^{n-1} P_i \to \prod_{i\in I} P_i$ denote the projection map. Of course, π_I depends on n and P_i, $i < n$, but these will always be clear from the context. A function f whose domain contains $\prod_{i=0}^{n-1} P_i$ is *canonical* on $\prod_{i=0}^{n-1} P_i$ if there is $I_f \subseteq n$ and a homeomorphic embedding $g_f\colon \prod_{i\in I_f} P_i \to \text{range}(f)$ such that $f = g_f \circ \pi_{I_f}$ on $\prod_{i=0}^{n-1} P_i$, that is, the following diagram commutes.



The following lemma, which is the key to the proof of Theorem 2.4, is an immediate consequence of Lefmann's theorem. With some extra work, it can be proved by using the polarized theorem of Blass ([15, p. 274]).

Lemma 2.2 *If X is a metric space and $f\colon \prod_{i=0}^{n-1} 2^{\mathbb{N}} \to X$ is Baire-measurable, then there are perfect sets $P_i \subseteq 2^{\mathbb{N}}$, $i < n$, such that f is canonical on $\prod_{i=0}^{n-1} P_i$.*

PROOF. By Lefmann's theorem find a perfect Q, $I \subseteq n$ and $J \subseteq n-1$ such that $f(\vec{x}) = f(\vec{y})$ if and only if $\vec{x}|I = \vec{y}|I$ and $D_J(\vec{x}) = D_J(\vec{y})$ for all $\vec{x}, \vec{y}$ in $[Q]^n_{\prec}$. Find k such that the k-th level of the subtree of $2^{<\mathbb{N}}$ whose branches are elements of Q has at last n distinct elements, $t_0 <_{\text{Lex}} t_1 <_{\text{Lex}} \dots <_{\text{Lex}} t_{n-1}$. Let $P_i = Q \cap [t_i]$ where $[t_i] = \{x \in 2^{\mathbb{N}} : t_i \text{ is an initial segment of } x\}$. Then $\prod_{i=0}^{n-1} P_i \subseteq [Q]^n_{\prec}$. Also, $D_J(\vec{x}) = D_J(\vec{y})$ for all $\vec{x}, \vec{y} \in \prod_{i=0}^{n-1} P_i$. Therefore $f(\vec{x}) = f(\vec{y})$ if and only if $\vec{x}|I = \vec{y}|I$. Then $g(\vec{x}|I) = f(\vec{x})$ is well-defined and one-to-one, and it witnesses that f is canonical.

Assume P_i, $i < n$, are perfect subsets of a Polish space X and $\mathcal{F}$ is a set of functions whose domains include $\prod_{i=0}^{n-1} P_i$. Then $\mathcal{F}$ is *canonical* on $\prod_{i=0}^{n-1} P_i$ if each $f \in \mathcal{F}$ is canonical on $\prod_{i=0}^{n-1} P_i$ and for any $f_1, f_2 \in \mathcal{F}$ with different restrictions to $\prod_{i=0}^{n-1} P_i$ the sets $f_1[\prod_{i=0}^{n-1} P_i]$ and $f_2[\prod_{i=0}^{n-1} P_i]$ are disjoint.

Lemma 2.3 *Assume P_i, $i < n$, are perfect subsets of a Polish space X and $\mathcal{F}$ is a finite set of continuous maps from X^n into X. Then there are perfect $Q_i \subseteq P_i$, $i < n$, such that $\mathcal{F}$ is canonical on $\prod_{i=0}^{n} Q_i$.*

PROOF. First, let $\mathcal{F} = \{f_1, f_2\}$. By applying Lemma 2.2 twice, we can assume that both f_1 and f_2 are canonical and continuous. If $f_1 \equiv f_2$ on $\prod_{i=0}^{n-1} P_i$, we are done. Otherwise, there is $\vec{x} \in \prod_{i=0}^{n-1} P_i$ such that $f_1(\vec{x}) \neq f_2(\vec{x})$. Since f_1 and f_2 are continuous, there are perfect $Q_i \subseteq P_i$ $(i < n)$ such that f_1 and f_2 map $\prod_{i=0}^{n-1} Q_i$ into disjoint sets. Since a set $\mathcal{F}$ is canonical if every $\{f_1, f_2\} \subseteq \mathcal{F}$ is canonical, the general case follows by induction.

We can now proceed to prove the main combinatorial result of this section.

Theorem 2.4 *Assume X is an uncountable Polish space and $f_n \colon X^{m_n} \to X$, $m_n \in \mathbb{N}$, are Baire-measurable or universally measurable functions. Then there is a perfect $P \subseteq X$ such that for each $i \in \mathbb{N}$ we can find $k_i \in \mathbb{N}$, relatively clopen non-empty $U_i \subseteq P^{k_i}$, compact $V_i \subseteq X$, and a family of homeomorphisms $g_i \colon U_i \to V_i$ such that*

(i) *$V_i \cap V_j = \emptyset$ for all $i \neq j$;*
(ii) *for every $A \subseteq P$ we have $\bigcup_{n=0}^{\infty} f_n[A^{m_n}] = \bigcup_{n=0}^{\infty} g_n[U_n \cap A^{k_n}]$;*
(iii) *if for some n_0, $f_{n_0} : X \to X$ is the identity map, then for some n_1, $g_{n_1} : U_{n_1} \to V_{n_1} = U_{n_1}$ is the identity.*

PROOF. We may assume that $2^{\mathbb{N}}$ is a subspace of X and that for each f_n and permutation π of m_n the function $(x_0, \ldots, x_{m_n-1}) \mapsto f_n(x_{\pi(0)}, \ldots, x_{\pi(m_n-1)})$ is listed as some f_k. By Mycielski's theorems ([16] [17]), we can assume that the functions $f_n|2^{\mathbb{N}}$ are continuous.

For a tree $T \subseteq 2^{<\mathbb{N}}$, let $T \restriction n = \{t \in T : |t| < n\}$. For $t \in 2^{<\mathbb{N}}$ let $[t] = \{x \in 2^{\mathbb{N}} : t \subseteq x\}$. Construct perfect trees T_i, $i \in \mathbb{N}$, and an increasing sequence n_i so that for all i we have:

(α) $T_{i+1} \restriction n_i = T_i \restriction n_i$.
(β) Every node of T_i of height n_i has two distinct extensions of height n_{i+1} in T_{i+1}.
(γ) Let t^i_s, $s \in 2^i$, be all the nodes of the n_i-th level of T_i enumerated so that $t^i_{s_1} <_{\text{Lex}} t^i_{s_2}$ precisely when $s_1 <_{\text{Lex}} s_2$. Let π^i_J be the projection

$\pi^i_J : \prod_{s\in 2^i}[t^i_s] \to \prod_{s\in J}[t^i_s]$, for $J \subseteq 2^i$. Then the family

$$\mathcal{F}_i = \{f_n \circ \pi^i_J : n \le i, |J| = m_n, J \subseteq 2^i\}.$$

is canonical on $\prod_{s\in 2^i}([t^i_s] \cap P_i)$ where $P_i = [T_i]$, the set of all branches of T_i.

This is easily achieved by a recursive construction using Lemma 2.3. Let $T = \bigcap_{i=0}^{\infty} T_i$ and $P = [T]$. By (α) and (β), P is perfect.

Let $N_s = [t^i_s] \cap P$ for $s \in 2^i$. By a partial homeomorphism on level i we mean a homeomorphic embedding $g : \prod_{s\in I} N_s \to X$ with $I \subseteq 2^i$. In this case put $I = R(g)$. Each function in each $\mathcal{F}_i$, as in (γ), is canonical on $\prod_{s\in 2^i}([t^i_s]\cap P_i) = \prod_{s\in 2^i} N_s$ and this is witnessed by a partial homeomorphism on level i. For each i, let $\mathcal{B}_i$ be a finite family of partial homeomorphisms on level i witnessing that $\mathcal{F}_i$, as in (γ), is canonical on $\prod_{s\in 2^i} N_s$. The $\mathcal{B}_i$s have the following properties (here $\pi_j : \prod_{s\in 2^i} N_s \to \prod_{s\in J} N_s$, for $J \subseteq 2^i$, are the projections):

(a) $\forall g_1, g_2 \in \mathcal{B}_i \ (g_1 \ne g_2 \Rightarrow \mathrm{rng}(g_1) \cap \mathrm{rng}(g_2) = \emptyset)$;
(b) $\forall n \le i \forall J \subseteq 2^i, |J| = m_n \Rightarrow \exists g \in \mathcal{B}_i \ (f_n \circ \pi_J = g \circ \pi_{R(g)})$;
(c) $\forall g \in \mathcal{B}_i \exists n \le i \exists J \subseteq 2^i, |J| = m_n \ (f_n \circ \pi_J = g \circ \pi_{R(g)})$;
(d) $\forall g \in \mathcal{B}_i \exists B \subseteq \mathcal{B}_{i+1} \ (g = \bigcup_{g'\in B} g')$.

Properties (a), (b) and (c) follow immediately from the definition of canonization. To see (d), note that if $g \in \mathcal{B}_i$ canonizes $f_n \circ \pi^i_J$, then g is the union of all those $g' \in \mathcal{B}_{i+1}$ which canonize $f_n \circ \pi^{i+1}_{J'}$ for $J' \subseteq 2^{i+1}$ such that $|J'| = m_n$ and $J = \{s|i : s \in J'\}$. Note that for $A \subseteq P$ we have

$$\bigcup_n f_n[A^{m_n}] = \bigcup_i \bigcup_{g\in\mathcal{B}_i} g[\textstyle\prod_{s\in R(g)}(A \cap N_s)]. \tag{2.1}$$

The inclusion $\supseteq$ follows from (c). To see $\subseteq$ let $\vec{x} = \{x_0, \dots, x_{m_n-1}\}_{<_{\mathrm{Lex}}}$ be in $[A]^{m_n}$. Let i be such that the set $\{\Delta(x_j, x_{j+1}) : j < m_n\}$ is included in n_i and $n \le i$. Put $J = \{s \in 2^i : \exists j < m_n \ x_j \in [t^i_s]\}$. By the choice of n_i, $|J| = m_n$. Now (b) guarantees the existence of $g \in \mathcal{B}_i$ such that

$$f_n(\vec{x}) = f_n \circ \pi^i_J(\vec{x}) = f_n \circ \pi_J(\vec{x}) = g \circ \pi_{R(g)}(\vec{x}) \in g[\textstyle\prod_{s\in R(g)}(A \cap N_s)].$$

Define $\mathcal{A}_0 = \mathcal{B}_0$ and $\mathcal{A}_{i+1} = \{g' \in \mathcal{B}_{i+1} : \forall g \in \mathcal{B}_i \ g' \not\subseteq g\}$. From these definitions it follows that

$$\bigcup_i \bigcup_{g\in\mathcal{B}_i} g[\textstyle\prod_{s\in R(g)}(A \cap N_s)] = \bigcup_i \bigcup_{g\in\mathcal{A}_i} g[\prod_{s\in R(g)}(A \cap N_s)]. \tag{2.2}$$

Enumerate now $\bigcup_i \mathcal{A}_i$ as $\{g_n : n \in \mathbb{N}\}$. We claim that the sequence g_n, $n \in \mathbb{N}$, is as required with U_n equal to the domain of g_n and V_n to the range of g_n. Requirement (ii) follows directly from (2.1) and (2.2). To prove (i) it suffices to show that for each i

$$\mathcal{A}_i \subseteq \{g' \in \mathcal{B}_i : \forall j < i \forall g \in \mathcal{B}_j \operatorname{rng}(g') \cap \operatorname{rng}(g) = \emptyset\}.$$

Let $g' \in \mathcal{B}_i$ be such that for some $j < i$ and $g^0 \in \mathcal{B}_j$, $\operatorname{rng}(g') \cap \operatorname{rng}(g^0) \neq \emptyset$. We need to prove that $g' \notin \mathcal{A}_i$. Applying (d) $i - j$ times, we find $g^1 \in \mathcal{B}_i$, $g^2 \in \mathcal{B}_{i-1}$ such that $\operatorname{rng}(g') \cap \operatorname{rng}(g^1) \neq \emptyset$ and $g^1 \subseteq g^2 \subseteq g^0$. It follows by (a) that $g^1 = g'$, so $g' \subseteq g^2$ whence $g' \notin \mathcal{A}_i$. Now, to show (iii), note that if $f_n = \mathrm{id}$, then after letting $J = \{s\}$ for some $s \in 2^n$, point (b) for $i = n$ produces a $g \in \mathcal{B}_n$ such that $\mathrm{id}|N_s = g \circ \pi_{R(g)}$. Clearly $R(g) = \{s\}$, the domain of g is N_s, and g is the identity map on it.

The following consequence of Theorem 2.4 belongs to *Borel model theory*, initiated by H. Friedman (see [18]). A structure is a *Borel structure* if its underlying set is a Polish space and all of its relations and functions are Borel-measurable. It is *completely Borel* if, moreover, all definable subsets of all of its finite powers are Borel.

Theorem 2.5 *Every uncountable Borel structure has submodels that are $\mathbf{\Sigma}^0_\alpha$-complete, $\alpha \geq 2$, and $\mathbf{\Pi}^0_\alpha$-complete, $\alpha \geq 3$.*

If this structure is either completely Borel or if the Projective Determinacy holds, then the submodel in question can be chosen to be elementary.

PROOF. Let X denote the underlying Polish space. Let f_n $(n \in \mathbb{N})$ be an enumeration of all iterations of functions in X, so that f_1 is the identity map. If X is completely Borel, then by Jankov, von Neumann uniformization theorem we can find C-measurable Skolem functions and add them to the list. If PD holds, we can find projective Skolem functions and add them to the list. Note that in either case all f_n are Baire-measurable. Apply Theorem 2.4 to these functions to obtain a perfect subset P of X, relatively clopen sets $U_i \subseteq P^{k_i}$, pairwise disjoint compact sets $V_i \subseteq X$, for $i, k_i \in \mathbb{N}$, and a family of homeomorphisms $g_i \colon U_i \to V_i \subseteq X$ such that for every $A \subseteq P$ we have

$$\bigcup_{n=0}^{\infty} f_n[A^{m_n}] = \bigcup_{n=0}^{\infty} g_n[U_n \cap A^{k_n}]. \tag{2.3}$$

Since the identity function is one of the f_ns, for some n_1, $U_{n_1} = V_{n_1}$ and g_{n_1} is the identity. It follows that for any $A \subseteq U_{n_1}$, we have $\langle A \rangle \cap U_{n_1} = A$.

Let $\Gamma \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta : \alpha \geq 2, \beta \geq 3\}$. Let $A \subseteq U_{n_1}$ be Γ-complete. (Such an A exists since U_{n_1} is a non-empty clopen subset of the Cantor set.) Then the submodel $\langle A \rangle$ of X generated by A satisfies the following two conditions:

(a) $\langle A \rangle \cap U_{n_1} = A$,

(b) If A is in Γ, then so is $\langle A\rangle$.

Property (a) holds because g_{n_1} is the identity map and the V_n' are disjoint. To show property (b), note that, by (2.3) and disjointness of the V_ns,

$$\langle A\rangle = \bigcup_n g_n[U_n \cap A^{k_n}] = \bigcap_m \Big(\bigcup_{n\leq m} g_n[U_n \cap A^{k_n}] \cup \bigcup_{n>m} V_n \Big). \tag{2.4}$$

If A is in any of the considered classes, then $g_n[U_n \cap A^{k_n}]$ is in it as well. If A is in $\mathbf{\Sigma}^0_\alpha$, $\alpha \geq 2$, then the first equality in (2.4) gives that $\langle A\rangle$ is in $\mathbf{\Sigma}^0_\alpha$. If A is in $\mathbf{\Pi}^0_\beta$, $\beta \geq 3$, then the second equality in (2.4) shows that $\langle A\rangle$ is in $\mathbf{\Pi}^0_\beta$. (This is because $\mathbf{\Pi}^0_\beta$ with $\beta \geq 3$ contains $\bigcup_{n>m} V_n$ which is $\mathbf{\Sigma}^0_2$ and is closed under finite unions.)

