AUTOMORPHISMS OF CORONA ALGEBRAS, AND GROUP COHOMOLOGY

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Abstract. In 2007 Phillips and Weaver showed that, assuming the Continuum Hypothesis, there exists an outer automorphism of the Calkin algebra. (The Calkin algebra is the algebra of bounded operators on a separable complex Hilbert space, modulo the compact operators.) In this paper we establish that the analogous conclusion holds for a broad family of quotient algebras. Specifically, we will show that assuming the Continuum Hypothesis, if A is a separable algebra which is either simple or stable, then the corona of A has nontrivial automorphisms. We also discuss a connection with cohomology theory, namely, that our proof can be viewed as a computation of the cardinality of a particular derived inverse limit.

1. Introduction

The Calkin algebra came into prominence as the ambient structure for the BDF theory in the seminal work of Brown–Douglas–Fillmore [3, 4] (see [5] for an exposition). In their latter article [4], the authors asked whether the Calkin algebra has any outer automorphisms. The question remained open for more than thirty years, when it was shown to be independent from the axioms of set theory by Phillips–Weaver [18] and the second author [10]. In this paper, we are interested in a generalization of this question to corona algebras (sometimes called outer multiplier algebras). Both the original question and the generalization naturally belong to the program of analyzing outer automorphism groups of quotient structures, a program pursued by the second author for over a decade (see [10], [9], and [8], among others).

Formally, if $A \subset B(\mathcal{H})$ and the annihilator of A is trivial, then the *multiplier algebra* of A is the set $M(A)$ consisting of all $m \in B(\mathcal{H})$ such that both mA and Am are contained in A. It is well-known $M(A)$ does not depend on the representation of A. Then A is a two-sided, norm-closed, ideal of $M(A)$ and the *corona* of A is simply the quotient $M(A)/A$. In this paper, the corona of A will be denoted by $Q(A)$ in order to avoid confusion with an algebra of continuous functions on a compact Hausdorff space X, denoted by $C(X)$.

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The corona construction can be viewed as a noncommutative analogue of the Čech–Stone remainder of a topological space, since if $A = C(X)$ then the corona of A is given by $C(\beta X \smallsetminus X)$. (Here, $\beta X \smallsetminus X$ is the Čech–Stone remainder of X). Corona algebras also generalize the Calkin algebra, since it is easily seen that the corona of the algebra $\mathcal{K}$ of compact operators is exactly $Q(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}$. Like the Calkin algebra, coronas have played a role in the literature; for instance they provide the ambient structure for the Busby invariant for extensions of C*-algebras [2, §II.8.4.4] (see also [17]).

We wish to generalize the result that the Calkin algebra can have outer automorphisms to the case of more general corona algebras. In our results, we will generalize "outer" to the more restrictive notion of "nontrivial". In fact, the definition of "trivial" below is arguably the most comprehensive definition that is reasonable (see Section 7 for discussion). The only property of trivial automorphisms that we shall need in our main results is that if A is a separable C*-algebra, then $Q(A)$ has at most $2^{\aleph_0}$ trivial automorphisms.

Definition 1.1. An automorphism Φ of a separable C*-algebra A is said to be *trivial* if the set

$$\Gamma_\Phi = \{(a, b) \in M(A)^2 : \Phi(a/A) = b/A\}$$

is Borel, where $M(A)$ is endowed with the strict topology.

As we shall see in Section 7, trivial automorphisms form a group, all inner automorphisms are trivial, and in case of the Calkin algebra all trivial automorphisms are inner. However in some coronas there are trivial automorphisms which are not inner. For example, if $A = C_0(X)$ is an abelian C*-algebra then the corona $Q(A)$ has no inner automorphisms. However, many trivial automorphisms arise from homeomorphisms between cocompact subsets of X. Conjecturally no other automorphisms of such $Q(A)$ can be constructed without use of additional set-theoretic axioms such as the Continuum Hypothesis (see [7, §4]).

We now arrive at the question of whether corona algebras have nontrivial automorphisms. In addition to the case of the Calkin algebra, the analogous question has been answered in several other categories (see the survey in [9]). Based on these answers, it is natural to consider the following two conjectures.

Conjecture 1.2. *The Continuum Hypothesis implies that the corona of every separable, non-unital C*-algebra has nontrivial automorphisms.*

Conjecture 1.3. *Forcing axioms imply that the corona of every separable, non-unital C*-algebra has only trivial automorphisms.*

Of course, Conjecture 1.3 can be made stronger with a more restrictive definition of "trivial automorphism." Still, a confirmation even of this weak form would be a remarkable achievement. In this paper we will give a confirmation of Conjecture 1.2 for a large class of corona algebras.

Theorem 1.4. *Assume that the Continuum Hypothesis holds. Let A be a σ-unital C^*-algebra of cardinality $2^{\aleph_0}$ such that:*

1. *A is simple, or*
2. *A is stable, or*
3. *$A \cong B \otimes C$, where C is non-unital and simple, or*
4. *A has a non-unital, σ-unital quotient with a faithful irreducible representation.*

Then the corona of A has $2^{2^{\aleph_0}}$ many automorphisms.

In the case when A is separable, the corona of A has at most $2^{\aleph_0}$ many trivial automorphisms. By the classical result of Cantor that $\kappa < 2^\kappa$ for every cardinal κ, and we have the following consequence.

