CAUCHY NETS AND OPEN COLORINGS

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ABSTRACT. The Open Coloring Axiom, OCA, (see [6]) is used to prove that $\mathbb{R}^{\mathbb{N}}$ equipped with a natural uniform structure is complete, answering a question from [3].

1. Cauchy nets

The Open Coloring Axiom, OCA, was introduced by Todorcevic in [6] in the course of studying the *Hausdorff completeness* (i.e. gap spectrum) of the structure $\mathbb{N}^{\mathbb{N}}$ of all integer-valued sequences (see [6, Theorem 8.6], also [1, Chapter IV]). In this note we use the same axiom to deduce another, rather different, completeness property of $\mathbb{N}^{\mathbb{N}}$. To express the new kind of completeness, we need to define an abstract distance on the set $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers. First let

$$\mathcal{F} = \{g : \mathbb{N} \to \mathbb{N} : \lim_{i\to\infty} g(i) = \infty\}.$$

For $g \in \mathcal{F}$ let

$$U_g = \{\langle x, y\rangle \text{ in } \mathbb{R}^{\mathbb{N}} : |x - y| < g + M \text{ for some fixed } M \in \mathbb{N}\}.$$

The symbol $|x - y| < g + M$ is interpreted as pointwise inequality, namely as

$$|x(n) - y(n)| < g(n) + M \qquad \text{for all } n.$$

If x, y are such that $|x - y|$ is uniformly bounded by some fixed $M \in \mathbb{N}$, we write $x \sim y$, and say that in this case x and y are *equivalent*. The following simple fact explains this terminology.

Lemma 1.1. *$\langle x, y\rangle \in U_g$ for all $g \in \mathcal{F}$ if and only if $x \sim y$.* □

Therefore, $\mathcal{U} = \{A \subseteq (\mathbb{R}^{\mathbb{N}})^2 : A \supseteq U_g \text{ for some } g \in \mathcal{F}\}$ is a pseudouniformity on $\mathbb{R}^{\mathbb{N}}$ (see [4]), which can naturally be identified with a uniformity on the quotient $\mathbb{R}^{\mathbb{N}}/\sim$. Let us note that $\mathbb{N}^{\mathbb{N}}$ intersects every $\sim$-equivalence class, and therefore $\mathcal{U}$ can be consider as a pseudouniformity on this space. One can think of the elements of index-set $\mathcal{F}$ as different orders of the infinity: Note that the set U_g gets finer (i.e. smaller) as the growth rate of g gets slower, and therefore only the slow-growing functions from $\mathcal{F}$ matter in the study of $\mathcal{U}$. Since this is exactly the opposite from the way the structure $\langle \mathcal{F}, <^*\rangle$ is usually considered (see e.g. [6]), let us define a transformation $\Phi \colon \mathcal{F} \to \mathcal{F}$ by

$$\Phi(f)(n) = \max\{k : f(k) \le n\}.$$

Recall that for $x, y \in \mathbb{R}^{\mathbb{N}}$ and $m \in \mathbb{N}$ we write

(1) $x <^m y$ if $x(n) < y(n)$ for all $n \ge m$.

Date: August 28, 2011.
1991 *Mathematics Subject Classification.* 54E15, 54A20, 03E35.

(2) $x <^* y$ if $x <^m y$ for some $m \in \mathbb{N}$.

Then $f <^* g$ implies that $\Phi(f) >^* \Phi(g)$ and for every $h \in \mathcal{F}$ there is some $g \in \mathcal{F}$ such that $\Phi(g) <^* h$. This fact shows that the slow-growing functions behave the same way as the well-studied fast growing functions, and it will be used later (see Claim 2.3).

A *Cauchy net* in $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ is a sequence of the form $\langle x_a : a \in D \rangle$ for some directed set $\langle D, < \rangle$ such that for every $g \in \mathcal{F}$ there is $d(g) \in D$ such that

$$\langle x_a, x_b \rangle \in U_g$$

for all $a, b \geq d(g)$ in D. This net *converges* to some $x_\infty \in \mathbb{R}^{\mathbb{N}}$ if for every $g \in \mathcal{F}$ there is $a \in D$ such that $\langle x_\infty, x_b \rangle \in U_g$ for all $b > a$ (see [4]). We say that a uniform space $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ is *complete* if every Cauchy net in this space converges.

The natural question whether the space $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ is complete was raised by T. Kaufhold in [3], and S. Watson ([8]) proved that under the Continuum Hypothesis the answer is negative by constructing an $\langle \omega_1, \omega_1 \rangle$-gap one of whose sides is a (necessarily divergent) Cauchy net. Then J. Steprans ([5]) proved that Proper Forcing Axiom, PFA, implies that every Cauchy net in $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ converges. The purpose of this note is to prove that the Open Coloring Axiom, OCA, gives the same conclusion. Since OCA is a consequence of PFA, this strengthens Steprans' result and also shows that the statement "Every Cauchy net in $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ converges" does not have any large cardinal strength.