It follows from (a) and (b) that if A is a Γ-complete subset of P, then $\langle A\rangle$ is a Γ-complete submodel of X, and it is moreover elementary if X is completely Borel or PD holds.

While in the case when X is a group it is possible to find a $\mathbf{\Pi}^0_1$-complete subgroup, it is easy to see that this does not apply to Borel structures in general. Also, it is sometimes impossible to find a $\mathbf{\Pi}^0_1$-complete elementary submodel of a given group. For example, every elementary submodel of $(\mathbb{R}, +)$ is dense.

Lemma 2.2 has an equivalent reformulation in terms of finite powers of Sacks forcing, $\mathcal{S}$. It says that for every real number r in the extension by $\mathcal{S}^n$ there is $I \subseteq n$ such that $V[r] = V[s_j : j \in I]$. We are grateful to A.R.D. Mathias for informing us that this form of Lemma 2.2 was proved in 1970s by Jensen (unpublished). It is natural to ask whether a similar lemma could be proved for every forcing that adds a real of minimal degree. The negative answer in the case of Silver forcing was obtained by Adamowicz [19].

3 Polishable subgroups

As stated in the introduction all Polishable subgroups are Borel. Therefore, both bor and pol ranks are defined for them. The main theorem of this section asserts that the two ranks essentially coincide on Polishable subgroups of Polish groups. They differ by 1s added from left and/or right. Note, however, that the first 1 in the formulas in Theorem 3.1 (i) and (ii) stems from the fact that the values of the Borel rank start at 1 while the values of the Polishable rank at 0. The second 1 in (i) is more consequential and it is responsible for Corollary 3.4.

Before stating the theorem comparing the two ranks, we recall some facts concerning the sequence G^p_α, $\alpha \leq \alpha_0$, described in the introduction, which can be associated with any Polishable subgroup and which is used to define the pol rank. The following remarks, whose proofs can be found in [7, Theorem 2.1], describe more precisely how $G^p_{\alpha+1}$ is obtained given H and G^p_α. Let H be a Polishable subgroup of a Polish group G'. Let τ be the Polish group topology on H. Define

$$H^{G'} = \{g \in G' : \forall 1 \in U \in \tau \exists h_1, h_2 \in H \; g \in h_1\overline{U} \cap \overline{U}h_2\}. \tag{3.1}$$

with the closure $\overline{U}$ taken in G' (see [7, p.351]). Then for each $\alpha < \alpha_0$,

$$G^p_{\alpha+1} = H^{G^p_\alpha}, \tag{3.2}$$

where G^p_α on the right hand side of the equality is taken with the Polish group topology τ_α on it. If α is a limit ordinal, let $G^p_\alpha = \bigcap_{\beta<\alpha} G^p_\beta$. By [7, Claim 2, p.352], for $x \in G^p_{\alpha+1}$ a neighborhood basis of x in $\tau_{\alpha+1}$ is given by $\overline{Wx} \cap \overline{xW} \cap G^p_{\alpha+1}$ where $W \subseteq H$ is an open neighborhood of 1 and the closure is taken in τ_α. It follows from [7, Claim 3, p.352] that, in fact, already the sets $\overline{Wx} \cap G^p_{\alpha+1}$ with $W \subseteq H$ an open neighborhood of 1 form a basis of $\tau_{\alpha+1}$ at x. Again the closure $\overline{Wx}$ is taken in τ_α.

Theorem 3.1 *Let G be a Polish group, and let H be a Polishable subgroup of G.*

(i) If $\mathrm{pol}(H,G)$ *is a successor, then* $\mathrm{bor}(H,G) = 1 + \mathrm{pol}(H,G) + 1$.
(ii) If $\mathrm{pol}(H,G)$ *is* 0 *or limit, then* $\mathrm{bor}(H,G) = 1 + \mathrm{pol}(H,G)$.

PROOF. Put $\alpha_0 = \mathrm{pol}(H,G)$. Let G^p_α, $\alpha \leq \alpha_0$, be the sequence of Polishable subgroups with Polish group topologies τ_α on them as in the definition of $\mathrm{pol}(H,G)$.

We first show the inequalities $\leq$ in (i) and (ii). The proof is by induction on $\alpha \leq \alpha_0$ which is straightforward once we realize that we should prove a bit more than we need. We show the following:

if α is a successor, then G^p_α is $\mathbf{\Pi}^0_{1+\alpha+1}$ and $\tau_\alpha \subseteq \mathbf{\Sigma}^0_{1+\alpha}|G^p_\alpha$;
if α is 0 or limit, then G^p_α is $\mathbf{\Pi}^0_{1+\alpha}$ and $\tau_\alpha \subseteq \mathbf{\Sigma}^0_{1+\alpha}|G^p_\alpha$.

In the above formulas, $\mathbf{\Sigma}^0_{1+\alpha}|G^p_\alpha$ stands for the family of intersections of sets in $\mathbf{\Sigma}^0_{1+\alpha}$ with G^p_α. Clearly, for $\alpha = \alpha_0$ the above formulas imply inequalities $\leq$ in (i) and (ii).

The case of $\alpha = 0$ is obvious. To go from α to $\alpha + 1$ note that $G^p_{\alpha+1}$ is a $\mathbf{\Pi}^0_3(\tau_\alpha)$ subset of G^p_α and $\tau_{\alpha+1} \subseteq \mathbf{\Sigma}^0_2(\tau_\alpha)|G^p_{\alpha+1}$. (Here $\mathbf{\Pi}^0_3(\tau_\alpha)$ and $\mathbf{\Sigma}^0_2(\tau_\alpha)$ stand for the appropriate families of sets defined with respect to τ_α.) By inductive assumption on τ_α, we get that

$$\tau_{\alpha+1} \subseteq \mathbf{\Sigma}^0_2(\tau_\alpha)|G^p_{\alpha+1} \subseteq \mathbf{\Sigma}^0_{1+\alpha+1}|G^p_{\alpha+1}.$$

and that $G^p_{\alpha+1}$ is in $\mathbf{\Pi}^0_{1+\alpha+2}|G^p_\alpha$. By our inductive assumption on G^p_α, we see that $G^p_{\alpha+1}$ is in $\mathbf{\Pi}^0_{1+\alpha+2}$. Thus, we are done for $\alpha+1$. To obtain the case of α limit, notice simply that G^p_α is the intersection of G^p_β with $\beta < \alpha$ and τ_α is generated by the intersections of sets in τ_β with G^p_α for $\beta < \alpha$ and apply the inductive assumption.

Now we prove the inequalities $\geq$ in (i) and (ii). The proof will use Vaught transforms with respect to the Polish group topology on H. These are defined as follows. For $A \subseteq G$ and $U \subseteq H$ open, we write

$$A^{\triangle U} = \{g \in G : \{h \in H : hg \in A\} \text{ is non-meager in } U\}\}$$
$$A^{*U} = \{g \in G : \{h \in H : hg \in A\} \text{ is comeager in } U\}.$$

For $V \subseteq H$ open and $A \subseteq G$ we have the following identity

$$A^{\triangle V} = \bigcup\{(A^{*W})^{\triangle U} : 1 \in W \text{ and } WU \subseteq V,\ U, W \text{ open in } H\}. \qquad (3.3)$$

We will leave checking it to the reader. For more on Vaught transforms see [3].

We first show the following claim.

Claim 3.2 *For each $\alpha < \alpha_0$, if $A \subseteq G^p_\alpha$ is τ_α-closed, then for any open $U \subseteq H$, $A^{\triangle U}$ is $\tau_{\alpha+1}$-open.*

PROOF. Recall that $C \subseteq G^p_{\alpha+1}$ is $\tau_{\alpha+1}$-open if for any $x \in C$ there exists $W \subseteq H$, an open neighborhood of $1 \in H$ such that $\overline{Wx} \cap G^p_{\alpha+1} \subseteq C$ where the closure of Wx, as well as all the closures below, is taken with respect to τ_α. Fix therefore $x_0 \in A^{\triangle U}$. Since A is τ_α-closed, there exists $V \subseteq U$ open non-empty and such that

$$\overline{Vx_0} \subseteq A. \qquad (3.4)$$

Pick now an open set $V_1 \subseteq H$ and an open set $W \subseteq H$ containing 1 and such that $V_1 W \subseteq V$. Then, using (3.4), for any $x \in \overline{Wx_0}$, we obtain

$$V_1 x \subseteq V_1\overline{Wx_0} \subseteq \overline{Vx_0} \subseteq A$$

which implies $x \in A^{\triangle U}$. Thus, $\overline{Wx_0} \subseteq A^{\triangle U}$ as required.

Now we prove by induction the following claim.

Claim 3.3 *For $A \subseteq G$ for any $\alpha \leq \alpha_0$ and any open $U \subseteq H$*

$$A \text{ in } \mathbf{\Sigma}^0_{1+\alpha} \text{ implies } A^{\triangle U} \cap G^p_\alpha \text{ is } \tau_\alpha\text{-open.}$$

PROOF. For $\alpha = 0$ this is obvious since $A^{\triangle U}$ is open whenever A is. Assume that Claim 3.3 holds for all ordinals $< \alpha$. If α is limit, then $A = \bigcup_n A_n$ where $A_n \in \mathbf{\Sigma}^0_{1+\alpha_n}$ with $\alpha_n < \alpha$. Then

$$A^{\triangle U} = \bigcup_n A_n^{\triangle U}$$

where, by inductive assumption, $A_n^{\triangle U} \cap G^p_{\alpha_n}$ is τ_{α_n}-open and so $A_n^{\triangle U} \cap G^p_\alpha$ is τ_α-open. It follows that $A^{\triangle U} \cap G^p_\alpha$ is τ_α-open.

Now assume α is a successor, say $\alpha = \beta + 1$. Note that our inductive assumption for β is equivalent to saying that $B^{*U} \cap G^p_\beta$ is τ_β-closed for any B in $\mathbf{\Pi}^0_{1+\beta}$. Let A be $\mathbf{\Sigma}^0_{1+\alpha}$. Then $A = \bigcup_n A_n$ with A_n in $\mathbf{\Pi}^0_{1+\beta}$. Using (3.3), we get

$$A^{\triangle U} = \bigcup_n A_n^{\triangle U} = \bigcup_n \bigcup \{(A_n^{*W})^{\triangle V} : 1 \in W \text{ and } WV \subseteq U,\ V, W \text{ open in } H\}.$$

Note that, by induction, each A_n^{*W} in the above formula is τ_β-closed when intersected with G^p_β; thus, by Claim 3.2, $(A_n^{*W})^{\triangle V} \cap G^p_{\beta+1} = (A_n^{*W} \cap G^p_{\beta+1})^{\triangle V}$ is $\tau_{\beta+1}$-open, i.e., τ_α-open which makes $A^{\triangle U} \cap G^p_\alpha$ τ_α-open.

We are now in a position to finish the proof of the theorem. It is not difficult to convince oneself that it will be sufficient to show that for any λ limit or 0 and $n \in \omega$

$$\text{bor}(H, G) \leq 1 + \lambda + n + 1 \ \Rightarrow\ \text{pol}(H, G) \leq \lambda + n.$$

So assume H is $\mathbf{\Pi}^0_{1+\lambda+n+1}$. We will prove that $H = G^p_{\lambda+n}$. To see this, it suffices to show that H is comeager in $G^p_{\lambda+n}$. Let $H = \bigcap_k A_k$ with $A_k \in \mathbf{\Sigma}^0_{1+\lambda+n}$. Then

$$H = H^{*H} = \bigcap_k A_k^{*H} = \bigcap_k \bigcap_U A_k^{\triangle U} \tag{3.5}$$

where the last intersection is taken over a countable basis of the Polish group topology on H. By Claim 3.3, each $A_k^{\triangle U}$ is $\tau_{\lambda+n}$-open. Since it contains H, it is also $\tau_{\lambda+n}$-dense in $G^p_{\lambda+n}$ and, therefore, comeager. It follows from (3.5) that H is comeager in $G^p_{\lambda+n}$ with $\tau_{\lambda+n}$.

In [20, Corollary 3.9], it is proved that if H is Polish nilpotent or Polish and admitting a two-sided invariant metric, then for any continuous action of H on a Polish space any $\mathbf{\Pi}^0_{\lambda+1}$ orbit is actually $\mathbf{\Pi}^0_\lambda$. The corollary below shows that in the action by left translations of H on a Polish group G into which it continuously embeds $H \times G \ni (h,g) \to hg \in G$, we get the same conclusion without any additional assumptions on H. Note that there are continuous actions of Abelian Polish groups on Polish spaces with a $\mathbf{\Pi}^0_2$ orbit which is not $\mathbf{\Pi}^0_1$ ([20, Example 4.2]).

Corollary 3.4 *Let H be a Polishable subgroup of a Polish group G. Let λ be equal to 1 or be a limit ordinal. Then if H is $\mathbf{\Pi}^0_{\lambda+1}$, then H is $\mathbf{\Pi}^0_{\lambda}$.*

PROOF. So assume that H is $\mathbf{\Pi}^0_{\lambda+1}$, that is, $\mathrm{bor}(H,G) \leq \lambda + 1$. A look at Theorem 3.1 convinces us that $\mathrm{bor}(H,G)$ cannot be the successor of a limit ordinal. Thus, $\mathrm{bor}(H,G) \leq \lambda$ and we are done.

4 Maximal divisible subgroups

The theory of maximal divisible subgroups and Ulm rank (= Ulm length) was classically developed for discrete Abelian groups, see [10]. In this section, we go beyond this context and consider maximal divisible subgroups of Polish Abelian groups. The theorem below shows that, when the Abelian group is assumed to be Polish, the procedure of obtaining the sequence of Ulm subgroups is descriptive set theoretic in nature and that the obvious estimate $\mathrm{ulm}(G) < |G|^+$ (where $|G|^+$ is the successor cardinal of the cardinality of G) on the Ulm rank of G can be improved.

Theorem 4.1 *(i) Let G be a standard Borel Abelian group with $d(G)$ Borel. Then* $\mathrm{ulm}(G) < \omega_1$.
(ii) If G is Polish Abelian, then $d(G)$ is Polishable so it is Borel. In particular, $\mathrm{ulm}(G) < \omega_1$.

PROOF. (i) Let $B = G \setminus d(G)$. Consider the relation on $B^{<\mathbb{N}}$ defined by letting $(g_0, \ldots, g_n) < (h_0, \ldots, h_m)$ if and only if

$$m < n, h_i = g_i \text{ for } i \leq m, \text{ and } (i+1)g_{i+1} = g_i \text{ for } i \leq n.$$

Note that the relation is Borel. Also it is well-founded. Indeed, a descending sequence with respect to $<$ would produce an infinite sequence $(g_i)_{i\in\mathbb{N}}$ such that $g_0 \notin d(G)$ but for all i, $(i+1)g_{i+1} = g_i$. This last condition however insures that $g_0 \in d(G)$. By the Kunen-Martin theorem, the rank of $<$, $\rho(<)$ is countable. (For the definition of ρ and a proof of the Kunen-Martin theorem see [3, Theorem 31.5].) Define for $A \subseteq B^{<\mathbb{N}}$,

$$D(A) = \{s : \exists t \; s > t \text{ and } t \in A\}.$$

Define $A_0 = B^{<\mathbb{N}}$, $A_{\alpha+1} = D(A_\alpha)$, and $A_\lambda = \bigcap_{\alpha<\lambda} A_\alpha$. By the usual simple calculation, we get that for any $s \in B^{<\mathbb{N}}$ and any ordinal α,

$$\rho(s) \geq \alpha \Leftrightarrow s \in A_\alpha. \tag{4.1}$$

We also have that for any ordinal α

$$A_{\omega\alpha} = (G^u_\alpha \setminus d(G))^{<\mathbb{N}}. \tag{4.2}$$

We check this by induction on α. Only the successor stage needs to be checked. So assume that (4.2) holds for α. Let s be in $B^{<\mathbb{N}}$ with length of s equal to p, that is, $s = (s_0, s_1, \ldots, s_{p-1})$. Then,

$$s \in A_{\omega\alpha+\omega} = \bigcap_n D^n(A_{\omega\alpha})$$

which happens precisely when

$$\forall n \ \ \exists g_1, \ldots, g_n \ s > sg_1 > \cdots > sg_1 \cdots g_n \in A_{\omega\alpha}.$$

By our inductive assumption, this is equivalent to

$$\forall n \ \exists g_1, \ldots, g_n \ \ s > sg_1 > \cdots > sg_1 \cdots g_n \in (G^u_\alpha \setminus d(G))^{<\mathbb{N}}.$$

Unravelling the definition of $<$, we see that this last statement is equivalent to saying

$$\forall n \ \exists g_n \in G^u_\alpha \setminus d(G) \ s_{p-1} = p(p+1) \cdots n g_n$$

hence to

$$\forall n \ \exists h_n \in G^u_\alpha \setminus d(G) \ s_{p-1} = n h_n.$$

Since G^u_α is a group and since, by assumption, $s_i \in B$ for all $i < p$, the last condition is equivalent to

$$s \in G^u_{\alpha+1}.$$

It follows from (4.2) that $G^u_\alpha = d(G)$ precisely when $A_{\omega\alpha} = \emptyset$. By (4.1) this last condition holds for all α with $\omega\alpha \geq \rho(<)$, that is, for some countable α.