Corollary 1.5. *Suppose that the Continuum Hypothesis holds, and that A is a separable, non-unital, C^*-algebra satisfying any of (1)–(4) above. Then the corona of A has nontrivial automorphisms.*

Before discussing the proof of Theorem 1.4, let us put it in context by reviewing the cases in which Conjectures 1.2 and 1.3 have been answered. The first result is due to W. Rudin [19], who confirmed Conjecture 1.2 for the corona algebra $C(\beta\mathbb{N}\setminus\mathbb{N})$. Of course this conclusion is only in hindsight; in fact, Rudin established the topological reformulation: there exist nontrivial homeomorphisms of the Čech–Stone remainder of a locally compact Polish space.

Next, suppose that A has an orthogonal sequence r_i of projections whose partial sums form an approximate unit. Then Conjecture 1.2 has been verified in two extreme cases. First, if $r_i A r_j \neq \{0\}$ for all $i \neq j$, then it follows from our methods in Section 4. On the other hand, if $r_i A r_j = \{0\}$ for all $i \neq j$, then $A = \bigoplus_j r_j A r_j$. In this situation the corona of A turns out to be *countably saturated* as a metric structure, and the conclusion of Conjecture 1.2 follows from results of [11] and [1]. However, it is not difficult to construct a C*-algebra with an orthogonal sequence of projections whose partial sums form an approximate unit for which neither method can be applied.

Another case in which Conjecture 1.2 has been confirmed is that of an algebra A such that for every separable subalgebra B of $M(A)$, there is a B-quasicentral approximate unit for A consisting of projections and the center of $Q(A)$ is separable. Here, an approximate unit a_λ is *B-quasicentral* if $[b, a_\lambda] \to 0$ for every $b \in B$ (see [11, Corollary 2.14]). The final case is in the projectionless domain, where the topological reformulation was confirmed in the case when $A = C_0([0,1))$ by a result of J. C. Yu (see [13, §9]).

The problem of establishing Conjecture 1.3 for coronas of separable C*-algebras is much more interesting (and more challenging!), and has only been verified in a few special cases. The case of $C(\beta\mathbb{N}\setminus\mathbb{N})$ was established by Shelah in [20]. Once again, this result is only in hindsight: Shelah was working in a Boolean-algebraic reformulation given by Stone duality in the real rank zero case. Following this, the second author handled the case of several other abelian algebras in [7]. Most recently, as we have mentioned, he settled the case of the Calkin algebra in [10].

We now turn to an outline of the proof of Theorem 1.4, and a discussion of how it is organized throughout the coming sections. In Section 2, we construct an inverse system of abelian groups, each of which consists of (equivalence classes of) elements of the infinite torus $\mathbb{T}^{\mathbb{N}}$. We then show, by building a complete binary tree consisting of partial threads through this inverse system, that the inverse limit will have many elements which are nontrivial in the sense that they do not arise from constant threads. This sort of construction is familiar in category theory, and in Section 3 we elaborate upon this connection.

Next, in Section 4, we construct a map from the inverse limit built in Section 2 into the automorphism group of $Q(A)$. This is done by stratifying $Q(A)$ into layers, and identifying elements of the abelian groups of Section 2 with automorphisms of the layers. In Section 4 we also isolate a technical condition on the C*-algebra A to ensure that the resulting map is one-to-one. In Section 5, we give a second, weaker technical condition which again ensures that the map is one-to-one. Finally, in Section 6, we conclude the proof of Theorem 1.4 by verifying that each of its alternative hypotheses (1)–(4) implies that one of the technical assumptions of Sections 4 or 5 is satisfied.

This argument is similar in one aspect to the proof in the case of the Calkin algebra found in [18]. As in that proof, we end up with a complete binary tree of height $\aleph_1$ consisting of partial automorphisms of $Q(A)$, and the branches of the tree determine distinct automorphisms. However, in [18] the partial automorphisms are defined on separable subalgebras, and most of the difficulty in the argument lies in showing that they can be extended at limit stages. (It requires ensuring that the automorphisms are asymptotically inner.) In our approach, based on [10], we stratify $Q(A)$ into nonseparable layers, and as we shall see, this makes the limit stages much easier to handle.

Another difference with the proof in [18] is that we will not require the full strength of the Continuum Hypothesis (which in the future we will abbreviate CH). As a consequence, we can conclude that the combinatorics of our proof are quite different from the essentially model-theoretic methods used in Rudin's proof in the case of $C(\beta\mathbb{N}\smallsetminus\mathbb{N})$. Indeed, in the forcing extension constructed in [12, Corollary 2] all automorphisms of $C(\beta\mathbb{N}\smallsetminus\mathbb{N})$ are trivial, while $\mathfrak{d} = \aleph_1$ and $2^{\aleph_0} < 2^{\aleph_1}$. And as we shall see in the proof, the latter two assumptions suffice to establish Theorem 1.4 and its corollary.

Lastly, we would like to mention the important and related question of whether there exists an automorphism of the Calkin algebra which is K-theory reversing. Unfortunately, our techniques cannot be used to answer it, since the automorphisms we construct are locally trivial. An answer to this question would require an extension of the model-theoretic methods of [11].