I would like to thank Stevo Todorcevic for several remarks which have considerably improved this note.

2. Cauchy nets under OCA

Let us first recall the statement of OCA (see [6, §8]).

OCA: If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is a partition such that K_0 is an open subset of $[X]^2 = \{\{x, y\} : x \neq y,\ x, y \in X\}$ then one of the following applies:

(a) There is an uncountable $Y \subseteq X$ which is K_0-*homogeneous*, i.e., $[Y]^2 \subseteq K_0$
(b) X can be covered by countably many sets, each of which is K_1^n-homogeneous (X is σ-K_1-*homogeneous*).

Our proof of the following theorem should be compared with the proof of [6, Theorem 8.7].

Theorem 2.1. *OCA implies that the uniform space $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ is complete.*

Proof. We need to prove that every Cauchy net in $\langle \mathbb{R}^{\mathbb{N}}, \mathcal{U} \rangle$ converges. Let us start by giving an alternative definition of the pseudouniformity $\mathcal{U}$. For $g \in \mathcal{F}$ define a subset of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ as follows:

$$U_g^* = \{\langle x, y \rangle : |x - y| <^* g\},$$

and consider the pseudouniformity $\mathcal{U}^* = \{A \subseteq (\mathbb{R}^{\mathbb{N}})^2 : A \supseteq U_g^* \text{ for some } g \in \mathcal{F}\}$.

Lemma 2.2. *The uniformities $\mathcal{U}$ and $\mathcal{U}^*$ coincide.*

Proof. It suffices to note that

$$U_{g/2} \subseteq U_g^* \subseteq U_g.$$

To see the left-hand side inequality, note that $|x-y| < g/2+M$ implies $|x-y| <^k g$, where k is large enough so that $g(k) > 2M$. For the right-hand side inequality, observe that $|x-y| <^k g$ implies $|x-y| < g + \max\{g(i) : i \leq k\}$. □

Fix a Cauchy net $\langle x_a : a \in D\rangle$. For $g \in \mathcal{F}$ pick $d(g) \in D$ such that

$$|x_a - x_b| <^* g$$

for $a, b \geq d(g)$. For $x \in \mathbb{R}^{\mathbb{N}}$ and $g \in \mathcal{F}$ let

$$B_g(x) = \{y : |x-y| < g\}$$

Define a partition $[\mathcal{F}]^2 = K_0 \cup K_1$ by letting $\{f, g\} \in K_0$ if

$$x_{d(f)}(m) - x_{d(g)}(m) > 2f(m) + 2g(m) + 2$$

for some $m \in \mathbb{N}$. (Or equivalently, $\{f, g\} \in K_1$ if $x_{d(g)} \in \overline{B_{2f+2g+2}(x_{d(f)})}$.) If we consider $\mathcal{F}$ as a subspace of $\mathcal{F} \times \mathbb{R}^{\mathbb{N}}$ by identifying $g \in \mathcal{F}$ with the pair $\langle g, x_{d(g)}\rangle$, then K_0 becomes an open subset of $[\mathcal{F}]^2$.

Assume first that $\mathcal{F}$ can be covered by countably many K_1-homogeneous subsets. Since the poset $\langle \mathcal{F}, >^*\rangle$ is σ-directed, one of these subsets, call it $\mathcal{H}$, is *cofinal* in this poset (namely, for every $g \in \mathcal{F}$ there is $g' \in \mathcal{H}$ such that $g >^* g'$). Consider the intersection

$$\bigcap_{g \in \mathcal{H}} \overline{B_{2g+2}(x_{d(g)})}.$$

We claim that this set is nonempty. To see this, it will suffice to prove that for every m the intersection of family of intervals

$$I_{g,m} = [x_{d(g)}(m) - 2g(m) - 2, x_{d(g)}(m) + 2g(m) + 2] \qquad (g \in \mathcal{H}),$$

is nonempty. But by the K_1-homogeneity we have

$$|x_{d(f)}(m) - x_{d(g)}(m)| \leq 2f(m) + 2g(m) + 4,$$

and therefore every two intervals from the family intersect, and by Helly's theorem every finite subfamily of this family has the nonempty intersection. By compactness, the intersection of the whole family, $\bigcap_{g \in h} I_{g,m}$, is nonempty for every m, and therefore we can pick

$$x_\infty \in \prod_{m=1}^{\infty} \bigcap_{g \in \mathcal{H}} I_{g,m} = \bigcap_{g \in h} \overline{B_{2g+2}(x_{d(g)})}.$$

Towards proving that our Cauchy net converges to x_∞, fix $g \in \mathcal{F}$ and find $g' <^* g/2 - 2$ in $\mathcal{H}$. Then for every $a \geq d(g')$ we have

$$|x_\infty - x_a| < |x_\infty - x_{d(g')}| + |x_{d(g')} - x_a| <^* 2g' + 2 <^* g.$$

Now assume $\mathcal{F}$ can not be covered by countably many K_1-homogeneous sets. Since K_0 is open, OCA implies that $\mathcal{F}$ includes an uncountable K_0-homogeneous set, g_ξ $(\xi < \omega_1)$.