(ii) Consider $H = \{(g_n) \in G^{\mathbb{N}} : \forall n \ g_n = (n+1)g_{n+1}\}$. Clearly H is a closed subgroup of the Polish group $G^{\mathbb{N}}$, so it is Polish with the inherited topology. It is easy to see that $d(G) = \pi[H]$ where $\pi : G^{\mathbb{N}} \to G$ is the projection on the 0-th coordinate. Since π is a continuous homomorphism, there exists a continuous isomorphism from $H/\ker(\pi)$ onto $d(G)$. Since $H/\ker(\pi)$ is Polish, we transfer the Polish topology on it to $d(G)$ to see that this last group is Polishable.

Since $d(G)$ is Polishable, the question of comparing $\mathrm{ulm}(G)$ and $\mathrm{pol}(d(G), G)$ for a Polish Abelian group G naturally arises. A classical theorem of Zippin, see

[10, Corollary 76.2] or §5, implies that there exist countable groups G_α, $\alpha < \omega_1$, such that $d(G_\alpha) = 0$ and $\mathrm{ulm}(G_\alpha) = \alpha$. Such G_αs are trivially Polish, actually, second countable, locally compact, and $\mathrm{pol}(d(G_\alpha), G_\alpha) = 0 < \mathrm{ulm}(G_\alpha)$. However, we show below that if $d(G)$ is dense in G, then the inequality $\mathrm{pol}(d(G), G) < \mathrm{ulm}(G)$ cannot occur.

Theorem 4.2 *Let G be a Polish Abelian group which is densely divisible. Then*

$$\mathrm{ulm}(G) \leq \mathrm{pol}(d(G), G).$$

PROOF. We start with the following claim.

Claim 4.3 *Let G_1 be a Polish Abelian densely divisible group. Let $\bar{g} \in \bigcap_k kG_1$ and let $0 \in U$ be an open set in the Polish group topology τ on $H = d(G_1)$. Then for some $f \in H$, $\bar{g} \in f + \overline{U}$.*

PROOF. Let $R = \{(g_n) \in G_1^{\mathbb{N}} : \forall n (n+1) g_{n+1} = g_n\}$, and let π be the projection from R onto the 0-th coordinate. The mapping π is a continuous homomorphism from the Polish group R into G_1, so it is Borel as a function from R to (H, τ) and, therefore, it is continuous as a function from R to (H, τ) (see [3, 9.10]). Let $U' = \pi^{-1}(U)$. U' is open in R and contains the sequence constantly equal to 0. It will be convenient to fix an invariant metric ρ on G_1 compatible with its Polish group topology. Such a metric exists by [21]. For some n, U' contains a set of the form $(V \times \cdots \times V \times G_1^{\mathbb{N}}) \cap R$ with V taken n-times, where $V = \{g \in G_1 : d(0, g) < \epsilon\}$ for some $\epsilon > 0$. Since $\bar{g} \in \bigcap_k kG_1$, given n, there exist $g_0, \ldots, g_n$ such that $\bar{g} = g_0$, $g_i = (i+1) g_{i+1}$ for $i < n$. Since H is dense, we can find a sequence of elements of H which converges to g_n. This allows us to pick $f \in H$ so that not only $\rho(f, g_n) < \epsilon$ but also

$$\forall i < n \; \rho((i+1) \cdots (n-1) n f, g_i) < \epsilon. \tag{4.3}$$

We claim that $n! f$ works. Indeed, let us pick a sequence $h_k \in H$, for $k \in \mathbb{N}$, so that $h_k \to g_n$ with $k \to \infty$. Then obviously $n! h_k \to \bar{g}$ with $k \to \infty$. The argument below is done for large enough k. By easy metric considerations involving (4.3), for each $i < n$, $\rho((i+1)(i+2) \cdots n f, (i+1)(i+2) \cdots n h_k) < \epsilon$. It follows that, for each $i < n$,

$$r_i^k = -(i+1)(i+2) \cdots n f + (i+1)(i+2) \cdots n h_k \in V.$$

Since $f - h_k \in H$, we can find a sequence (r_i^k), $i = n, n+1, \ldots$, so that $r_n^k = f - h_k$ and $r_m^k = (m+1) r_{m+1}^k$ if $m \geq n$. Then

$$(r_0^k, \ldots, r_{n-1}^k, r_n^k, \ldots) \in (V \times \cdots \times V \times G_1^{\mathbb{N}}) \cap R$$

and so $r_0^k \in U$. Since $n! f + r_0^k \to \bar{g}$ when $k \to \infty$, the claim is proved.

Let $H = d(G)$. Consider two transfinite sequences of subgroups

$$(G^u_\alpha)_{\alpha \leq \alpha_0},\ (G^p_\alpha)_{\alpha \leq \alpha_1}$$

the first one of which consists of the Ulm subgroups of G with $G_{\alpha_0} = H$ while the second one is obtained from the process of recovering the Polish group topology on H. To show that $\alpha_0 \leq \alpha_1$, it suffices to prove that, for each $\alpha \leq \alpha_0$, $G^u_\alpha \subseteq G^p_\alpha$. We do this by induction. Only the successor stage requires an argument. Note that for each $\alpha \leq \alpha_1$, $d(G^p_\alpha) = H$ because G^p_α is a subgroup of G containing H. Thus, by the claim applied to $G_1 = G^p_\alpha$ taken with its Polish group topology τ_α as in the definition of the sequence $(G^p_\alpha)_{\alpha \leq \alpha_0}$ and by the formulas (3.1) and (3.2) given at the beginning of Section 3 (note that G is Abelian here), we get

$$\bigcap_{k \in \mathbb{N}} kG^p_\alpha \subseteq G^p_{\alpha+1}. \tag{4.4}$$

Now

$$G^u_{\alpha+1} = \bigcap_{k \in \mathbb{N}} kG^u_\alpha \subseteq \bigcap_{k \in \mathbb{N}} kG^p_\alpha \subseteq G^p_{\alpha+1}$$

where the equality holds by definition of $G^u_{\alpha+1}$, the first inclusion by our inductive assumption, and the second inclusion by (4.4). Therefore, the theorem is proved.

As a consequence of Theorems 4.2 and 3.1, we get the following corollary.

Corollary 4.4 *Let G be a Polish Abelian group which is densely divisible. Then*

$$1 + \mathrm{ulm}(G) \leq \mathrm{bor}(d(G), G).$$

In relation to the above corollary, one should mention paper [22] in which Barker considers countable recursive reduced Abelian p-groups and investigates the complexity of the Ulm groups using the hyperarithmetic hierarchy. His Proposition 5.2 yields a lower bound on the hyperarithmetic class of the Ulm groups. However, this result, and certainly its proof, seem to be only loosely related to Corollary 4.4.

5 $d(G)$ can be arbitrarily complex

It is known that there exist second countable locally compact Abelian groups (all such groups are Polish) which are densely divisible but not divisible, see e.g., [9, 4.16] or [21, (24.44)]. In light of Theorem 4.2, it is important to ask if a strengthening of this fact holds, that is, are there densely divisible Polish, or even second countable locally compact Abelian groups, with arbitrarily high Ulm rank? (Note that by Theorem 4.2 this must necessarily be more difficult than getting $\mathrm{ulm}(G)$ arbitrarily large without assuming that $d(G)$ is dense.) All the densely divisible Polish groups in literature have the Ulm rank ≤ 1.

Here is a standard example of a densely divisible Polish group that is not divisible.

Example 5.1 *Consider $\mathbb{Q}$, the group of rationals, with the discrete topology. Let G be the subgroup of all those $(x_n) \in \mathbb{Q}^{\mathbb{N}}$ for which x_n is an integer for large enough n. To see that G carries a Polish group topology note that $\mathbb{Z}^{\mathbb{N}}$, which is Polish in the product topology, is a subgroup of G with $G/\mathbb{Z}^{\mathbb{N}}$ countable. Furthermore, it is easy to see that $d(G)$ consists of all $(x_n) \in G$ such that for each $m \in \mathbb{N}$, m divides x_n for n large enough. This is easily seen to be a proper and dense (in the Polish group topology on G) subgroup of G.*

Recall that the ideal $\mathcal{Z}_0$ of sets of *asymptotic zero density* is defined by

$$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_n |A \cap n|/n = 0\}.$$

This ideal is a $\mathbf{\Pi}^0_3$-complete subgroup of $(\mathcal{P}(\mathbb{N}), \Delta)$ and

$$\mathbf{d}_{\mathcal{Z}_0}(A, B) = \sup_n |(A \Delta B) \cap n|/n$$

is a complete separable metric on $\mathcal{Z}_0$ compatible with the group operations.

Example 5.2 *A Polish group that is densely divisible and such that $d(G)$ is a $\mathbf{\Pi}^0_3$-complete subgroup. For $x \in \mathbb{Q}^{\mathbb{N}}$ let $D_x = \{n : x(n) \notin \mathbb{Z}\}$ and let*

$$G_{\mathcal{Z}_0} = \{x \in \mathbb{Q}^{\mathbb{N}} : D_x \in \mathcal{Z}_0\}.$$

Define a metric on $G_{\mathcal{Z}_0}$ by

$$\mathbf{d}(x, y) = \frac{1}{\min\{n : x(n) \neq y(n)\} + 1} + \mathbf{d}_{\mathcal{Z}_0}(D_x, D_y).$$

This is a complete separable metric on $G_{\mathcal{Z}_0}$ compatible with the group operations. It is not difficult to check that

$$d(G_{\mathcal{Z}_0}) = \{x \in \mathbb{Q}^{\mathbb{N}} : (\forall k \in \mathbb{N})\{n : x(n) \notin k\,\mathbb{Z}\} \in \mathcal{Z}_0\}$$

and therefore $\mathrm{bor}(d(G_{\mathcal{Z}_0}), G_{\mathcal{Z}_0}) = 3$.

Clearly, if $\mathcal{I}$ is a Borel ideal on $\mathbb{N}$ that is a $\mathbf{\Pi}^0_\alpha$-complete Polishable subgroup of $(\mathcal{P}(\mathbb{N}), \Delta)$ then $G_{\mathcal{I}}$ is a densely divisible Polish group such that $\mathrm{bor}(d(G_{\mathcal{I}}), G_{\mathcal{I}}) = \alpha$. However, all Polishable Borel ideals on $\mathbb{N}$ are $\mathbf{\Pi}^0_3$ ([8]).

This section is devoted to the proof of the following theorem.

Theorem 5.3 *For every countable ordinal α there is a Polish group K_α which is Abelian, densely divisible, and such that*

$$\operatorname{ulm}(K_\alpha) = \alpha.$$

Moreover, if α is a successor ordinal, there exists a locally compact, second countable group K_α as above.

The groups used to prove the above theorem will be defined in terms of parameters which we call forests. Theorem 5.3 will follow from Theorem 5.16, proved below, which gives an upper and a lower estimate on the Ulm rank in terms of natural derivatives associated with the forests.

5.1 Forests and derivations

Let

$$T = \mathbb{N}^{<\mathbb{N}} = \{t\colon \{0, 1, \ldots, n\} \to \mathbb{N} : \text{ for some } n \in \mathbb{N}\}.$$

Then T ordered by the end-extension, $\subseteq$, is a tree. We will think of T as 'growing downwards' so that if $s \subseteq t$, then t is 'below' s. For $s, t \in T$, we write $s \perp t$ if neither $s \subseteq t$ nor $t \subseteq s$. For $t \in T$ let $|t|$ denote its length. For $s, t \in T$ with $s \subseteq t$ let

$$d(t, s) = |t| - |s|$$

and for $A \subseteq T$ let

$$d(A, s) = \min\{d(t, s) : t \in A \text{ and } s \subseteq t\}$$

where it is understood that $\min \emptyset = \infty$. We say that a set $A \subseteq T$ is *bounded* if for some M, $d(t, s) \leq M$ for all $t, s \in A$ with $s \subseteq t$. In case we need to specify the constant M, we will say that A is bounded by M.

We now introduce the notion of a forest and other notions related to it. A set $P \subseteq T$ is called *forest* if for all $t_1, t_2 \in P$ and s with $t_1 \subseteq s \subseteq t_2$, we have $s \in P$. A $t \in P$ is called a *top node* in P if it does not properly extend any element of P. Note that in a forest each element extends precisely one top node. A forest is called a *tree* if it has precisely one top node. A forest P is called *well-founded* if there is no $x \in \mathbb{N}^\mathbb{N}$ such that $x|n \in P$ for infinitely many $n \in \mathbb{N}$. If P, Q are forests we write $P \sqsubseteq Q$ if $P \subseteq Q$ and for any $t \in P$ and $s \in Q$ with $t \subseteq s$ we have $s \in P$. If P is a forest, let

$$T_P = \{t \in T : \exists s \in P\ s \subseteq t\}.$$

5.1.1 Well-founded derivation

Let $P \subseteq T$ be a forest. For $\alpha < \omega_1$, recursively define sets $P^{(\alpha)}$ as follows. Let $P^{(0)} = P$, $P^{(\lambda)} = \bigcap_{\alpha<\lambda} P^{(\alpha)}$ if λ is limit, and

$$P^{(\alpha+1)} = \{t \in P^{(\alpha)} : \forall n \exists s \in P^{(\alpha)}\ t \subseteq s \text{ and } d(s,t) \geq n\}.$$

Now put

$$|P|_{wf} = \min\{\alpha :\ P^{(\alpha+1)} = P^{(\alpha)}\}.$$

5.1.2 Splitting derivation

For $\alpha < \omega_1$, recursively define sets $P_{(\alpha)}$ as follows. Set $P_{(0)} = P$, $P_{(\lambda)} = \bigcap_{\alpha<\lambda} P_{(\alpha)}$, for λ limit, and

$$P_{(\alpha+1)} = \{t \in P_{(\alpha)} : \exists s_1, s_2 \in P_{(\alpha)}\ s_1 \perp s_2,\ t \subseteq s_1,\ t \subseteq s_2\}.$$

We call this process the *splitting derivation*. Note that if P is well-founded, then for each ordinal α

$$P^{(\alpha)} \subseteq P_{(\alpha)}.$$

Define further $r_P : \{-1\} \cup \omega_1 \to \omega_1$, where $\{-1\} \cup \omega_1$ is well ordered in the natural way, by letting $r_P(-1) = 0$ and

$$r_P(\alpha) = \begin{cases} \sup_{\beta<\alpha} r_P(\beta), & \text{if } \exists \beta \in \omega_1\ \beta < \alpha \text{ and } P_{(\beta)} \setminus P_{(\alpha)} \text{ is bounded;} \\ (\sup_{\beta<\alpha} r_P(\beta)) + 1, & \text{if } \forall \beta \in \omega_1\ \beta < \alpha \text{ implies } P_{(\beta)} \setminus P_{(\alpha)} \text{ is not bounded.} \end{cases}$$

In particular, $r_P(0) = 1$. Let $\alpha_P = \min\{\alpha \in \{-1\} \cup \omega_1 :\ P_{(\alpha+1)} = P_{(\alpha+2)}\}$, and define

$$|P|_{sp} = r_P(\alpha_P).$$

For example, let $P = \{\langle n\rangle^k : 1 \leq k \leq n, n \in \mathbb{N}\}$, where $\langle n\rangle^k$ stands for the sequence of length k whose all entries are equal to n. Then $|P|_{wf} = 1$, $P_{(1)} = \emptyset$, $r_P(0) = 1$, $r_P(1) = 2$, $\alpha_P = 0$ and $|P|_{sp} = 1$. If $S = \{\langle n\rangle^k : k \leq n, n \in \mathbb{N}\} \cup \{(\langle n\rangle^n)^\frown\langle m\rangle : n \in \mathbb{N}, m \in \mathbb{N}\}$ then $S_{(1)} = P$ and $|S|_{sp} = 2$.