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2. Building blocks for automorphisms

Following [10, §1], we begin by constructing coherent sequences (of uncountable length) of elements of the infinite torus $\mathbb{T}^{\mathbb{N}}$. In later sections, elements of $\mathbb{T}^{\mathbb{N}}$ will feature prominently as the nonzero entries of diagonal unitary matrices, and the coherent sequences constructed here will ultimately be patched together to define automorphisms of C*-algebras.

To explain what is meant by "coherent", let us introduce a family of pseudometrics Δ_I on $\mathbb{T}^{\mathbb{N}}$, where I ranges over the finite subsets of $\mathbb{N}$. The connection between Δ_I and automorphisms of C*-algebras will become clear in Lemma 4.4, where it will be shown that each Δ_I corresponds to the distance between two automorphisms when restricted to a certain fragment of the corona algebra.

Initially, we define Δ_I when $I = \{\, i, j \,\}$ consists of just two elements. For $\alpha, \beta \in \mathbb{T}^{\mathbb{N}}$ we let

$$\Delta_{\{\, i,j \,\}}(\alpha, \beta) = \left| \alpha(i)\overline{\alpha(j)} - \beta(i)\overline{\beta(j)} \right| \ .$$

Then, for I a finite subset of $\mathbb{N}$ we write

$$\Delta_I(\alpha, \beta) = \max_{i,j \in I} \Delta_{\{\, i,j \,\}}(\alpha, \beta)$$

It is clear that the Δ_I satisfy the triangle inequality

$$\Delta_I(\alpha, \gamma) \leq \Delta_I(\alpha, \beta) + \Delta_I(\beta, \gamma) \ ,$$

and therefore Δ_I is a pseudometric on $\mathbb{T}^{\mathbb{N}}$. In most cases, we will only need to evaluate $\Delta_I(\alpha, 1)$, and this is easily seen to be the diameter of the set of values of α on I.

We have the following inequality relating two different Δ_I's.

Lemma 2.1. *Let I and J be finite subsets of $\mathbb{N}$. Then for any fixed $i_0 \in I$ and $j_0 \in J$ we have*

$$\Delta_{I \cup J}(\alpha, \beta) \leq \Delta_I(\alpha, \beta) + \Delta_J(\alpha, \beta) + \Delta_{\{\, i_0, j_0 \,\}}(\alpha, \beta)$$

Proof. We begin by observing that $\Delta_{\{\, i,j \,\}}(\alpha, \beta)$ can also be written

$$\Delta_{\{\, i,j \,\}}(\alpha, \beta) = \left| \alpha(i)\overline{\beta(i)} - \alpha(j)\overline{\beta(j)} \right| \ .$$

It follows that for fixed α, β, we have that $\Delta_{\{\, \cdot,\cdot \,\}}(\alpha, \beta)$ satisfies the following triangle inequality in the indices:

$$\Delta_{\{\, i,k \,\}}(\alpha, \beta) \leq \Delta_{\{\, i,j \,\}}(\alpha, \beta) + \Delta_{\{\, j,k \,\}}(\alpha, \beta) \ .$$

Thus, for $i \in I$ and $j \in J$, we have

$$\Delta_{\{i,j\}}(\alpha,\beta) \leq \Delta_{\{i,i_0\}}(\alpha,\beta) + \Delta_{\{i_0,j_0\}}(\alpha,\beta) + \Delta_{\{j_0,j\}}(\alpha,\beta) ,$$

and the desired inequality follows. □

For an infinite subset $X \subset \mathbb{N}$, we will need the following notation. First, for $j \in \mathbb{N}$ let $n(X,j)$ denote the j^{th} element in the increasing enumeration of $\{0\} \cup X$. Next, let

$$I(X,j) = \big[\, n(X,j), n(X,j+1) \,\big)$$

denote the j^{th} interval in the natural partition of $\mathbb{N}$ into finite intervals with endpoints in X. We are now prepared to make our key definition.

Definition 2.2. For $X \subset \mathbb{N}$, we let $\mathbf{F}_X$ denote the subgroup of $\mathbb{T}^{\mathbb{N}}$ defined by

$$\mathbf{F}_X = \left\{ \alpha \in \mathbb{T}^{\mathbb{N}} \;\middle|\; \lim_{j\to\infty} \Delta_{I(X,j)\cup I(X,j+1)}(\alpha,1) = 0 \right\} ,$$

and let $\mathbf{G}_X$ denote the quotient

$$\mathbf{G}_X = \mathbb{T}^{\mathbb{N}}/\mathbf{F}_X \ .$$

(We will consider these groups as discrete.)

As we shall see in Lemma 4.5, the elements of $\mathbf{F}_X$ will give rise to automorphisms of a corona algebra which are trivial on a certain fragment of that algebra. For future convenience, we presently note that Lemma 2.1 implies that $\mathbf{F}_X$ can also be written as

$$\mathbf{F}_X = \left\{ \alpha \in \mathbb{T}^{\mathbb{N}} \;\middle|\; \begin{array}{l} \lim_{j\to\infty} \Delta_{I(X,j)}(\alpha,1) = 0, \text{ and} \\ \lim_{j\to\infty} \Delta_{\{n(X,j),n(X,j+1)\}}(\alpha,1) = 0 \end{array} \right\} .$$

Note that if $Y \subset X$ then every $I(Y,j)$ can be written as $\bigcup_{k\in L} I(X,k)$ for some finite set L. Since we have $\Delta_{I(Y,j)} \geq \max_{k\in L} \Delta_{I(X,k)}$, it follows that $\mathbf{F}_Y \subset \mathbf{F}_X$. Moreover, if the symmetric difference of Y and Z is finite then $\mathbf{F}_Y = \mathbf{F}_Z$. Therefore, we have the following result.

