# EVERY COMPACTUM THAT MAPS ONTO ITS OWN SQUARE MAPS ONTO ITS OWN COUNTABLE INFINITE PRODUCT

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**Theorem.** *A compact space $X$ maps onto $X^2$ if and only if it maps onto $X^{\mathbb{N}}$.*

*Proof.* We prove only the nontrivial direction. Let $f_i\colon X \to X$ ($i \in \{1,2\}$) be continuous maps such that $x \mapsto (f_1(x), f_2(x))$ maps $X$ onto $X^2$. Define $g_n\colon X \to X$ ($n \in \mathbb{N}$) as follows:

$$\begin{aligned} g_1 &= f_1 \\ g_2 &= f_1 \circ f_2 \\ g_3 &= f_1 \circ f_2 \circ f_2 \\ g_n &= f_1 \circ \underbrace{f_2 \circ \cdots \circ f_2}_{n-1 \text{ times}}, \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

We claim that $x \mapsto (g_n(x))_{n=1}^{\infty}$ maps $X$ onto $X^{\mathbb{N}}$. Since $X$ is compact, it suffices to show that the image of $X$ under this map is dense in $X^{\mathbb{N}}$, and it turn it suffices to show that for every $n \in \mathbb{N}$ and $\vec{y} = (y_i)_{i=1}^{n}$ in $X^n$ there is an $x \in X$ such that $\vec{z} = f(x)$ satisfies $z_i = y_i$ for all $i \le n$. Equivalently, we need to prove that for every $n$-tuple $(y_1, \dots, y_n)$ there is an $x \in X$ such that $g_i(x) = y_i$ for $i \le n$. We prove this by induction on $n$.

For $n = 2$, find $x' \in X$ such that $f_1(x') = y_2$ (possible, since $f_1$ is onto), and then, by using the fact that $x \mapsto (f_1(x), f_2(x))$ maps $X$ onto $X^2$, find $x \in X$ such that $f_1(x) = y_1$ and $f_1(x) = x'$. Then we have $g_2(x) = f_1(f_2(x)) = f_1(x') = y_2$.

Assume the claim is true for $n$ and prove it for $n+1$. By the induction hypothesis, find $x' \in X$ such that $g_i(x') = y_{i+1}$, for $1 \le i \le n$. Now pick $x$ such that $f_1(x) = y_1$ and $f_2(x) = x'$. Then for $2 \le i \le n+1$ we have $g_i(x) = g_{i-1}(f_2(x)) = g_{i-1}(x') = y_i$, and this completes the proof. □

By simple cardinal arithmetic, Theorem implies that a compactum of size strictly less than the continuum does not map onto its own square. The compactness assumption cannot be omitted from Theorem since $X = \mathbb{Q}$ maps onto $\mathbb{Q}^2$ but not onto $\mathbb{Q}^{\mathbb{N}}$. J. van Mill points out that the version of this result in which $X^2$ and $X^{\mathbb{N}}$ are required to be homeomorphic to $X$ is false, by [1].

## References

[1] J. van Mill. A Peano continuum homeomorphic to its own square but not to its countable infinite product. *Proceedings of the American Mathematical Society*, 80:703–705, 1980.

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