We note a simple lemma.

Lemma 5.4 *Let P, Q be forests with $P \sqsubseteq Q$. Then for each ordinal α*

(i) $P_{(\alpha)} = Q_{(\alpha)} \cap P$ *and*
(ii) $r_P(\alpha) \leq r_Q(\alpha)$.

PROOF. We only note that (i) is proved by induction on α. Point (ii) is proved by induction using (i). We leave details to the reader.

Recall the following particular case of a result of Zippin (see [10, Corollary 76.2]):

> For every countable ordinal α there is a countable Abelian group G such that the Ulm rank of G is equal to α.

Our construction of K_α will use this result. In fact, we will need to use the inner workings of Zippin's groups. Therefore, we will recast the construction of these groups in a way that is suitable for our applications. We will define the groups to be sets of finite subsets of certain trees. This is different from what is done in [10] but the difference lies only in presentation. In fact, in [23] trees were used, in a manner similar to what we do here, to handle a generalization of Ulm's classification theorem.

Let H_2 be the set of all finite $a \subseteq T$. For $a \in H_2$ define its 'upwards closure' by

$$\widehat{a} = \{t \in T : s \supseteq t \text{ for some } s \in a\}.$$

Let us denote the set of $\supseteq$-maximal elements of a by a^+, thus

$$a^+ = \{t \in a : (\forall s \in a) t \subseteq s \text{ implies } t = s\}.$$

We define addition on H_2 as follows. If $a_1, \ldots, a_m$ are in H_2, let $x = \widehat{\bigcup_{i=1}^m a_i}$ and recursively

$$\begin{aligned} x_1 &= x^+, \\ x_{n+1} &= \left(x \setminus \bigcup_{i=1}^n x_i\right)^+, \qquad \text{for } n \geq 1. \end{aligned}$$

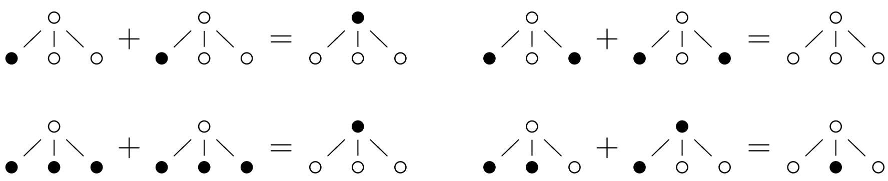
Since x is finite, there is the maximal k such that x_k is nonempty. We define functions $f_i \colon x_i \to \omega$ and $c_i \colon x_i \to \omega$ $(1 \leq i \leq k)$ recursively as follows (by $\lfloor q \rfloor$ we denote the largest integer not bigger than q).

$$\begin{aligned} f_1(t) &= |\{i : t \in a_i\}|, \\ c_1(t) &= 0, \\ c_{n+1}(t) &= \sum \lfloor \{f_n(s) : t \subseteq s\}/2 \rfloor, \qquad \text{for } n \geq 1, \\ f_{n+1}(t) &= |\{i : t \in a_i\}| + c_{n+1}(t). \end{aligned}$$

Finally let $f = \bigcup_{i=1}^k f_i$ and

$$\sum_{i=1}^m a_i = \{t \in x : f(t) \text{ is odd}\}.$$

If T consisted of a single branch, then this would be binary addition. In a sense, it is just a 'fan-out' version of binary addition. Here are some examples of the addition defined as above. (The top point in the diagrams is the top node of T.)



If P is a forest, define

$$H_2(P) = \{a \in H_2 : a \subseteq P\}$$

with addition given by

$$a +_P b = (a + b) \cap P.$$

Note that $H_2 = H_2(T)$.

Lemma 5.5 *The structure $(H_2(P), +_P)$ is an Abelian group for every forest P.*

PROOF. $H_2(P)$ is closed under $+_P$. The operation $+_P$ is clearly commutative, and the neutral is $\emptyset$. The equality $(a_1 +_P a_2) +_P a_3 = \sum_{i=1}^{3} a_i \cap P$ follows easily from the definition, hence the addition is also associative. To see that every $a \in H_2(P)$ has an inverse, note that $-\{t\} = \{s \in P : s \subseteq t\}$. Thus $-a = \sum_{t \in a}\{s \in P : s \subseteq t\}$.

From this point on we will drop the subscript in $+_P$.

Lemma 5.6 *If $Q \sqsubseteq P$, then the mapping $a \to a \cap Q$ is a homomorphism of $H_2(P)$ onto $H_2(Q)$.* $\square$

If $Q \sqsubseteq P$ then $H_2(Q)$ is included in $H_2(P)$. However, $H_2(Q)$ is not necessarily a subgroup of $H_2(P)$ in this situation. For example, if t is a top node of Q but not a top node of P, then $t + t = 0$ holds in $H_2(Q)$ but not in $H_2(P)$.

A $B \subseteq P$ is a *path* if it is upwards closed in P and for all s, t in B we have either $s \subseteq t$ or $t \subseteq s$.

Lemma 5.7 *Let P be a forest. If B is a finite path in P, then $H_2(B) = \{a \in H_2(P) : a \subseteq B\}$ is isomorphic to the cyclic group of rank 2^n, where n is the number of elements in B.*

PROOF. Let t_0 be the shortest element in B. To show that $H_2(B)$ is isomorphic to $\mathbb{Z}/2^n$, map $a \in H_2(B)$ to the coset of $\sum_{t\in a} 2^{d(t_0,t)}$.

An *end node* of a forest P is a $t \in P$ such that no $s \in P$ properly extends t.

Lemma 5.8 *Let $P \subseteq T$ be a forest.*

(i) For all $n \in \mathbb{N}$ and $a \in H_2(P)$ the equation $(2n+1)x = a$ has a solution in $H_2(P)$.
(ii) The equation $2x = a$ has a solution in $H_2(P)$ if and only if a does not contain any end nodes of P.
(iii) The group $H_2(P)$ is divisible.

PROOF. In the case when $a \in H_2(P)$ is included in a single path of P, clauses (i) and (ii) follow by Lemma 5.7. But for every $a \in H_2(P)$ there is $k \in \mathbb{N}$ and a_i $(i \leq k)$ in $H_2(P)$ so that $a = \sum_{i=1}^{k} a_i$ and each a_i is included in a single path, and this implies the general case of (i) and (ii). Clause (iii) is an immediate consequence of (i) and (ii).

Lemma 5.9 *If P is a forest, then* $\mathrm{ulm}(H_2(P)) = |P|_{wf}$.

PROOF. The equation $(2n+1)x = a$ has a solution in $H_2(P)$ for all n and a, by Lemma 5.8(i). The equation $2x = a$ has a solution in $H_2(P)$ if and only if a contains no end nodes of P. Hence for an $a \in H_2(P)$, $a \in H_2(P)^u_1$ if and only if the equation $kx = a$ has a solution in $H_2(P)$ for every k which is if and only if, for every $t \in a$ and every $n \in \mathbb{N}$, there is an $s \in P$ such that $t \subseteq s$ and $d(s,t) \geq n$. That is, the Ulm-derivative of $H_2(P)$ is equal to $H_2(P^{(1)})$. By induction it follows that $H_2(P)^u_\alpha = H_2(P^{(\alpha)})$ and the lemma follows.

For every ordinal γ there is a well-founded tree whose rank is equal to γ. This, together with Lemma 5.9, implies Zippin's result quoted at the beginning of this subsection.

5.3 Densely divisible groups–definitions

We describe now the construction of a family of groups among which we will find our K_αs. Fix a system of subsets A_s, $s \in \mathbb{N}^{<\mathbb{N}}$, of $\mathbb{N}$ satisfying the following for every s:

(1) $A_{\langle\rangle} = \mathbb{N}$,

(2) A_s is infinite,
(3) $A_{s\hat{}i}$, $1 \leq i < \infty$, are pairwise disjoint, and
(4) $A_s = \bigcup_{s=1}^{\infty} A_{s\hat{}i}$.

Now define the intervals I_s^n ($n \in \mathbb{N}$) of $\mathbb{N}$ as follows: $I_s^n = [k_{n-1}, k_n)$, where $k_0 = 1$ and $\{k_n\}_{n=1}^{\infty}$ is the increasing enumeration of A_s. Then these intervals satisfy the following properties for all $t \supseteq s$:

(5) $\{I_s^n\}_{n=1}^{\infty}$ form a disjoint partition of $\mathbb{N}$,
(6) for every m there is (a unique) n such that $I_s^m \subseteq I_t^n$,
(7) for all n there is exactly one i such that $\max I_{s\hat{}\langle i\rangle}^k = \max I_s^n$ for some k.

These are the only properties of the family $\{I_s^n\}$ that we shall need.

Let P be a forest. We can naturally identify elements of the group $H_2(T_P)^{\mathbb{N}}$ with subsets of $T_P \times \mathbb{N}$. Let $H_2(T_P)^{<\mathbb{N}}$ be the subgroup of the product $H_2(T_P)^{\mathbb{N}}$ consisting of all finite $\mathbf{a} \subseteq T_P \times \mathbb{N}$.

5.3.1 Groups $G(P)$ and $G(P)_\infty$

Let $G(P)$ be the subgroup of $H_2(T_P)^{<\mathbb{N}}$ generated by elements of the form

$$\{t\} \times I_t^n$$

with $t \in T_P$. We will call such elements *generators.*

Lemma 5.10 *Let $P \subseteq T$ be a forest.*

(i) $G(P)$ consists of all $\mathbf{a} \subseteq T_P \times \mathbb{N}$ that are finite unions of generators.
(ii) $G(P)$ is divisible.

PROOF. (i) Obviously all finite unions of generators are in $G(P)$. Therefore, it will suffice to show that the set G' of all finite unions of generators is a group.

For $\mathbf{a}, \mathbf{b}$ in G' we prove that $\mathbf{a} + \mathbf{b} \in G'$. Fix a generator I_t^m and pick $i \in I_t^m$. Whether $(t, i) \in \mathbf{a} + \mathbf{b}$ or not depends only on the intersections of $\mathbf{a}$ and $\mathbf{b}$ with the set $\{(s, i) \in T_P \times \mathbb{N} : t \subseteq s\}$. Recall that $t \subseteq s$ implies $I_t^m \subseteq I_s^k$ or $I_t^m \cap I_s^k = \emptyset$; therefore, if $i, j \in I_t^m$ for some m, we have $\{s \in T_P : (s, i) \in \mathbf{a}\} = \{s \in T_P : (s, j) \in \mathbf{a}\}$; and the same holds for $\mathbf{b}$. Hence I_t^m is either included in or disjoint from $\mathbf{a} + \mathbf{b}$. Since I_t^m was arbitrary, $\mathbf{a} + \mathbf{b} \in G'$. Now pick $\mathbf{a} \in G'$. We need to prove that $-\mathbf{a}$ is in G'. It will suffice to prove this for the generators, but $-(\{t\} \times I_t^n) = \{s \in T_P : s \subseteq t\} \times I_t^n$ and (i) is established.

(ii) For every $\mathbf{a} \in G(P)$ and $m \in \mathbb{N}$ we need to find $\mathbf{b} \in G(P)$ such that $m\mathbf{b} = \mathbf{a}$. If m is odd, this follows immediately from the fact that the order of each element of $H_2^{<\mathbb{N}}$ is a power of 2.

It remains to prove that for every $\mathbf{a} \in G(P)$ there is a $\mathbf{b} \in G(P)$ such that $2\mathbf{b} = \mathbf{a}$. It suffices to treat the case when $\mathbf{a} = \{t\} \times I_t^n$, $t \in T_P$. We will prove this by induction on n. Assume first that $n = 1$, thus $I_t^1 = [1, m]$ for some m. Let i be the unique natural number such that $I^1_{t\hat{}\langle i\rangle} = I_t^1$, as guaranteed by (7) and (6). Since $t \in T_P$, $t\hat{}\langle i\rangle \in T_P$. Thus, $\mathbf{b}_1 = I^1_{t\hat{}\langle i\rangle} \times \{t\hat{}\langle i\rangle\}$ is in $G(P)$, and $2\mathbf{b}_1 = \mathbf{a}$. Now let $\mathbf{a} = \{t\} \times I_t^n$ and assume that for each $m < n$ there is a $\mathbf{b}_m$ in $G(P)$ such that $2\mathbf{b}_m = \{t\} \times I_t^m$. Let i, k be the unique pair of natural numbers such that $\max I^k_{t\hat{}\langle i\rangle} = \max I_t^n$, as guaranteed by (7). Then $t\hat{}\langle i\rangle \in T_P$ and $I^k_{t\hat{}\langle i\rangle} = \bigcup_{j=l}^n I_t^j$ for some l. Let

$$\mathbf{b}_n = \{t\hat{}\langle i\rangle\} \times I^k_{t\hat{}\langle i\rangle} - \sum_{k=l}^{n-1} \mathbf{b}_k.$$

Then $2\mathbf{b}_n = \mathbf{a}$ and $\mathbf{b}_n \in G(P)$. This completes the inductive proof.

Let $G(P)_\infty$ be the closure of $G(P)$ inside $H_2(T_P)^{\mathbb{N}}$, in its product topology, where $H_2(T_P)$ is taken with the discrete topology. Obviously, $G(P)_\infty$ is a subgroup of $H_2(T_P)^{\mathbb{N}}$.

An element $\mathbf{a}$ of $G(P)_\infty$ is *symmetric* if it is of the form $a \times \mathbb{N}$ for some $a \in H_2$.

If P is a forest, we let

$(T_P)_{\leq i} = \{t \in T_P :\ d(t, t_0) \leq i$ where t_0 is the unique topnode of P with $t_0 \subseteq t\}$.

Similarly,

$(T_P)_{\geq i} = \{t \in T :\ d(t, t_0) \geq i$ where t_0 is the unique topnode of P with $t_0 \subseteq t\}$.

Lemma 5.11 *Let P be a forest. Assume that $\mathbf{a} \in G(P)_\infty$ is symmetric. Let $n \in \mathbb{N}$. Then*

(i) Every solution to the equation $(2n+1)x = \mathbf{a}$ in $G(P)_\infty$ is also symmetric.
(ii) Every solution to the equation $2^n x = \mathbf{a}$ in $G(P)_\infty$ is of the form $\mathbf{c} + \mathbf{d}$, where $\mathbf{c}$ is symmetric and $\mathbf{d} \subseteq (T_P)_{\leq n-1} \times \mathbb{N}$.

PROOF. We use Δ to denote the symmetric difference of sets. We shall also apply other set-operations to members of H_2.