Proposition 2.3. *If $Y \subset^* X$ then $\mathbf{F}_Y \subset \mathbf{F}_X$. Hence, also $\mathbf{G}_X$ is a quotient of $\mathbf{G}_Y$.*

(Here, $\subset^*$ denotes the *almost inclusion* relation, that is, $Y \subset^* X$ if and only if $Y \setminus X$ is finite.)

What is more, it is easy to see that for $Y \subset^* X$ we have the following commutative diagram whose rows are exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{F}_Y \arrow[r] \arrow[d] & \mathbb{T}^{\mathbb{N}} \arrow[r] \arrow[d] & \mathbf{G}_Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{F}_X \arrow[r] & \mathbb{T}^{\mathbb{N}} \arrow[r] & \mathbf{G}_X \arrow[r] & 0 \end{tikzcd} \tag{1}$$

Here, the arrow $\mathbf{F}_Y \longrightarrow \mathbf{F}_X$ is the inclusion map, the arrow $\mathbb{T}^{\mathbb{N}} \longrightarrow \mathbb{T}^{\mathbb{N}}$ is the identity, and the arrow $\mathbf{G}_Y \longrightarrow \mathbf{G}_X$ is the quotient map.

For the remainder of this section, we will work with a family $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ consisting of infinite sets, which we will assume has the following properties.

Hypothesis 2.4.

- $\mathcal{U}$ is closed under finite intersections and under finite modifications of its elements.
- $\mathcal{U}$ is $\aleph_1$-generated, that is, there exist $X_\xi \in \mathcal{U}$, $\xi < \omega_1$, such that for any $X \in \mathcal{U}$ there exists ξ with $X_\xi \subset^* X$.
- The family of enumerating functions $n(X, \cdot)$ of elements $X \in \mathcal{U}$ forms a *dominating family.* That is, for any $f \in \mathbb{N}^{\mathbb{N}}$ there exists $X \in \mathcal{U}$ such that $f(i) \leq n(X, i)$ for all but finitely many $i \in \mathbb{N}$.

We remark that the assumption that such a family $\mathcal{U}$ exists follows from CH. In fact, it follows from the axiom $\mathfrak{d} = \aleph_1$, which means: the least cardinality of a dominating family is exactly $\aleph_1$. This axiom is strictly weaker than CH—see [10, p. 629] for a discussion.

We are now ready to present the main technical result of this section. Recall that by Proposition 2.3 and the discussion following it, the groups $\mathbf{G}_X$, for $X \in \mathcal{U}$, form an inverse system indexed by $\mathcal{U}$ with respect to the reverse almost inclusion ordering $\supset^*$. The main result is simply to count the elements of $\varprojlim_{X \in \mathcal{U}} \mathbf{G}_X$.

Theorem 2.5. *If $\mathcal{U}$ satisfies Hypothesis 2.4, then $\varprojlim_{X \in \mathcal{U}} \mathbf{G}_X$ has cardinality $2^{\aleph_1}$.*

As a consequence, CH implies that most of the elements of $\varprojlim_{X \in \mathcal{U}} \mathbf{G}_X$ are *nontrivial*, in the sense that they do not arise from a constant thread.

Corollary 2.6. *Suppose that CH holds, and that $\mathcal{U}$ satisfies Hypothesis 2.4. Then the quotient $\varprojlim_{X \in \mathcal{U}} \mathbf{G}_X / \operatorname{im}(\mathbb{T}^{\mathbb{N}})$ is nontrivial, where $\operatorname{im}(\mathbb{T}^{\mathbb{N}})$ denotes the image of $\mathbb{T}^{\mathbb{N}}$ under the diagonal map $\alpha \mapsto ([\alpha]_{\mathbf{F}_X})_{X \in \mathcal{U}}$.*

We remark that to establish the Corollary, it is enough to replace the assumption of CH with the so called *weak Continuum Hypothesis*—the statement that $2^{\aleph_0} < 2^{\aleph_1}$.

In later sections, we will show how to construct automorphisms of a corona algebra from elements of the inverse limit $\varprojlim_{X \in \mathcal{U}} \mathbf{G}_X$. There, we will use Theorem 2.5 it to show the analog of Corollary 2.6 that most of the automorphisms we construct will be *nontrivial automorphisms.*

Proof of Theorem 2.5. Fix a strictly $\subset^*$-decreasing chain X_ξ, for $\xi < \omega_1$ that generates $\mathcal{U}$. We must construct $\alpha_s \in \mathbb{T}^{\mathbb{N}}$, for $s \in 2^\xi$ and $\xi < \omega_1$, by recursion so that (writing $\mathbf{F}_\xi$ for $\mathbf{F}_{X_\xi}$)

- $s \sqsubset t$ implies $\alpha_s \alpha_t^{-1} \in \mathbf{F}_\xi$, where $\xi = \operatorname{dom}(s)$, and
- $\alpha_{s^\frown 0} \alpha_{s^\frown 1}^{-1} \notin \mathbf{F}_{\xi+1}$, where $\xi = \operatorname{dom}(s)$.