Claim 2.3. There is a g_{ω_1} in $\mathcal{F}$ such that $g_{\omega_1} <^* g_\xi$ for all ξ.

Proof. Let $\Phi \colon \mathcal{F} \to \mathcal{F}$ be the function defined in the introduction. Pick f_ξ such that $\Phi(f_\xi) <^* g_\xi$. Recall that OCA implies that there is an $f_{\omega_1} >^* f_\xi$ for all ξ (see [6, Theorem 8.7]). Then $g_{\omega_1} = \Phi(f_{\omega_1})$ is as required. □

Consider functions $x_\xi = x_{d(g_\xi)}$ ($\xi \le \omega_1$). Since D is directed, for every ξ there is $a_\xi > d(g_\xi), d(g_{\omega_1})$, and by the definition of $d(\cdot)$ we have

$$|x_\xi - x_{\omega_1}| < |x_\xi - x_{a_\xi}| + |x_{a_\xi} - x_{\omega_1}| <^* g_\xi + g_{\omega_1} <^* 2g_\xi.$$

Let $n_\xi \in \mathbb{N}$ be such that $|x_\xi - x_{\omega_1}| <^{n_\xi} 2g_\xi$. By going to an uncountable subset, we can assume that $n_\xi = \bar{n}$ and that $\lceil x_\xi(i) \rceil = \lceil x_\eta(i) \rceil = \bar{s}_\xi(i)$ for all $i \le \bar{n}$. for all ξ and some fixed $\bar{n} \in \mathbb{N}$ and $\bar{s}\colon \{1, \dots, \bar{n}\} \to \mathbb{N}$. Therefore if $m > \bar{n}$ we have

$$|x_\xi(m) - x_\eta(m)| < 2g_\xi(m) + 2g_\eta(m)$$

for all $\xi, \eta < \omega_1$. Moreover, for $m \le \bar{n}$ we have $|x_\xi(m) - x_\eta(m)| < 2$ for all $\xi, \eta < \omega_1$, and therefore $|x_\xi - x_\eta| < 2g_\xi + 2g_\eta + 2$, or equivalently, $\{g_\xi, g_\eta\} \in K_1$. But this is a contradiction, and it completes the proof. □

3. OCA$_\infty$

Our first proof of Theorem 2.1 used a strengthening of OCA which was extracted from Steprans' proof ([5]). Although it turned out that Theorem 2.1 already follows from Todorcevic's OCA, this axiom may turn out to be interesting in its own right.

If X is a topological space, then

$$[X]^2 = K_0^n \cup K_1^n, \qquad n \in \mathbb{N}$$

is a *decreasing sequence of open partitions* if every K_0^n is an open subset of $[X]^2$ (in a natural topology induced by the topology on X) and $K_0^n \supseteq K_0^{n+1}$ for all n.

OCA$_\infty$: If X is a separable metric space and $[X]^2 = K_0^n \cup K_1^n$, $n \in \mathbb{N}$, is a decreasing sequence of open partitions then one of the following applies:

(a) $X = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is K_1^n-homogeneous (in this case we say that X is *σ-K_1^*-homogeneous*).

(b) There is an uncountable $Y \subseteq X$ which is covered by countably many K_0^n-homogeneous sets (Y is *σ-K_0^n-homogeneous*) for every n.

Note that the requirement that sequence K_0^n is decreasing is, in some sense, necessary, for if $\bigcap_{i \in s} K_0^i = \emptyset$ for some finite set s then clearly no uncountable set can be simultaneously K_0^i-homogeneous for all $i \in F$.

We do not know whether OCA$_\infty$ (or its stronger version, see (b′) below) follows from OCA. We can consider a strengthening of OCA$_\infty$, obtained when (b) is replaced by:

(b′) There is an uncountable $Z \subseteq \{0,1\}^{\mathbb{N}}$ and $y_\alpha \in X$ ($\alpha \in Z$) such that (recall that $\Delta(\alpha, \beta)$ is the minimal n such that $\alpha(n) \ne \beta(n)$):

$$\{y_\alpha, y_\beta\} \in K_0^{\Delta(\alpha,\beta)}$$

for all $\alpha, \beta \in Z$.