Point (i) follows immediately from the fact that each element of H_2 has order a power of 2 so in it solutions to equations $(2n+1)x = a$ are unique.

We argue for (ii). For $\mathbf{a} \in G(P)_\infty$ let $\mathbf{a}(i) = \{t \in T : (t,i) \in \mathbf{a}\}$. First we show that, for any $\mathbf{a} \in G(P)_\infty$, for all $i \in \mathbb{N}$ the symmetric difference of $\mathbf{a}(i)$ and $\mathbf{a}(i+1)$ is included in a single path of T. Fix, therefore, $i \in \mathbb{N}$. Keep in mind that Lemma 5.10 implies that each element of $G(P)_\infty$ is a (not necessarily finite) union of generators. If $s,t \in \mathbf{a}(i)\Delta\mathbf{a}(i+1)$, then $i = \max(I_s^n)$ for some n and $i = \max(I_t^m)$ for some m. But (6) implies $i = \max(I_{s\wedge t}^k)$, and by (7) we have $s = s\wedge t$ or $t = s\wedge t$. Since s,t were arbitrary, all elements of $\mathbf{a}(i)\Delta\mathbf{a}(i+1)$ are $\subseteq$-comparable.

Now we get (ii) from the following observation concerning H_2. Assume a, b are elements of H_2 which are such that $a\Delta b$ is contained in one branch of T. Then $2^n a = 2^n b$ if and only if $a\Delta b \subseteq T_{\leq n-1}$. The implication from right to left is obvious. To prove the other implication, let $t \in a\Delta b$ extend all elements of $a\Delta b$. (Such a t exists since $a\Delta b$ is included in a single path of T_P.) Let t_0 be the top node of P with $t_0 \subseteq t$. Put $k = d(t,t_0)$, and let $a' = (T_P)_{\geq k} \cap a$ and $b' = (T_P)_{\geq k} \cap b$. We may assume $t \in a$; thus $a' = \{t\} \cup c$ and $b' = c$ for some c. It follows that $(2^n a')\Delta(2^n b') \subseteq \{s \in T_P : s \subseteq t\}$. Since $(2^n a' \Delta 2^n b') \cap T_{\geq k-n} = (2^n a \Delta 2^n b) \cap T_{\geq k-n}$, the set $2^n a \Delta 2^n b$ will be nonempty if $d(t,t_0) = k \geq n$, so $d(t,t_0) \leq n-1$.

5.3.2 Groups $L(P)$, $K(P)$, $K_0(P)$

Fix T_l, $l \in \mathbb{N}$, an increasing sequence of finite subtrees of T such that $T = \bigcup_{l=1}^{\infty} T_l$ and $\emptyset \in T_0$. Consider the group

$$L(P) = \{\mathbf{a} \in H_2(T_P)^{\mathbb{N}} : \forall^\infty n \; \mathbf{a}(n) \subseteq P \cap T_n\}.$$

It contains the compact (with the product topology inherited from $H_2(T_P)^{\mathbb{N}}$) subgroup $\{\mathbf{a} \in H_2(T_P)^{\mathbb{N}} : \forall n \; \mathbf{a}(n) \subseteq P \cap T_n\}$ which has countable index in it. Therefore, $L(P)$ is a locally compact, second countable topological group with the topology τ which coincides with the product topology on the subgroup $\{\mathbf{a} \in H_2(T_P)^{\mathbb{N}} : \forall n \; \mathbf{a}(n) \subseteq P \cap T_n\}$ and makes each of its cosets closed and open. Note that $G(P) \subseteq L(P)$. Let $K(P)$ be the closure in τ of $G(P)$ in $L(P)$, that is,

$$K(P) = \overline{G(P)}^{\tau}.$$

We will show that, with appropriate choice of P, these groups provide examples which prove Theorem 5.3. Define

$$K_0(P) = \{\mathbf{a} \in K(P) : \forall n \; \mathbf{a}(n) \subseteq P \cap T_n\}.$$

We will leave it to the reader to check that $K_0(P)$ is a compact subgroup of $K(P)$ with countable index in it. Here is a diagram illustrating inclusions between the groups involved in our construction (where $K \to G$ means that

K is a subgroup of G).

$$\begin{array}{ccccc} H_2(T_P)^{\mathbb{N}} & \longleftarrow & G(P)_\infty & \longleftarrow & G(P) \\ \uparrow & & \uparrow & \swarrow & \\ L(P) & \longleftarrow & K(P) & \longleftarrow & K_0(P) \end{array}$$

The following lemma collects some basic properties of $K(P)$. First, however, we need to introduce a new notion. If P is a forest and $\mathbf{f}$ an element of $K(P)$, by a *divisible sequence for* $\mathbf{f}$, we mean a sequence $\mathbf{f}_n \in K(P)$, $n \in \mathbb{N}$, with $\mathbf{f} = \mathbf{f}_0$ and $\mathbf{f}_n = 2\mathbf{f}_{n+1}$ for all n.

Lemma 5.12 *Let P be a forest.*

(i) The group $K(P)$ is locally compact, second countable, Abelian and densely divisible.

(ii) Let $\mathbf{a}$ be a subset of $T_P \times \mathbb{N}$. Then $\mathbf{a} \in K(P)$ if and only if the following three conditions hold

(a) $\mathbf{a} \cap (T_P \times \{n\})$ is finite for any $n \in \mathbb{N}$;

(b) $\mathbf{a} \setminus \bigcup_n((P \cap T_n) \times \{n\})$ is finite;

(c) $\mathbf{a}$ is the union of a family of generators $\{t\} \times I_t^n$ with $t \in T_P$, $n \in \mathbb{N}$.

(iii) Let $\mathbf{a} \in K(P)$ and $m \in \mathbb{N}$. There exists $\mathbf{b} \in K(P)$ with $\mathbf{a} = (2m+1)\mathbf{b}$. In particular, $\mathbf{a} \in d(K(P))$ precisely when there exists a divisible sequence for $\mathbf{a}$.

PROOF. (i) Since $K(P)$ is a closed subgroup of $L(P)$, it is second countable, locally compact, and Abelian. The group $G(P)$ is divisible by Lemma 5.10(ii) and it is dense in $K(P)$.

(ii) To see $\Rightarrow$, assume $\mathbf{a} \in K(P)$. Then (a) and (b) follow immediately from the fact that $\mathbf{a}$ is an element of $L(P)$ and (c) holds since $\mathbf{a}$ is in the closure, with respect to τ, of $G(P)$.

To see $\Leftarrow$, fix $a \subseteq T_P \times \mathbb{N}$ such that (a), (b) and (c) hold. Note that (c) implies that $\mathbf{a}$ is the union of a disjoint family of generators. If this family is finite, $\mathbf{a}$ is in $G(P)$ so also in $K(P)$ and we are done. Otherwise, let $\mathbf{g}_i$, $i \in \mathbb{N}$, be a 1-to-1 listing of this family. In the sum $\sum_{i \in \mathbb{N}} \mathbf{g}_i$, all the partial sums are in $G(P)$ and, by (a) and (b), they converge in $L(P)$ to $\mathbf{a}$ which puts $\mathbf{a}$ in $K(P)$.

(iii) Each element of $K(P)$ is the sum of an element of $G(P)$ and one of $K_0(P)$. Since $G(P)$ is divisible by Lemma 5.10(ii), it will be enough to show the claim for $\mathbf{a} \in K_0(P)$.

A generator $\{t\} \times I_t^n$ of $G(P)$ generates a cyclic subgroup of order $2^{|t|-|t_0|+1}$, where t_0 is the unique top node of P extended by t. So there exists a (unique modulo $2^{|t|-|t_0|+1}$) $k_t \in \mathbb{N}$ such that

$$\{t\} \times I_t^n = (2m+1)k_t(\{t\} \times I_t^n).$$

Thus, for each such generator there exists $\mathbf{b}_{n,t}$ such that

(α) $\mathbf{b}_{n,t} \in G(P)$;
(β) $\mathbf{b}_{n,t} \subseteq \{s \in T : s \subseteq t\} \times I_t^n$;
(γ) $(2m+1)\mathbf{b}_{n,t} = \{t\} \times I_t^n$.

Now $\mathbf{a} \in K_0(P)$ is a disjoint union of generators $\{t_i\} \times I_{t_i}^{n_i}$, $i \in \mathbb{N}$, belonging to $K_0(P)$, that is, with $t_i \in T_l$ for all $l \in I_{t_i}^{n_i}$. Now, (α) and (β) imply that $\mathbf{b}_{n_i,t_i}(l) \subseteq P \cap T_l$ for all $l \in \mathbb{N}$; thus, $\mathbf{b}_{n_i,t_i} \in K_0(P)$. Define

$$\mathbf{b} = \sum_{i=0}^{\infty} \mathbf{b}_{n_i,t_i}.$$

It is easy to see, using (ii), that the series converges in the compact group topology on $K_0(P)$. Moreover, by (γ),

$$(2m+1)\mathbf{b} = \sum_{i=0}^{\infty}(2m+1)\mathbf{b}_{n_i,t_i} = \sum_{i=0}^{\infty}\{t_i\} \times I_{t_i}^{n_i} = \mathbf{a}$$

which proves (iii).

5.4 Densely divisible groups–results

5.4.1 The lower bound on the Ulm rank

The following theorem gives a lower bound for the Ulm rank of groups of the form $K(P)$. Recall that $|P|_{wf}$ is the well-founded rank of a forest P.

Theorem 5.13 *Let P be a forest. Then* $\mathrm{ulm}(K(P)) \geq |P|_{wf}$.

PROOF. The group $H_2(P)$ is isomorphic to the subgroup S of $K(P)$ consisting of all symmetric elements in $K(P)$ via the isomorphism sending a to the symmetric element of $K(P)$ constantly equal to a. Now observe the obvious relationship between the Ulm sequence S_α^u, $\alpha < \mathrm{ulm}(S)$, of S and the Ulm sequence $K(P)_\alpha^u$, $\alpha < \mathrm{ulm}(K(P))$ of K: $S_\alpha^u \subseteq S \cap K(P)_\alpha^u$. On the other hand, note that since τ is stronger than the product topology, $K(P) \subseteq G(P)_\infty$ and, therefore, Lemma 5.11 implies that $d(S) = S \cap d(K(P))$. The conclusion follows.

5.4.2 The upper bound on the Ulm rank and end-extending trees

Let P be a forest. In order to control the Ulm rank of $K(P)$ from the above we need to assume P is of a special kind. Recall the definitions of the derivations $P^{(\alpha)}$ and $P_{(\alpha)}$ and of $|P|_{wf}$ and $|P|_{sp}$ from Subsection 5.1. We call P *end-extending* if for any ordinal α

$$\forall t \in P \setminus P_{(\alpha)} \forall s_1, s_2 \in P_{(\alpha)} \setminus P_{(\alpha+1)} \ (s_1 \subseteq s_2 \text{ and } s_1 \subseteq t \Rightarrow s_2 \subseteq t).$$

For example, the forest $\{\langle 0\rangle^k : k \in \mathbb{N}\} \cup \{(\langle 0\rangle^k)\hat{}\langle 1\rangle : k \in \mathbb{N}\}$ is not end-extending. Take, for example, $s_1 = \langle 0\rangle$, $s_2 = \langle 00\rangle$ and $t = \langle 01\rangle$. On the other hand, $P = \{\langle n\rangle^k : k \leq n, n \in \mathbb{N}\} \cup \{(\langle n\rangle^n)\hat{}\langle m\rangle : n \in \mathbb{N}, m \in \mathbb{N}\}$ is end-extending; note that $P_{(1)} = \{\langle n\rangle^k : k \leq n, n \in \mathbb{N}\}$.

Lemma 5.14 *For every $\alpha \in \{-1\} \cup \omega_1$ there is an end-extending forest $P(\alpha)$ such that*

$$|P(\alpha)|_{wf} = |P(\alpha)|_{sp} = \alpha + 1.$$

PROOF. Inductively define forests $P(\alpha)$ for $\alpha \in \{-1\} \cup \omega_1$. Let $P(-1) = \emptyset$. If $P(\beta)$ for $\beta < \alpha$ are defined then let β_n $(n \in \mathbb{N})$ be a sequence constantly equal to $\alpha - 1$ if α is 0 or a successor and if α is limit let it be a sequence cofinal in α. Define

$$P(\alpha) = \{\langle n\rangle^k : k, n \in \mathbb{N} \text{ and } 1 \leq k \leq n+1\} \cup \bigcup_{n \in \mathbb{N}} \{(\langle n\rangle^{n+1})\hat{}s : s \in P(\beta_n)\},$$

where $\langle n\rangle^m$ stands for the sequence of length m whose all entries are equal to n and $(\langle n\rangle^{n+1})\hat{}s$ stands for the sequence obtained by concatenating $\langle n\rangle^{n+1}$ and s. Note that $\emptyset \notin P(\alpha)$.

It is easily checked that $P(\alpha)$ is a well-founded forest. A straightforward induction shows $|P(\alpha)|_{wf} = \alpha + 1$ and $\min\{\gamma : P(\alpha)_{(\gamma+1)} = \emptyset\} = \alpha$. Using this and the definition of the forests $P(\alpha)$, we see that each $P(\alpha)$ is end-extending and that $r_{P(\alpha)}(\gamma) = \gamma + 1$ for $\gamma \leq \alpha$. It follows, therefore, that $|P(\alpha)|_{sp} = r_{P(\alpha)}(\alpha) = \alpha + 1$.

We record the following simple lemma.

Lemma 5.15 *Let P and Q be forests with $P \sqsubseteq Q$. If Q is end-extending, then so is P.*

PROOF. This is an immediate corollary of Lemma 5.4(i).

The next theorem provides us with an upper bound for the Ulm rank of groups of the form $K(P)$, where P is well-founded and end-extending. It complements therefore Theorem 5.13. We do not know whether the assumption that P is end-extending is necessary here.

Theorem 5.16 *Let P be a forest. If P is well-founded and end-extending, then* $\mathrm{ulm}(K(P)) \leq |P|_{sp}$.

PROOF. The proof of this theorem is long. We split it, therefore, into three parts.

Part I: Introductory claims. Let $Q \sqsubseteq P$ be forests. For $\mathbf{a} \in K(P)$, let $\mathbf{a}|T_Q$ be the element of $K(Q)$ defined by

$$\mathbf{a}|T_Q = \mathbf{a} \cap (T_Q \times \mathbb{N}).$$

Similarly to Lemma 5.6, the mapping $\mathbf{a} \to \mathbf{a}|T_Q$ defines a homomorphism from $K(P)$ onto $K(Q)$. Since T_Q is a subset of T_P and elements of $K(Q)$ and $K(P)$ are subsets of $T_Q \times \mathbb{N}$ and $T_P \times \mathbb{N}$, respectively, we can naturally identify $K(Q)$ with a subset $K(P)$. Let i denote the inclusion. We call this i the *natural embedding* of $K(Q)$ into $K(P)$. Note that $i(\mathbf{a})|T_Q = \mathbf{a}$ for any $\mathbf{a} \in K(Q)$. If Q has a top node t that is not a top node of P, then i is not a homomorphism, since then $\mathbf{a} = \{t\} \times \mathbb{N}$ satisfies $2\mathbf{a} = 0$ in $K(Q)$ but not in $K(P)$.

Claim 5.17 *Let $Q \sqsubseteq P$ be forests. Assume that for some $M \in \mathbb{N}$, for each $s \in P$*

$$(\exists t \in Q\ s \subseteq t) \Rightarrow d(Q, s) \leq M.$$

(i) If $\mathbf{a}$ and $\mathbf{b}$ are in $K(Q)$ and are such that $2\mathbf{a} = \mathbf{b}$, then $2^{M+1}i(\mathbf{a}) = 2^M i(\mathbf{b})$.

(ii) $\mathbf{b} \in K(Q)$ is divisible in $K(Q)$ iff some $\mathbf{b}' \in K(P)$, with $\mathbf{b}'|T_Q = \mathbf{b}$ and

$$\mathbf{b}' \subseteq \{t \in T_P : (\exists s \in T_Q)\, t \subseteq s\} \times \mathbb{N},$$

is divisible in $K(P)$.