For the successor stage, it suffices to show that $\mathbf{F}_{\xi+1} \subsetneq \mathbf{F}_\xi$. For this, we initially thin out the sequence X_ξ to suppose that for all m there exists $n(m)$ such that at least m intervals of X_ξ are contained in the interval $I(X_{\xi+1}, n(m))$. This can be done without any loss of generality by our third assumption in Hypothesis 2.4 (although it would also suffice to assume that $\mathcal{U}$ is a nonprincipal ultrafilter). Now, it is easy to construct an element $\alpha \in \mathbb{T}^{\mathbb{N}}$ which is constant on each $I(X_\xi, j)$, satisfies $\alpha(n(X_\xi, j+1)) = e^{i\pi/m}\alpha(n(X_\xi, j))$ whenever $I(X_\xi, j) \subset I(X_{\xi+1}, n(m))$, and satisfies $\alpha(n(X_\xi, j+1)) = \alpha(n(X_\xi, j))$ otherwise. Clearly $\alpha \in \mathbf{F}_\xi$, but for each m we have $\Delta_{I(X_{\xi+1},n(m))}(\alpha, 1) = 2$ and so $\alpha \notin \mathbf{F}_{\xi+1}$.

Next, we suppose that $s \in 2^\xi$ and $s = \bigcup s_n$ where α_{s_n} have been defined for all n. Since s_n and s are the only sequences we are interested in, we may simplify the notation and write α_n, X_n, and $\mathbf{F}_n$ for α_{s_n}, $X_{|s_n|}$, and $\mathbf{F}_{X_{|s_n|}}$ respectively. We will choose X_∞ to be a subset of X_ξ which is so sparse that it satisfies

$$\text{(2)} \quad \begin{array}{l} \text{whenever } n \leq k \text{ and } I(X_n, j) \subset I(X_\infty, k), \text{ we have} \\ \Delta_{I(X_n,j)}(\alpha_k, \alpha_n) < 1/k. \end{array}$$

For this, we inductively choose $I(X_\infty, k-1)$ large enough to include all of the remaining $I(X_n, j)$ such that $n \leq k$ and $\Delta_{I(X_n,j)}(\alpha_k, \alpha_n) \geq 1/k$. This can be done because for each fixed $n \leq k$, our first bulleted inductive hypothesis implies that $\Delta_{I(X_n,j)}(\alpha_k, \alpha_n) \to 0$. Using exactly the same reasoning, we can also suppose that

$$\text{(3)} \quad \begin{array}{l} \text{whenever } n \leq k \text{ and } I(X_n, j) \subset I(X_\infty, k), \text{ we have} \\ \Delta_{\{\, n(X_n,j), n(X_n,j+1)\,\}}(\alpha_k, \alpha_n) < 1/k. \end{array}$$

Next, we must define α_s so that for each n, we have $\alpha_s\alpha_n^{-1} \in \mathbf{F}_n$. In other words, we will need to satisfy both:

$$\lim_{j\to\infty} \Delta_{I(X_n,j)}(\alpha_s, \alpha_n) = 0 \tag{4}$$

$$\lim_{j\to\infty} \Delta_{\{\, n(X_n,j), n(X_n,j+1)\,\}}(\alpha_s, \alpha_n) = 0 \tag{5}$$

For (4) it would be sufficient to let $\alpha_s(i) = \alpha_n(i)$ whenever $i \in I(X_\infty, n)$. However, to establish (5) we shall need to be a little more careful. Specifically, we define $\alpha_s(i)$ inductively such that for all $i \in I(X_\infty, n) \cup \{n(X_\infty, n+1)\}$ we have $\alpha_s(i) = \gamma_n\, \alpha_n(i)$, where $\gamma_n \in \mathbb{T}$ is a uniquely determined constant.

Now to verify (4), let $k(j)$ be such that $n \leq k(j)$ and $I(X_n, j) \subset I(X_\infty, k(j))$. Then using the definition of α_s, together with the fact that constant multiples do not have an effect on the value of Δ_I, we see that

$$\Delta_{I(X_n,j)}(\alpha_s, \alpha_n) = \Delta_{I(X_n,j)}(\gamma_{k(j)}\alpha_{k(j)}, \alpha_n) = \Delta_{I(X_n,j)}(\alpha_{k(j)}, \alpha_n)\ .$$

By (2), the latter term is $< 1/k(j)$. Since $k(j) \to \infty$ as $j \to \infty$, this establishes (4). But now (5) is similar, because using the same reasoning we have

$$\Delta_{\{\,n(X_n,j),n(X_n,j+1)\,\}}(\alpha_s,\alpha_n)=\Delta_{\{\,n(X_n,j),n(X_n,j+1)\,\}}(\alpha_{k(j)},\alpha_n)\ ,$$

and the latter term is $< 1/k(j)$ by property (3). □

3. Connection with cohomology

In this section we explore a connection between Theorem 2.6 and cohomology theory. Note that a similar connection exists in the work of the second author (see the discussion in [7, §2]) as well as Talayco [21]. We assume the reader is familiar with the most basic categorical notions, such as short exact sequences.