If (b′) is satisfied, then set $Y = \{y_\alpha : \alpha \in Z\}$ satisfies a strong form of (b), because for every $s \in \{0,1\}^n$ the set

$$Y_s = \{y_\alpha : \alpha \in [s]\}$$

is K_0^n-homogeneous, and therefore Y is covered by 2^n many K_0^n-homogeneous sets for every n.

The consistency of OCA$_\infty$, or its stronger version defined above, can be proved in the same way as the consistency of OCA (see [6]), by using the following lemma instead of [6, Theorem 4.4].

Lemma 3.1. *Assume CH, let X be a separable metric space and let $[X]^2 = K_0^n \cup K_1^n$ be a decreasing sequence of open partitions. If X is not σ-K_1^*-homogeneous, then there is a ccc poset $\mathcal{P} = \mathcal{P}(X, K_0^*)$ which forces (b′)*

Proof. By using a diagonalization argument of [6, Theorem 4.4], we can find an uncountable subset Y of X such that poset of finite K_0^n-homogeneous subsets of Y is powerfully ccc (i.e. every its finite power is ccc) for every n. Define poset $\mathcal{P}$ so that its typical condition is $p = \langle F^p, g^p\rangle = \langle F, s\rangle$, where

(1) F is a finite subtree of $\{0,1\}^{<\mathbb{N}}$,
(2) g is an injection of top-nodes of F into Y
(3) $\{g(s), g(t)\} \in K_0^{\Delta(s,t)}$ for every pair s, t of distinct top-nodes of F.

The ordering on $\mathcal{P}$ is defined by letting $\langle F^p, g^p\rangle \le \langle F^q, g^q\rangle$ if

4. $F^p \supseteq F^q$ and $g^p(t) = g^q(s)$ whenever $s \subseteq t$.

To see that $\mathcal{P}$ is ccc, fix $\langle F^\xi, g^\xi\rangle$ $(\xi < \omega_1)$ in $\mathcal{P}$. We can assume that $F^\xi = \bar{F}$ for some fixed tree $\bar{F}$ and all ξ. Let $\bar{n}$ be the height of $\bar{F}$ and let $t_1, \dots, t_{\bar{k}}$ be its top-nodes. Finite sets $\{g^\xi(t_1), \dots, g^\xi(t_{\bar{k}})\}$ can be considered as conditions in k-th power of a poset of finite $K_0^{\bar{n}}$-homogeneous subsets of Y. Since this poset is powerfully ccc, we can find $\xi < \eta$ such that

$$\{g^\xi(t_i), g^\eta(t_i)\} \in K_0^{\bar{n}}$$

for all $i = 1, \dots, \bar{k}$. Now define an extension $\langle F, g\rangle$ of $\langle \bar{F}, g^\xi\rangle$ and $\langle \bar{F}, g^\eta\rangle$ as follows: Let F be the end-extension of $\bar{F}$ obtained by adding nodes t_i^ξ and t_i^η above each t_i, so that t_i^j $(i \le k, j = \xi, \eta)$ are the top-nodes of F. Define g by letting

$$g(t_i^\zeta) = g^\zeta(t_i), \qquad \text{for } \zeta = \xi, \eta \text{ and } i = 1, \dots \bar{k}.$$

The choice of $\bar{n}$ assures that $\Delta(t_i^\xi, t_i^\eta) \ge \bar{n}$ and therefore condition 3. is satisfied. Therefore $\langle F, g\rangle$ extends both $\langle F^\xi, g^\xi\rangle$ and $\langle F^\eta, g^\eta\rangle$, and poset $\mathcal{P}$ is ccc.

If G is a $\mathcal{P}$-generic filter, let $Y_G = \{g^p(t) : p \in G\}$, and for $y \in Y_G$ let

$$\alpha_G(y) = \bigcup \{t \in \{0,1\}^{<\mathbb{N}} : g^p(t) = y \text{ for some } p \in G\}.$$

Note that, by 4., the set on the right-hand side will be a branch (infinite, by genericity) of $\{0,1\}^{<\mathbb{N}}$, and therefore $Z_G = \{\alpha_G(y) : y \in Y_G\}$ is a subset of a Cantor space. If p is a condition in $\mathcal{P}$ which forces Y to be uncountable; then p forces that Z_G and Y_G satisfy (b′). □

Remark 3.2. One of the reasons why can OCA be considered as a natural axiom is the fact that it has a definable version (see [7], [2]). Let us note that a definable version of OCA_∞ is also true. Namely, if $\mathcal{A}$ is an analytic subset of some Polish space and $\{K_0^n\}$ is a decreasing sequence of open partitions of $[\mathcal{A}]^2$, then either $\mathcal{A}$ is σ-K_1^*-homogeneous or there is a continuous embedding g of a Cantor space into $\mathcal{A}$ such that $\{g(\alpha), g(\beta)\} \in K_0^{\Delta(\alpha,\beta)}$ for all distinct α, β.

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