PROOF. (i) If $2\mathbf{a} = \mathbf{b}$, then

$$2i(\mathbf{a})\Delta i(\mathbf{b}) \subseteq \{t \in P \setminus Q : (\exists s \in Q) t \subseteq s\} \times \mathbb{N},$$

and by our assumption every $\mathbf{c} \in K(P)$ included in this set satisfies $2^M\mathbf{c} = 0$.

(ii) The implication from right to left is obvious since the mapping $\mathbf{f} \to \mathbf{f}|T_Q$ from $K(P)$ into $K(Q)$ is a homomorphism, so a divisible element $\mathbf{b}'$ is mapped to a divisible element $\mathbf{b} = \mathbf{b}'|T_Q$.

To prove the other implication, let $\mathbf{b}_n \in K(Q)$, $n \in \mathbb{N}$, be a divisible sequence for $\mathbf{b}$. Let $i : K(Q) \to K(P)$ be the natural embedding. Define

$$\mathbf{b}' = 2^M i(\mathbf{b}_M).$$

Since $\mathbf{b}_M \subseteq T_Q \times \mathbb{N}$, $\mathbf{b}' \subseteq \{t \in T_P : (\exists s \in T_Q)\, t \subseteq s\} \times \mathbb{N}$. Also $\mathbf{b}' \in K(P)$ is divisible as witnessed by the sequence $\mathbf{b}'_n = 2^M i(\mathbf{b}_{M+n})$, $n \in \mathbb{N}$ and Claim 5.17(i). Moreover, we have in $K(Q)$

$$\mathbf{b}'|T_Q = (2^M i(\mathbf{b}_M))|T_Q = 2^M(i(\mathbf{b}_M)|T_Q) = 2^M \mathbf{b}_M = \mathbf{b}.$$

Part II: Statement (*). Note that for each β, $P \setminus P_{(\beta)}$ is a forest and $P \setminus P_{(\beta)} \sqsubseteq P$. Set

$$P_\beta = P \setminus P_{(\beta)}.$$

To prove the lemma, we will establish the following statement by induction on $\gamma \in \omega_1$:

(∗) For any well-founded, end-extending forest P and $\beta \in \{-1\} \cup \omega_1$, if $r_P(\beta) \leq \gamma$, then $K(P_{\beta+1})^u_\gamma \subseteq d(K(P_{\beta+1}))$.

The lemma follows from the above statement. Indeed, fix a well-founded, end-extending forest P. Let α_P be the ordinal involved in the definition of $|P|_{sp}$ so that $|P|_{sp} = r_P(\alpha_P)$. Since P is well-founded, $P_{(\alpha_P+1)} = \emptyset$, so $P_{\alpha_P+1} = P$. Now applying statement (∗) to $\gamma = r_P(\alpha_P)$, we get $K(P)^u_{r_P(\alpha_P)} \subseteq d(K(P))$, hence $d(K(P)) = K(P)^u_{r_P(\alpha_P)}$. Since $r_P(\alpha_P) = |P|_{sp}$, we get $\mathrm{ulm}(K(P)) \leq |P|_{sp}$.

The proof of (∗) will occupy the rest of the proof of Theorem 5.16. If $\gamma = 0$ and $r_P(\beta) \leq \gamma = 0$, then $\beta = -1$ and the statement holds since $P_0 = \emptyset$ so $K(P_0)$ has only the empty set as its element (which is its zero) since $G(P_0)$ consists of the empty set only. Thus, trivially $K(P_0) = d(K(P_0))$ holds. Assume now (∗) holds for all $\gamma < \gamma_0$. Note that it follows from the definition of r_P that, for any forest P, all values of r_P which are bigger than 0 are successors. Therefore, if γ_0 is limit, (∗) holds for γ_0. Thus, we can assume that

$$\gamma_0 = \gamma' + 1 \text{ for some } \gamma' \in \omega_1. \tag{5.1}$$

Part III: The successor stage of statement (∗). We now proceed with the proof of (∗) for γ_0 as in (5.1). This will take a while.

For the rest of this proof, we fix a well-founded, end-extending forest P.

It will suffice to show that, for any β with $r_P(\beta) = \gamma' + 1$, $K(P_{\beta+1})^u_{\gamma'+1} \subseteq d(K(P_{\beta+1}))$. Let

$$\beta_0 = \min\{\beta : r_P(\beta) = \gamma' + 1\}.$$

Let us notice

$$\forall \alpha < \omega_1 \text{ limit } (\exists \xi < \alpha\ P_{(\xi)} = P_{(\alpha)} \text{ or } \forall \xi < \alpha\ P_{(\xi)} \setminus P_{(\alpha)} \text{ is unbounded}). \quad (5.2)$$

This holds because for $t \in P_{(\xi)}$ the condition

$$\sup\{d(s,t) : s \supseteq t \text{ and } s \in P_{(\xi)}\} < n$$

implies $t \notin P_{(\xi+n)}$. Now, since $\beta_0+\omega$ is limit, (5.2) together with the definition of r_P imply that either $P_{(\beta_0+k)} = P_{(\beta_0+\omega)}$ for some $k < \omega$ or $r_P(\beta_0+\omega) > r_P(\beta_0)$. In the former case there exists a $k_0 < \omega$ with $\alpha_P \leq \beta_0+k_0$; in the latter case the set of all β with $r_P(\beta) = \gamma'+1$ is included in the interval $[\beta_0, \beta_0+\omega)$. Our goal is, therefore, to prove that for each $k \in \omega$ with $r_P(\beta_0+k) = \gamma'+1$, $K(P_{\beta_0+k+1})^u_{\gamma'+1} \subseteq d(K(P_{\beta_0+k+1}))$.

For the rest of this proof, in addition to P, we also fix $k \in \omega$ with $r_P(\beta_0+k) = \gamma'+1$ and $\mathbf{a} \in K(P_{\beta_0+k+1})^u_{\gamma'+1}$.

By Lemma 5.12(iii), it will suffice to produce a divisible sequence for $\mathbf{a}$ in the group $K(P_{\beta_0+k+1})$. (The notion of a divisible sequence is defined in the paragraph preceding Lemma 5.12.)

The remainder of the proof will be spent on constructing a divisible sequence for $\mathbf{a}$ in $K(P_{\beta_0+k+1})$. Two main steps of this construction are contained in Sublemmas 5.19 and 5.21. Before stating the sublemmas we fix some notation. Let $w_0, w_1, \ldots$ be the top nodes of P_{β_0+k+1}. From this point on we will write

$$Q = P_{\beta_0+k+1},$$
$$Q^{\leq n} = \{s \in Q : d(P_{\beta_0+k}, s) \leq n\}, \quad \text{for } n \in \mathbb{N}.$$

If R is a forest and $w \in R$, let

$$R[w] = \{s \in R : s \supseteq w\}.$$

It is easy to check that $R[w]$ is a forest—in fact, a tree—and that $R[w] \sqsubseteq R$.

From the very definition of the splitting derivation and of end-extending forests we have the following claim which provides a description of $Q[w_m]$, $m \in \mathbb{N}$. We leave the proof of the claim to the reader.

Claim 5.18 *Let $s, t \in Q[w_m]$ for some m.*

(i) If $s, t \notin P_{\beta_0+k}$, then $w_m \subseteq s \subseteq t$ or $w_m \subseteq t \subseteq s$.
(ii) If $s \notin P_{\beta_0+k}$ and $t \in P_{\beta_0+k}$, then $w_m \subseteq s \subseteq t$. □

Claim 5.18 easily implies that $Q^{\leq n}$ is a forest and $Q^{\leq n} \sqsubseteq Q$. We therefore have

$$P_{\beta_0} \sqsubseteq P_{\beta_0+k} \sqsubseteq Q^{\leq n} \sqsubseteq Q.$$

Perhaps a more detailed indication of how to picture $Q[w_m]$ and $Q^{\leq n}$ will be helpful in understanding the remainder of the proof. By Claim 5.18, the top part of $Q[w_m]$, that is, $Q[w_m] \setminus P_{\beta_0+k}$, is a finite set linearly ordered by $\subseteq$ whose shortest element is w_m. If z_m is the longest element of this linear part, all elements of $P_{\beta_0+k} \cap Q[w_m]$ extend z_m. The forest $Q^{\leq n}$ is the union of pairwise disjoint forests $Q^{\leq n} \cap Q[w_m]$ which are trees if $n \geq 1$. The top part of $Q^{\leq n} \cap Q[w_m]$ consists of n longest elements extending w_m and extended by z_m. Its bottom part is $P_{\beta_0+k} \cap Q[w_m]$.

Let R be a forest. For $\mathbf{f} \in K(R)$ and $i \in \mathbb{N}$, we say that $\mathbf{f}$ is *i-far* if $\mathbf{f} \cap (T_R \times \{0, \ldots, i-1\}) = \emptyset$. Note that if $\mathbf{f}$ is i-far, then for each m, $m\mathbf{f}$ is i-far as well.

Sublemma 5.19 *For each $n \in \mathbb{N}$, $\mathbf{a}|T_{Q^{\leq n}}$ is divisible in $K(Q^{\leq n})$.*

PROOF. Since $r_P(\beta_0) = r_P(\beta_0 + k)$, $P_{(\beta_0)} \setminus P_{(\beta_0+k)}$ is bounded. Let $N \in \mathbb{N}$ be such that $P_{(\beta_0)} \setminus P_{(\beta_0+k)}$ is bounded by N. Since $\mathbf{a} \in K(Q)^u_{\gamma'+1}$, we can find $\mathbf{c}_l \in K(Q)$ such that

$$2^{n+N+l}\mathbf{c}_l = \mathbf{a} \text{ and } \mathbf{c}_l \in K(Q)^u_{\gamma'}. \tag{5.3}$$

Let

$$Q' = \{t \in Q^{\leq n} : \exists s \in Q^{\leq n} \; s \subseteq t \text{ and } d(t,s) \geq N + n\}.$$

(Observe that Q' depends both on n and N, hence implicitly on k.) We leave it to the reader to check that Q' is a forest with $Q' \sqsubseteq P_{\beta_0}$. So we have

$$Q' \sqsubseteq P_{\beta_0} \sqsubseteq P_{\beta_0+k} \sqsubseteq Q^{\leq n} \sqsubseteq Q.$$

Since $P_{\beta_0} \sqsubseteq Q^{\leq n}$, we have the natural embedding $i : K(Q') \to K(Q^{\leq n})$.

Our plan is first to reduce showing that $\mathbf{a}|T_{Q^{\leq n}}$ is divisible to showing

(a) $2^{N+n} i(\mathbf{c}_0|T_{Q'})$ is divisible in $K(Q^{\leq n})$,
(b) proving that $2^{N+n} i(\mathbf{c}_0|T_{Q'})$ is divisible in $K(Q^{\leq n})$ provided $\mathbf{c}_0|T_{Q'}$ is divisible in $K(Q')$, and finally
(c) establishing divisibility of $\mathbf{c}_0|T_{Q'}$ directly by producing a divisible sequence for it using the inductive hypothesis $(*)$ for γ'.

Step (a).

This is accomplished by proving the following formula:

$$\mathbf{a}|T_{Q^{\leq n}} - 2^{N+n} i(\mathbf{c}_0|T_{Q'}) \in d(K(Q^{\leq n})). \tag{5.4}$$

The definition of Q' gives

$$\mathbf{c}_0|T_{Q'} \subseteq \mathbf{c}_0 \cap (\{t \in T_{Q^{\leq n}} : \exists s \in Q^{\leq n} \; s \subseteq t \text{ and } d(t,s) \geq N + n\} \times \mathbb{N}).$$

Moreover, the set theoretic difference $\mathbf{d}$ of the two elements on the two sides of $\subseteq$ of the above formula is included in $\mathbf{c}_0 \setminus (Q^{\leq n} \times \mathbb{N})$ and, therefore, in $\mathbf{c}_0 \setminus (Q \times \mathbb{N})$. Thus, it is an element of $G(Q^{\leq n})$ and, therefore, it is divisible in $K(Q^{\leq n})$ by Lemma 5.10(ii). Additionally, using (5.3) for $l = 0$, we get

$$\begin{aligned}\mathbf{a}|T_{Q^{\leq n}} &= 2^{N+n}(\mathbf{c}_0|T_{Q^{\leq n}}) \\ &= 2^{N+n}(\mathbf{c}_0 \cap (\{t \in T_{Q^{\leq n}} : \exists s \in Q^{\leq n}\ s \subseteq t \text{ and } d(t,s) \geq N+n\} \times \mathbb{N})) \\ &= 2^{N+n} i(\mathbf{c}_0|T_{Q'}) + 2^{N+n}\mathbf{d}\end{aligned}$$

and (5.4) follows.

Equation (5.4) implies that in order to prove that $\mathbf{a}|T_{Q^{\leq n}} \in d(K(Q^{\leq n}))$, we only need to show that $2^{N+n} i(\mathbf{c}_0|T_{Q'}) \in d(K(Q^{\leq n}))$.

Step (b)

We assume $\mathbf{c}_0|T_{Q'}$ is divisible in $K(Q')$ and prove $2^{N+n} i(\mathbf{c}_0|T_{Q'})$ is divisible in $K(Q^{\leq n})$. By Claim 5.17(ii), there is a divisible element $\mathbf{b}$ in $K(Q^{\leq n})$ such that $\mathbf{c}_0|T_{Q'} = \mathbf{b}|T_{Q'}$ and

$$\begin{aligned}\mathbf{b} &\subseteq (\{t \in T_{Q^{\leq n}} : (\exists s \in T_{Q'})\, t \subseteq s\} \times \mathbb{N}) \\ &= (T_{Q'} \times \mathbb{N}) \cup (\{t \in Q^{\leq n} : (\exists s \in Q')\, t \subseteq s\} \times \mathbb{N}).\end{aligned}$$

From the above formula we get the equality in the formula below

$$d(K(Q^{\leq n})) \ni 2^{N+n}\mathbf{b} = 2^{N+n} i(\mathbf{c}_0|T_{Q'}).$$

Step (c).

It remains to see that $\mathbf{c}_0|T_{Q'} \in d(K(Q'))$. Let now $v_0, v_1, \ldots$ be the top nodes of Q'. First we will use the inductive hypothesis $(*)$ for γ' to prove the following

$$\mathbf{c}_l|T_{Q'[v_i]} \in d(K(Q'[v_i])) \text{ for all } i \text{ and } l. \tag{5.5}$$

To see this, fix i and l and note that since $Q'[v_i] \sqsubseteq Q$, the function $K(Q) \ni \mathbf{f} \to \mathbf{f}|T_{Q'[v_i]} \in K(Q'[v_i])$ is a homomorphism. Therefore, we have

$$\mathbf{c}_l|T_{Q'[v_i]} \in K(Q'[v_i])^u_{\gamma'}. \tag{5.6}$$

Note also that there is $\alpha < \beta_0$ such that

$$v_i \in P_{\alpha+1}. \tag{5.7}$$

Since $r_P(\alpha) < r_P(\beta_0) = \gamma' + 1$, we have

$$r_P(\alpha) \leq \gamma'. \tag{5.8}$$

Since $Q'[v_i] \sqsubseteq P$, (5.7) and (5.8) imply by Lemma 5.4 that $(Q'[v_i])_{(\alpha+1)} = \emptyset$ and $r_{Q'[v_i]}(\alpha) \leq \gamma'$. Since $Q'[v_i] \sqsubseteq P$, we have by Lemma 5.15 that $Q'[v_i]$ is end-extending and it is, of course, well-founded. Thus, by our inductive assumption $(*)$ for γ' and by (5.6), we have $\mathbf{c}_l|T_{Q'[v_i]} \in d(K(Q'[v_i]))$.