We begin by observing that the conclusion of the previous section, that $\varprojlim \mathbf{G}_X/\operatorname{im}(\mathbb{T}^{\mathbb{N}})$ can be nontrivial, stems from the fact that surjective maps need not remain so after passing to the inverse limit. Indeed, notice that each of the quotient maps $\mathbb{T}^{\mathbb{N}}\to\mathbf{G}_X$ is surjective, but Theorem 2.5 implies that the natural map $\mathbb{T}^{\mathbb{N}}\to\varprojlim\mathbf{G}_X$ is not surjective.

In the language of category theory, we say that $\varprojlim$ is a left-exact covariant functor which is not right-exact. This means that whenever we apply it to a short exact sequence of objects

$$0\longrightarrow\mathcal{A}\longrightarrow\mathcal{B}\longrightarrow\mathcal{C}\longrightarrow 0$$

we obtain an exact sequence

$$0\longrightarrow\varprojlim\mathcal{A}\longrightarrow\varprojlim\mathcal{B}\longrightarrow\varprojlim\mathcal{C}$$

but we can not in general add the last $\longrightarrow 0$. As we shall explain, it follows that $\varprojlim$ gives rise to a sequence of derived functors in the same way that $\operatorname{Hom}(A,\cdot)$ gives rise to the functors $\operatorname{Ext}^n(A,\cdot)$. The derived functors of $\varprojlim$ are denoted by $\varprojlim^n$. We refer the reader to [14] for a detailed introduction to the derived functors $\varprojlim^n$. What we will show is that the conclusion of Theorem 2.6 corresponds to the statement that the first derived inverse limit $\varprojlim^1\mathbf{F}_X$ is nontrivial.

Proceeding generally, let D be a directed set and work in the category of inverse systems of abelian groups which are indexed by D. In other words, an object is a sequence $\mathcal{A}=(A_d : d\in D)$ together with a system of maps $\pi_{de}\colon A_e\to A_d$ for $d<e$, satisfying the usual composition law: if $d<e<f$ then $\pi_{de}\circ\pi_{ef}=\pi_{df}$. Given such an $\mathcal{A}$, it is possible to find an *injective resolution*, that is, an exact sequence of injective objects:

$$0\longrightarrow\mathcal{A}\longrightarrow Q^0(\mathcal{A})\longrightarrow Q^1(\mathcal{A})\longrightarrow\ \cdots \tag{6}$$

Although we have no need for the details, we can find such a resolution explicitly as follows. First, for each A_d let M_d be an injective abelian group containing A_d. Then define $Q^0(\mathcal{A})$ to be the inverse system $(Q_d : d\in D)$ where $Q_d=\prod_{d'\le d}M_{d'}$, with respect to the natural restriction maps. It is not hard to check that $Q^0(\mathcal{A})$ is an injective object, and that $\mathcal{A}$ embeds into

$Q^0(\mathcal{A})$. One then continues the resolution inductively by letting $Q^n(\mathcal{A})$ be the Q^0 of the cokernel of the previous map.

We now we apply $\varprojlim$ to each term in the resolution in Equation (6) to obtain a sequence:

$$0 \longrightarrow \varprojlim \mathcal{A} \longrightarrow \varprojlim Q^0(\mathcal{A}) \longrightarrow \varprojlim Q^1(\mathcal{A}) \longrightarrow \cdots \tag{7}$$

Let us denote this sequence as a whole by $Q(\mathcal{A})$, and each map $\varprojlim Q^n(\mathcal{A}) \to \varprojlim Q^{n+1}(\mathcal{A})$ by d^n. Then $Q(\mathcal{A})$ is not necessarily exact, but it still has the property that $\operatorname{im} d^n \subset \ker d^{n+1}$. Such a sequence is called a *cochain*, and the maps d^n are called *coboundary* maps. The *cohomology* groups $H^n(Q(\mathcal{A})) = \ker d^{n+1} / \operatorname{im} d^n$ measure the inexactness of the cochain (here, $H^0(Q(\mathcal{A})) = \ker d^0$).

Definition 3.1. For each n, the derived functor $\varprojlim^n$ is defined by

$$\varprojlim{}^n \mathcal{A} = H^n(Q(\mathcal{A}))$$

It is a standard fact that this definition is independent of the choice of injective resolution $Q^n(\mathcal{A})$.

Since $\varprojlim$ is left-exact, the cochain in Equation (7) is exact at the term $\varprojlim Q^0(\mathcal{A})$. It follows that $\varprojlim^0 \mathcal{A}$ is precisely $\varprojlim \mathcal{A}$; in other words, the functor $\varprojlim^0$ is precisely $\varprojlim$. In many cases the higher $\varprojlim^n$ groups are trivial, and a number of conditions have been established to guarantee that $\varprojlim^1 \mathcal{A}$ vanishes. In this paper we only have need of the following.

Definition 3.2. An inverse system $\mathcal{A}$ is said to be *flasque* if for every downwards-closed $J \subset D$, every partial thread $(a_d)_{d\in J}$ can be extended to a thread $(a_d)_{d\in D}$.