We will now carefully glue together divisible sequences for $\mathbf{c}_l|T_{Q'[v_i]}$ to produce a divisible sequence for $\mathbf{c}_0|T_{Q'}$ thus showing that this element is divisible in $K(Q')$. Note first that for each l, $2^{N+n}(2^l\mathbf{c}_l) = \mathbf{a}$ and, therefore, $2^{N+n}(2^l\mathbf{c}_l - \mathbf{c}_0) = 0$ for all l. It follows now from Lemma 5.11(ii) that $2^l\mathbf{c}_l - \mathbf{c}_0$ is a sum of a symmetric element and an element included in $(T_Q)_{\leq N+n}$. Hence $(2^l\mathbf{c}_l - \mathbf{c}_0)|T_{Q'}$ is symmetric and, therefore, included in finitely many sets among $T_{Q'[v_i]} \times \mathbb{N}$, $i \in \mathbb{N}$. Thus,

$$\forall l\, \exists i_l^1\, \forall i \geq i_l^1\ (2^l\mathbf{c}_l)|T_{Q'[v_i]} = \mathbf{c}_0|T_{Q'[v_i]}. \tag{5.9}$$

Furthermore, since the intersection of $\mathbf{c}_l$ with any set of the form $T_Q \times \{n\}$ is finite and, since for each $\mathbf{c}_l$ for all but finitely many n, $\mathbf{c}_l \cap (T_Q \times \{n\}) \subseteq Q \times \{n\}$, we have

$$\forall l\, \exists i_l^2\, \forall i \geq i_l^2\ \mathbf{c}_l|T_{Q'[v_i]} \text{ is } l\text{-far and is in } K_0(Q'[v_i]). \tag{5.10}$$

We can obviously assume that in formulas (5.9) and (5.10), for each l, $i_l^1 = i_l^2$ and that after setting $i_l = i_l^1 = i_l^2$ we have $i_l < i_{l+1}$.

By (5.5), we can fix a divisible sequence $\mathbf{c}_{l,i,r}$, $r \in \mathbb{N}$, in $K(Q'[v_i])$ for $\mathbf{c}_l|T_{Q'[v_i]}$. Define $\mathbf{b}_p \in K(Q')$ by the following formula. For $i \in \mathbb{N}$ let

$$\mathbf{b}_p|T_{Q'[v_i]} = \begin{cases} 2^{l-p}\mathbf{c}_l|T_{Q'[v_i]}, & \text{if } i_l < i \leq i_{l+1} \text{ and } l \geq p; \\ \mathbf{c}_{l,i,p-l}, & \text{if } i_l < i \leq i_{l+1} \text{ and } l < p; \\ \mathbf{c}_{0,i,p}, & \text{if } i \leq i_0. \end{cases}$$

It is not difficult to check, using Lemma 5.12(ii), that (5.10) implies that $\mathbf{b}_p$ is an element of $K(Q')$. Moreover, a direct calculation shows that $2\mathbf{b}_{p+1} = \mathbf{b}_p$. Furthermore, (5.9) and the definition of $\mathbf{b}_0$ give that $\mathbf{b}_0 = \mathbf{c}_0|T_{Q'}$. Thus, $\mathbf{c}_0|T_{Q'}$ is divisible in $K(Q')$, and the sublemma is proved.

Since for every m we have $w_m \in Q^{\leq n}$ for a large enough n, Sublemma 5.19 implies that $\mathbf{a}|T_{Q[w_m]}$ is divisible in $K(Q[w_m])$ for each m. Note that $\mathbf{a}|T_Q = \bigcup_m \mathbf{a}|T_{Q[w_m]}$. The rest of the proof of $(*)$ is concerned with choosing divisible sequences for each $\mathbf{a}|T_{Q[w_m]}$ which together produce a divisible sequence for $\mathbf{a}$ in $K(Q)$.

Claim 5.20 *Let $m, n \in \mathbb{N}$ be given. Assume that $Q[w_m] \not\subseteq Q^{\leq n}$. Let $\mathbf{f}, \mathbf{g} \in K(Q[w_m])$ be such that $2^n\mathbf{f} = 2^n\mathbf{g}$ and $(\mathbf{f} - \mathbf{g})|T_{Q[w_m] \cap P_{\beta_0+k}}$ does not contain a symmetric non-empty element. Then*

$$\mathbf{f} - \mathbf{g} \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) < n\} \times \mathbb{N}.$$

PROOF. Note first that by Claim 5.18, $Q[w_m] \not\subseteq Q^{\leq n}$ implies $w_m \notin Q^{\leq n}$. Thus, by the definition of $Q^{\leq n}$, $d(P_{\beta_0+k}, w_m) > n$. It follows that $Q[w_m] \cap P_{\beta_0+k} \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) \geq n\}$. This is the inclusion which will be used in the proof.

Since $2^n(\mathbf{f} - \mathbf{g}) = 0$, Lemma 5.11(ii) implies that

$$(\mathbf{f} - \mathbf{g}) \cap \{t \in T_{Q[w_m]} : d(t, w_m) \geq n\} \times \mathbb{N} \text{ is symmetric.} \tag{5.11}$$

Thus, by our assumptions that $(\mathbf{f} - \mathbf{g}) \cap (T_{Q[w_m] \cap P_{\beta_0+k}} \times \mathbb{N})$ does not contain a nonempty symmetric element and that $Q[w_m] \cap P_{\beta_0+k} \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) \geq n\}$, we get

$$(\mathbf{f} - \mathbf{g}) \cap (T_{Q[w_m] \cap P_{\beta_0+k}} \times \mathbb{N}) = \emptyset. \tag{5.12}$$

Since $(\mathbf{f} - \mathbf{g}) \setminus (Q[w_m] \times \mathbb{N})$ is finite and since nonempty symmetric elements are infinite, by (5.11) we also have

$$\mathbf{f} - \mathbf{g} \subseteq (Q[w_m] \cup \{t \in T_{Q[w_m]} : d(t, w_m) < n\}) \times \mathbb{N}. \tag{5.13}$$

From (5.12) and (5.13), we obtain

$$\mathbf{f} - \mathbf{g} \subseteq ((Q[w_m] \setminus P_{\beta_0+k}) \cup \{t \in T_{Q[w_m]} : d(t, w_m) < n\}) \times \mathbb{N}. \tag{5.14}$$

This, together with $2^n(\mathbf{f} - \mathbf{g}) = 0$, implies that

$$2^n((\mathbf{f} - \mathbf{g}) \cap (\{t \in Q[w_m] \setminus P_{\beta_0+k} : d(t, w_m) \geq n\} \times \mathbb{N})) = 0. \tag{5.15}$$

By Claim 5.18, $Q[w_m] \setminus P_{\beta_0+k}$ is linearly ordered by $\subseteq$ with w_m as the shortest element. It follows, therefore, from (5.15) that

$$(\mathbf{f} - \mathbf{g}) \cap (\{t \in Q[w_m] \setminus P_{\beta_0+k} : d(t, w_m) \geq n\} \times \mathbb{N}) = \emptyset$$

and the conclusion of Claim 5.20 is a consequence of this and of (5.14).

Sublemma 5.21 *Given n, for all but finitely many m there exists a divisible sequence for $\mathbf{a}|T_{Q[w_m]}$ in $K(Q[w_m])$ whose n-th element is n-far and is in $K_0(Q[w_m])$.*

PROOF. Recall that for every m there exists l such that $Q[w_m] \subseteq Q^{\leq l}$ and in this situation $K(Q^{\leq l}) \ni \mathbf{f} \to \mathbf{f}|T_{Q[w_m]} \in K(Q[w_m])$ a homomorphism. Hence Sublemma 5.19 implies that $\mathbf{a}|T_{Q[w_m]}$ has a divisible sequence, and we only need to show that for all large enough m such a divisible sequence can be chosen so that its n-th element is n-far and in $K_0(Q[w_m])$.

Let n be given and fixed. By Sublemma 5.19, there exists $\mathbf{b} \in K(Q^{\leq n})$ which is divisible and such that

$$2^n\mathbf{b} = \mathbf{a}|T_{Q^{\leq n}}.$$

Let $\mathbf{b}'$ be an element of $K(Q)$ with

$$2^n \mathbf{b}' = \mathbf{a}. \tag{5.16}$$

The existence of $\mathbf{b}'$ is assured by our assumption $\mathbf{a} \in K(Q)^u_{\gamma'+1}$ and $\gamma' \geq 0$.

First consider all m with $Q[w_m] \subseteq Q^{\leq n}$. Note that by Lemma 5.12 (ii) for all such m with $m \geq m_0$, for some m_0, $\mathbf{b}|T_{Q[w_m]}$ is n-far and is an element of $K_0(Q[w_m])$. Also, for all such m, $\mathbf{b}|T_{Q[w_m]}$ is divisible as a homomorphic image of a divisible element. Thus, any divisible sequence for $\mathbf{b}|T_{Q[w_m]}$ gives rise to a divisible sequence of $\mathbf{a}|T_{Q[w_m]}$ with all the required properties.

Now we consider all m with $Q[w_m] \not\subseteq Q^{\leq n}$. In fact, to simplify notation, assume that this condition holds for all m. First we show that for all but finitely many m there exists $\mathbf{b}_m \in K(Q[w_m])$ such that

(a) $\mathbf{b}_m|T_{P_{\beta_0+k} \cap Q[w_m]} = \mathbf{b}|T_{P_{\beta_0+k} \cap Q[w_m]}$;
(b) $2^n \mathbf{b}_m = \mathbf{a}|T_{Q[w_m]}$;
(c) $\mathbf{b}_m \in K_0(Q[w_m])$;
(d) $\mathbf{b}_m \cap (\{t \in Q[w_m] : d(t, w_m) < n\} \times \{0, 1, \ldots, n-1\}) = \emptyset$;
(e) $\mathbf{b}_m$ divisible in $K(Q[w_m])$.

To see that this suffices, we define the divisible sequence of $\mathbf{a}|T_{Q[w_m]}$ sought for in the conclusion of Sublemma 5.21 to be a divisible sequence of $\mathbf{a}|T_{Q[w_m]}$ which includes $\mathbf{b}_m$ as its n-th element. Such a sequence exists by (b) and (e). Point (c) states that, for large enough m, $\mathbf{b}_m \in K_0(Q[w_m])$. Thus, it remains to check that for m large enough $\mathbf{b}_m$ is n-far, that is, that the intersection $\mathbf{b}_m \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\})$ is empty. From (d) we see that $\mathbf{b}_m \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\})$ does not intersect $\{t \in Q[w_m] : d(t, w_m) < n\} \times \mathbb{N}$ for large enough m. Therefore, it will suffice to show that for large m

$$\mathbf{b}_m \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\}) \subseteq \{t \in Q[w_m] : d(t, w_m) < n\} \times \mathbb{N}\,. \tag{5.17}$$

Since $\mathbf{b} \in K(Q^{\leq n})$, $(\mathbf{b}|T_{P_{\beta_0+k} \cap Q[w_m]}) \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\}) = \emptyset$ for all but finitely many m. From this and (a) and (c) it follows that

$$\mathbf{b}_m \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\}) \subseteq (Q[w_m] \setminus P_{\beta_0+k}) \times \mathbb{N} \tag{5.18}$$

for large enough m. It follows immediately from (5.16) and (b) that in $K(Q[w_m])$ for large m we have

$$2^n \mathbf{b}_m - 2^n(\mathbf{b}'|T_{Q[w_m]}) = 0. \tag{5.19}$$

Since $\mathbf{b}' \in K(Q)$ for large enough m we also have

$$(\mathbf{b}'|T_{Q[w_m]}) \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\}) = \emptyset. \tag{5.20}$$

Now from (5.19) and (5.20) we obtain for large m

$$2^n \mathbf{b}_m \cap (T_{Q[w_m]} \times \{0, \ldots, n-1\}) = \emptyset. \tag{5.21}$$

By Claim 5.18, $Q[w_m] \setminus P_{\beta_0+k}$ is linearly ordered by $\subseteq$. Therefore, (5.21) and (5.18) imply (5.17) as desired.

We proceed now with the construction of elements $\mathbf{b}_m$ with properties (a)–(e). Since $\mathbf{b}|T_{Q^{\leq n} \cap Q[w_m]}$ is divisible, by Claim 5.17(ii) for each m there exists $\mathbf{b}'_m \in K(Q[w_m])$ which is divisible in $K(Q[w_m])$ and is such that

$$\mathbf{b}'_m|T_{Q^{\leq n} \cap Q[w_m]} = \mathbf{b}|T_{Q^{\leq n} \cap Q[w_m]}. \tag{5.22}$$

Note that since $\mathbf{b}$ is an element of $K(Q^{\leq n})$, for some m_0 and for all $m \geq m_0$ we have $\mathbf{b}|T_{Q^{\leq n} \cap Q[w_m]} \in K_0(Q^{\leq n} \cap Q[w_m])$. So for such m there is $\mathbf{f} \in G(Q[w_m])$ such that

$$\mathbf{f} \subseteq \mathbf{b}'_m \setminus (T_{Q^{\leq n} \cap Q[w_m]} \times \mathbb{N})$$

and $\mathbf{b}'_m - \mathbf{f} \in K_0(Q[w_m])$. By Lemma 5.10(ii), $\mathbf{f}$ is divisible in $K(Q[w_m])$. Let us denote $\mathbf{b}'_m - \mathbf{f}$ again by $\mathbf{b}'_m$ and note that it is still divisible, fulfills (5.22) and

$$\forall m \geq m_0 \ \mathbf{b}'_m \in K_0(Q[w_m]). \tag{5.23}$$

Now we define auxiliary elements $\mathbf{d}_m \in K(Q[w_m])$ by letting

$$\mathbf{d}_m = \mathbf{a}|T_{Q[w_m]} - 2^n \mathbf{b}'_m. \tag{5.24}$$

We record some simple observations concerning the newly defined elements and the $\mathbf{b}'$ of (5.16). First note that, for each m, we have

$$2^n(\mathbf{b}'|T_{Q[w_m]} - \mathbf{b}'_m) = \mathbf{a}|T_{Q[w_m]} - 2^n \mathbf{b}'_m = \mathbf{d}_m. \tag{5.25}$$

Now notice that $\mathbf{a}|T_{Q[w_m]}$ is divisible in $K(Q[w_m])$ by Sublemma 5.19 applied to l large enough for $Q[w_m] \subseteq Q^{\leq l}$ to hold, because for such l and m, $\mathbf{a}|T_{Q^{\leq l}} \to \mathbf{a}|T_{Q[w_m]}$ is a homomorphism from $K(Q^{\leq l})$ to $K(Q[w_m])$. Since $\mathbf{b}'_m$ is also divisible, we see from the definition of $\mathbf{d}_m$ that

$$\mathbf{d}_m \text{ is divisible in } K(Q[w_m]). \tag{5.26}$$

Further note that for some $m_1 \geq m_0$ for all $m \geq m_1$, $\mathbf{b}'|T_{Q[w_m]} \in K_0(Q[w_m])$. This combined with (5.23) gives that

$$\forall m \geq m_1 \ \mathbf{b}'|T_{Q[w_m]} - \mathbf{b}'_m \in K_0(Q[w_m]) \tag{5.27}$$

Thus, for $m \geq m_1$, from (5.27) and (5.25) we obtain $\mathbf{d}_m \in K_0(Q[w_m])$. By $\mathbf{a}|T_{Q^{\leq n}} = 2^n \mathbf{b}$, (5.24), and (5.22), we have $\mathbf{d}_m|T_{Q^{\leq n} \cap Q[w_m]} = 0$. From these last two pieces of information about $\mathbf{d}_m$ we deduce that

$$\forall m \geq m_1 \ \mathbf{d}_m \subseteq (Q[w_m] \setminus Q^{\leq n}) \times \mathbb{N}\,. \tag{5.28}$$

From this point on we fix $m \geq m_1$. Claim 5.18 allows us to enumerate all elements of $Q[w_m] \setminus P_{\beta_0+k}$ in a sequence $(t_i)_{i\leq p}$, for some p which depends on m, so that $w_m = t_0 \subseteq t_1 \subseteq \cdots \subseteq t_p$, with $|t_{i+1}| = |t_i| + 1$. (Note that by our assumption $Q[w_m] \nsubseteq Q^{\leq n}$, we have $p \geq n$.) Now define $\mathbf{g}_m \subseteq \{t_0, \ldots, t_p\} \times \mathbb{N}$ via

$$(t, l) \in \mathbf{g}_m \Leftrightarrow \exists i \leq p \ (i \geq n,\ t = t_i \text{ and } (t_{i-n}, l) \in \mathbf{d}_m).$$

We claim the following hold:

(i) $\mathbf{g}_m \in K_0(Q[w_m])$;
(ii) $\mathbf{g}_m \subseteq (Q[w_m] \setminus P_{\beta_0+k}) \times \mathbb{N}$;
(iii) $2^n\mathbf{g}_m = \mathbf{d}_m$.

Assume for a moment that property (i) holds. Then point (ii) is a consequence of the definition of $\mathbf{g}_m$, and point (iii) is a consequence of (5.28) and elementary properties of the group H_2. To check property (i), we will need to go back to the definition of $G(Q[w_m])$ and $K_0(Q[w_m])$. It will suffice to show that for each $n \leq i \leq p$,

(α) $\{l \in \mathbb{N} : (t_i, l) \in \mathbf{g}_m\}$ is a union of the intervals $I^r_{t_i}$, $r \in \mathbb{N}$;
(β) $\forall l \in \mathbb{N}\ (t_i, l) \in \mathbf{g}_m$ implies $t_i \in T_l$.

By Lemma 5.10(i), (α) shows that $\mathbf{g}_m$ is in the closure of $G(Q[w_m])$ in $L(Q[w_m])$; thus, $\mathbf{g}_m \in K(Q[w_m])$. Point ($\beta$) along with (ii) give then $\mathbf{g}_m \in K_0(Q[w_m])$.

Before we start proving (α) and (β), note that by definition of $\mathbf{g}_m$, for $n \leq i \leq p$,

$$\{l : (t_i, l) \in \mathbf{g}_m\} = \{l : (t_{i-n}, l) \in \mathbf{d}_m\}. \tag{5.29}$$

Thus, to prove (α), it suffices to show that the set on the right hand side of (5.29) is a union of $I^r_{t_i}$s. To see this note that if $2^n\mathbf{f} = \mathbf{g}$, $\mathbf{f}, \mathbf{g} \in K(Q[w_m])$, then, for any $t \in T_{Q[w_m]}$, $\{l : (t, l) \in \mathbf{g}\}$ is a Boolean combination (in which we allow infinite unions and intersections) of sets of the form $\{l : (s, l) \in \mathbf{f}\}$ for $T_{Q[w_m]} \ni s \supseteq t$ and $d(s, t) \geq n$. Therefore, by (5.25), for each $n \leq i \leq p$, $\{l : (t_{i-n}, l) \in \mathbf{d}_m\}$ is a Boolean combination of the sets $\{l : (s, l) \in (\mathbf{b}'|T_{Q[w_m]}) - \mathbf{b}'_m\}$ where $t_{i-n} \subseteq s$ and $d(s, t_{i-n}) \geq n$. Hence, by (5.27), it is a Boolean combination of the sets $\{l : (s, l) \in (\mathbf{b}'|T_{Q[w_m]}) - \mathbf{b}'_m\}$ where $t_i \subseteq s$ since all elements of $Q[w_m]$ extending t_{i-n} and such that $d(s, t_{i-n}) \geq n$ extend t_i. By the definition of the families I^r_s, $r \in \mathbb{N}$, $s \in T$, such a Boolean combination is a union of intervals $I^r_{t_i}$, $r \in \mathbb{N}$.

Now we give an argument for (β). Again by (5.29) it suffices to see that for $n \leq i \leq p$ and any $l \in \mathbb{N}$, if $(t_{i-n}, l) \in \mathbf{d}_m$, then $t_i \in T_l$. If $(t_{i-n}, l) \in \mathbf{d}_m$, then, by (5.25), for some $s \in T$ with $t_{i-n} \subseteq s$ and $d(s, t_{i-n}) \geq n$, we have $(s, l) \in (\mathbf{b}'|T_{Q[w_m]}) - \mathbf{b}'_m$. By (5.27), this implies $s \in Q[w_m] \cap T_l$. Note that if $s \in Q[w_m]$, $t_{i-n} \subseteq s$, and $d(s, t_{i-n}) \geq n$, then $t_i \subseteq s$. Since T_l is a subtree of T with $\emptyset \in T_l$, we see that $t_i \in T_l$, so (β) holds. Thus, (i) is established.

By (5.26), $\mathbf{d}_m$ is divisible in $K(Q[w_m])$, hence there exists $\mathbf{g}'_m \in K(Q[w_m])$ which is divisible and $2^n \mathbf{g}'_m = \mathbf{d}_m$. We establish now the following formula

$$\mathbf{g}'_m - \mathbf{g}_m \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) < n\} \times \mathbb{N}. \tag{5.30}$$

In order to prove it, by (iii), $2^n \mathbf{g}'_m = \mathbf{d}_m$, and Claim 5.20, it will suffice to show that $(\mathbf{g}'_m - \mathbf{g}_m)|T_{Q[w_m] \cap P_{\beta_0+k}} = 0$. First we note that from (5.28) we get

$$2^n (\mathbf{g}'_m | T_{Q[w_m] \cap Q^{\leq n}}) = \mathbf{d}_m | T_{Q[w_m] \cap Q^{\leq n}} = 0.$$

It follows by Lemma 5.11(ii) that $\mathbf{g}'_m | T_{Q[w_m] \cap P_{\beta_0+k}}$ is symmetric. (We use here the inclusion $T_{Q[w_m] \cap P_{\beta_0+k}} \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) \geq n\}$ which follows from our assumption $Q[w_m] \not\subseteq Q^{\leq n}$.) The divisibility of $\mathbf{g}'_m$ in $K(Q[w_m])$ implies that $\mathbf{g}'_m | T_{Q[w_m] \cap P_{\beta_0+k}}$ is divisible in $K(Q[w_m] \cap P_{\beta_0+k})$. Since this last element is symmetric, if it were nomempty then by Lemma 5.11(ii) any divisible sequence for it would produce an infinite branch through $Q[w_m] \cap P_{\beta_0+k}$, contradicting well-foundedness of this tree. (This is true since, with notation as in Lemma 5.11(ii), if $n > 0$ and $\{s\} \times \mathbb{N} \subseteq \mathbf{a}$, then $\mathbf{c}$ contains $\{t\} \times \mathbb{N}$ for some $t \in P$ with $t \supsetneq s$.) Thus, we have $\mathbf{g}'_m | T_{Q[w_m] \cap P_{\beta_0+k}} = 0$. Combining this with (ii) we get

$$(\mathbf{g}'_m - \mathbf{g}_m) | T_{Q[w_m] \cap P_{\beta_0+k}} = \mathbf{g}'_m | T_{Q[w_m] \cap P_{\beta_0+k}} = 0$$

and (5.30) follows.

Now define

$$\mathbf{g}''_m = \mathbf{g}'_m \cap \bigcup_l (Q[w_m] \cap T_l) \times \{l\}$$

Since $\mathbf{g}''_m - \mathbf{g}'_m$ is finite, it is in $G(Q[w_m])$ and hence divisible. Thus, since $\mathbf{g}'_m$ is divisible, so is $\mathbf{g}''_m$. We claim that $\mathbf{g}''_m$ fulfills (i)–(iii). By Lemma 5.12(ii) and the definition of $K_0(Q[w_m])$, it obviously fulfills (i). Point (ii) follows from (5.30), $w_m \notin Q^{\leq n}$, and (ii) for $\mathbf{g}_m$. Since by (i) $\mathbf{g}_m \subseteq \bigcup_l (Q[w_m] \cap T_l) \times \{l\}$, it follows from (5.30) that

$$\mathbf{g}'_m \setminus \bigcup_l (Q[w_m] \cap T_l) \times \{l\} \subseteq \{t \in T_{Q[w_m]} : d(t, w_m) < n\} \times \mathbb{N}.$$

Now (iii) for $\mathbf{g}''_m$ follows from this, (5.30), and (iii) for $\mathbf{g}_m$.

Finally, we are able to define an element $\mathbf{b}_m$ with properties (a)–(e). (Recall that all this is done for $m \geq m_1$.) Let

$$\mathbf{b}_m = \mathbf{b}'_m + \mathbf{g}''_m.$$

Below we refer to properties (i)–(iii) with $\mathbf{g}_m$ replaced in them by $\mathbf{g}''_m$. Element $\mathbf{b}_m$ is an element of $K(Q[w_m])$ by (i). Properties (c) and (e) hold of $\mathbf{b}_m$ since they hold of both $\mathbf{b}'_m$ and $\mathbf{g}''_m$. Points (a) and (b) follow by (ii) and (5.22), and (5.24) and (iii), respectively. We now modify $\mathbf{b}_m$ by subtracting from it

$$\mathbf{b}_m \cap \bigcup \{\{t\} \times I_t^k : t \in Q[w_m],\, d(t, w_m) < n,\, k \leq K\} \tag{5.31}$$

where $K \in \mathbb{N}$ is chosen so that for each t with $d(t, w_m) < n$, we have $\{0, \ldots, n-1\} \subseteq \bigcup_{k \leq K} I_t^k$. These modified elements we again call $\mathbf{b}_m$. They obviously fulfill (a)–(d). Since the element (5.31) being subtracted is in $G(Q[w_m])$, so divisible, the modified $\mathbf{b}_m$s still fulfill (e) and Sublemma 5.21 is proved.

We now proceed with the proof that $\mathbf{a}|T_Q$ is divisible in $K(Q)$. Sublemma 5.21 allows us to define an increasing sequence $m_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that

$$\forall m \geq m_n \ \mathbf{a}|T_{Q[w_m]} \text{ has a divisible sequence whose } n\text{-th element is } n\text{-far and in } K_0(Q[w_m]). \tag{5.32}$$

For $m_{n+1} > m \geq m_n$, let $\mathbf{a}_i^m \in K(Q[w_m])$, $i \in \mathbb{N}$, be a divisible sequence for $\mathbf{a}|T_{Q[w_m]}$ as in (5.32). Additionally note that for each m, $\mathbf{a}|T_{Q[w_m]}$ is divisible in $K(Q[w_m])$ by Sublemma 5.19 applied to l with $Q[w_m] \subseteq Q^{\leq l}$. This allows us to find for $0 \leq m < m_0$, a divisible sequence $\mathbf{a}_i^m \in K(Q[w_m])$, $i \in \mathbb{N}$, for $\mathbf{a}|T_{Q[w_m]}$. Define now $\mathbf{a}_i \subseteq T_Q \times \mathbb{N}$, $i \in \mathbb{N}$, by letting

$$\mathbf{a}_i \cap (T_{Q[w_m]} \times \mathbb{N}) = \mathbf{a}_i^m. \tag{5.33}$$

We claim that each $\mathbf{a}_i$ is in $K(Q)$. Fix i. If $n \geq i$, then, by (5.32), for $m_{n+1} > m \geq m_n$, $\mathbf{a}_i^m = 2^{n-i}\mathbf{a}_n^m$ and $\mathbf{a}_n^m$ is n-far and in $K_0(Q[w_m])$. Therefore, for m such that $m_{n+1} > m \geq m_n$ with $n \geq i$, we have that $\mathbf{a}_i^m$ is n-far, hence i-far, and in $K_0(Q[w_m])$. Thus, in view of (5.33), for all $m \geq m_i$, $\mathbf{a}_i \cap (T_{Q[w_m]} \times \mathbb{N})$ is i-far and is an element of $K_0(Q[w_m])$. This conclusion combined with Lemma 5.12(ii) gives that $\mathbf{a}_i$ is an element of $K(Q)$.

A simple calculation shows that $(\mathbf{a}_i)_{i \in \mathbb{N}}$ is a divisible sequence for $\mathbf{a}$ which finishes our proof of Theorem 5.16.

5.4.3 Proof of Theorem 5.3

We will derive Theorem 5.3 from Theorems 5.13 and 5.16 which were established above.

PROOF. [Proof of Theorem 5.3] For $\alpha \in \{-1\} \cup \omega_1$ let $P(\alpha)$ be an end-extending forest as defined in Lemma 5.14. Let $K_{\alpha+1} = K(P(\alpha))$. These groups are locally compact, second countable, Abelian and densely divisible by Lemma 5.12(i). Theorems 5.13 and 5.16 imply now that $\mathrm{ulm}(K_{\alpha+1}) = \alpha + 1$, giving Theorem 5.3 for successor ordinals.

Let λ be a countable limit ordinal. Put $K_\lambda = \prod_n K_{\alpha_n+1}$ where α_n, $n \in \mathbb{N}$, lists all the ordinals smaller than λ. This is a Polish Abelian group being a countable product of such groups. A direct calculation shows that $\mathrm{ulm}(K_\lambda) = \lambda$.

6 Questions

We will raise here a few questions which are related to our results and which remain unanswered.

In Theorem 2.1, we showed that any uncountable Polish group contains a Borel subgroup of arbitrary Borel rank bigger than 2. A corresponding problem for Polishable subgroups remains open.

Question 6.1 *Does every uncountable Polish group G contain, for any given $\alpha < \omega_1$, a Polishable subgroup H with* $\mathrm{pol}(H, G) = \alpha$*?*

An affirmative answer to Question 6.1 would, in fact, strengthen Theorem 2.1 (for all $\mathbf{\Pi}^0_\alpha$ classes with $\alpha \geq 3$ and not a successor of a limit ordinal) since producing a Polishable subgroup with a given pol rank is equivalent to producing a Polishable subgroup of a corresponding bor rank given by Theorem 3.1. For Abelian G the answer to Question 6.1 is "yes" by [13].

From Theorem 4.2 we know that for a densely divisible Abelian Polish group G, $\mathrm{ulm}(G) \leq \mathrm{pol}(d(G), G)$. However, the following question is open.

Question 6.2 *Let G be a Polish Abelian densely divisible group. Is it true that* $\mathrm{ulm}(G) = \mathrm{pol}(d(G), G)$*? If not, is there a bound on α such that* $\mathrm{ulm}(G) + \alpha = \mathrm{pol}(d(G), G)$*?*

Let us consider the possibility of answering the above question in the negative. The simplest way of making the Ulm rank smaller than the Polishable rank would be to find a Polish Abelian torsion free group G, which guarantees that $\mathrm{ulm}(G) \leq 1$, with $d(G)$ having high Borel rank and then use Theorem 3.1. To make it work, we would need first to have an affirmative answer to the following question.

Question 6.3 *Given $\alpha < \omega_1$, does there exist a Polish Abelian torsion free group G with* $\mathrm{bor}(2G, G) \geq \alpha$*?*

Note that the group $G_{\mathcal{Z}_0}$ from Example 5.2 satisfies $\mathrm{bor}(2G_{\mathcal{Z}_0}, G_{\mathcal{Z}_0}) = 3$.

In the context of Theorem 5.3, it would be interesting to know if the construction given in this paper can produce locally compact, second countable Abelian groups which are densely divisible and whose Ulm rank is a limit ordinal. The natural candidate here is the group

$$K'_\lambda = \{(\mathbf{a}_n) \in \prod_n K(P(\alpha_n)) : \forall^\infty n\ \mathbf{a}_n \in K_0(P(\alpha_n))\}$$

where α_n, $n \in \mathbb{N}$, enumerate all ordinals smaller than the countable limit ordinal λ and the $P(\alpha)$s are the trees defined in the proof of Theorem 5.3.

Question 6.4 *Is it true that* $\mathrm{ulm}(K'_\lambda) = \lambda$*?*

Theorem 5.16 can be used to show that $\lambda \leq \mathrm{ulm}(K'_\lambda) \leq \lambda + 1$.

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