It is a basic result that if $\mathcal{A}$ is flasque, then $\varprojlim^n \mathcal{A} = 0$ for all $n \geq 1$ (see [14, Théorème 1.8]). Flasque systems are not uncommon; for instance if $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$, and the maps π_{ij} are all surjective, then $\mathcal{A}$ is flasque.

We now turn to a computation of the higher $\varprojlim^n$ groups for the inverse systems from the previous section. Let $\mathcal{F}$ denote the inverse system consisting of $(\mathbf{F}_X)_{X\in\mathcal{U}}$ with respect to the inclusion maps, and similarly let $\mathcal{G}$ denote $(\mathbf{G}_X)_{X\in\mathcal{U}}$ with respect to the quotient maps. Notice that $\mathcal{F}$ is rather degenerate as an inverse system; its "projection" maps are injective! Nevertheless, we now show that it's first derived inverse limit $\varprojlim^1 \mathcal{F}$ is nontrivial, and in fact it coincides precisely with the object of study of Theorem 2.6.

Proposition 3.3. *The group* $\varprojlim^1 \mathcal{F}$ *is precisely* $\varprojlim \mathcal{G} / \operatorname{im}(\mathbb{T}^{\mathbb{N}})$*, where* $\mathbb{T}^{\mathbb{N}} \to \varprojlim \mathcal{G}$ *is the diagonal map.*

Proof. Rather than proceed via a direct computation (which would involve the messy injective resolution), we will use the fact that $\varprojlim^1 \mathcal{F}$ appears in a long exact sequence. Let $\mathcal{T}$ denote the inverse system consisting of $(\mathbb{T}^{\mathbb{N}})_{X\in\mathcal{U}}$ together with the identity maps. Then we see from Equation (1) that in the category of inverse systems, we have an exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow \mathcal{G} \longrightarrow 0$$

Again letting $Q(\mathcal{A})$ denote the cochain constructed in Equation (7), it follows that there is an exact sequence of cochains

$$0 \longrightarrow Q(\mathcal{F}) \longrightarrow Q(\mathcal{T}) \longrightarrow Q(\mathcal{G}) \longrightarrow 0$$

By the zig-zag lemma, we get a long exact sequence consisting of the cohomology groups of $Q(\mathcal{F})$, $Q(\mathcal{T})$, and $Q(\mathcal{G})$. By Definition 3.1 and the following remark, this means that we have an exact sequence:

$$0 \longrightarrow \varprojlim \mathcal{F} \longrightarrow \varprojlim \mathcal{T} \longrightarrow \varprojlim \mathcal{G} \longrightarrow \varprojlim^1 \mathcal{F} \longrightarrow \varprojlim^1 \mathcal{T} \longrightarrow \cdots$$

Now, it is clear that that $\varprojlim T$ is just $\mathbb{T}^{\mathbb{N}}$. Second, it is easy to see that $\mathcal{T}$ is flasque, and hence that $\varprojlim^1(\mathcal{T}) = 0$. Hence, we have an exact sequence:

$$\mathbb{T}^{\mathbb{N}} \longrightarrow \varprojlim \mathcal{G} \longrightarrow \varprojlim^1 \mathcal{F} \longrightarrow 0$$

But this precisely means that $\varprojlim^1 \mathcal{F}$ is isomorphic to $\varprojlim \mathcal{G} / \operatorname{im}(\mathbb{T}^{\mathbb{N}})$, as desired. □

For completeness, it is worth showing that the object identified in Proposition 3.3 is the last interesting one in the picture.

Proposition 3.4. *The derived inverse limits* $\varprojlim^n \mathcal{F}$ *vanish for all* $n > 1$.

Proof. We shall use the fact [14, §2] that if J is cofinal in $\mathcal{U}$, then $\varprojlim^n \mathcal{F} = \varprojlim^n_J \mathcal{F}$ for all n. Let $J = \{ X_\xi : \xi < \omega_1 \}$ denote the subset of $\mathcal{U}$ consisting just of the elements of the generating tower. Then it is again clear that $\mathcal{T} \restriction J$ is flasque, but moreover $\mathcal{G} \restriction J$ is flasque as well. Indeed, if $X_\xi \in J$ then any partial thread $(g_{X_\alpha})_{\alpha<\xi}$ can be extended to a thread on all of J using the construction at the limit stage in the proof of Theorem 2.5. Therefore in the exact sequence

$$0 \longrightarrow \varprojlim{}_J \mathcal{F} \longrightarrow \varprojlim{}_J \mathcal{T} \longrightarrow \varprojlim{}_J \mathcal{G} \longrightarrow \varprojlim{}^1_J \mathcal{F} \longrightarrow \varprojlim{}^1_J \mathcal{T} \longrightarrow \cdots$$

we have that two out of every three terms beginning with $\varprojlim^1_J \mathcal{T}$ is equal to 0. It follows that the sequence vanishes at $\varprojlim^1_J \mathcal{T}$, and in particular $\varprojlim^n \mathcal{F} = \varprojlim^n_J \mathcal{F} = 0$ for $n > 1$. □

4. The main result

In this section we show how elements of the group $\varprojlim \mathbf{G}_X$ constructed in Section 2 give rise to automorphisms of the corona algebra of a C*-algebra A. For this, we shall need to impose the following hypothesis on A.

Hypothesis 4.1. We assume A has a sequence of orthogonal projections r_i, for $i \in \mathbb{N}$, such that:

- $r_i A r_j \neq 0$ for all i and j; and

- the sequence of partial sums $p_n = \sum_{i<n} r_i$ form an *approximate unit* for A. That is, for any $a \in A$ we have $p_n a \to a$.

The following is our first generalization of the result of Phillips and Weaver. In the next section, we shall strengthen it by slightly weakening Hypothesis 4.1.

Theorem 4.2. *Suppose that $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ satisfies Hypothesis 2.4 and that A satisfies Hypothesis 4.1. Then there is an embedding of $\varprojlim_{X\in\mathcal{U}} \mathbf{G}_X$ into the automorphism group of $Q(A)$.*

The proof we consist of a series of lemmas. We begin by showing how to stratify the corona $Q(A)$ into a family of systems of operators $C_X(A)$ indexed by elements $X \in \mathcal{P}(\mathbb{N})$. Each system $C_X(A)$ will consist of operators which are very near to being block diagonal, in the sense of the r_i. (Of course, we cannot hope to stratify the corona using multipliers which are actually block diagonal in this sense.)

For $I \subset \mathbb{N}$ let us define

$$p_I = \sum_{i\in I} r_i \ .$$

We then let

$$DD_X(A) = \left\{\, m \in M(A) : p_{I(X,i)}\, m\, p_{I(X,j)} = 0 \text{ whenever } \ |i-j| \geq 2 \,\right\} \ .$$

We shall see during the proof of Lemma 4.3 that every element $m \in DD_X(A)$ can be written as a sum of two multipliers which are block-diagonal in the sense of the r_i.

Now, we let $C_X(A) \subset Q(A)$ be the fragment which comes from the image of $DD_X(A)$ inside the corona:

$$C_X(A) = DD_X(A)/(A \cap DD_X(A)) \ .$$

The following lemma shows that these fragments do indeed stratify $Q(A)$. Special cases of it appeared during the initial segment of the proof of [6, Theorem 3.1] (where it was assumed that A is a UHF algebra) and [10, Lemma 1.2] (where $A = \mathcal{K}$).

Lemma 4.3. *For every $m \in M(A)$ there is a subset $X \subset \mathbb{N}$ such that m is represented in $C_X(A)$. More precisely, every $m \in M(A)$ can be written as $m = d + a$ with $d \in DD_X(A)$ and $a \in A$.*

Proof. Let $n(0) = 0$ and inductively define an increasing sequence $n(j)$ so that the following holds for $j \geq 1$:

(8) $\quad (1 - p_{n(j+1)})\, m\, p_{n(j)}$ and $(1 - p_{n(j+1)})\, m^*\, p_{n(j)}$ have norm $\leq 2^{-j}$.

This is possible since $m\, p_{n(j)} \in A$ and p_n form an approximate unit for A. We claim that if $X = \{\, n(j) : j \in \mathbb{N} \,\}$ then X is as desired.

For this, we let m_e and m_o be defined as follows:

$$m_e = \sum_i p_{I(X,2i)\cup I(X,2i+1)}\, m\, p_{I(X,2i)\cup I(X,2i+1)} \ ,$$

$$m_o = \sum_i p_{I(2i+1)}\, m\, p_{I(2i+2)} + p_{I(2i+2)}\, m\, p_{I(2i+1)} \ .$$

See Figure 1 for a clearer picture of how m_e and m_o are selected.

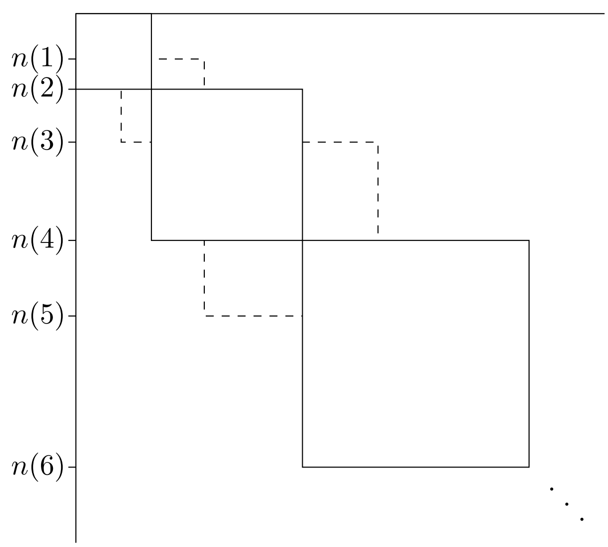

FIGURE 1. The solid square regions represent fragments of m captured in m_e; the dashed rectangular additions represent fragments of m captured in m_o. The details of the construction imply that the uncaptured region is summable in norm and therefore represents an element of A.

Now, $d = m_e + m_o$ makes up a very large portion of m; indeed, a straightforward computation shows that $a = m - m_e - m_o$ satisfies

$$\left\|(1 - p_{n(i)})\, a\right\| \leq 2^{-i+4}$$

for all i and therefore $a \in A$, as required.

Finally, we need only verify that $d = m_e + m_o$ lies in $DD_X(A)$. Indeed, considering each term in the definitions of both m_e and m_o, if $|i - j| \geq 2$ then the term is annihilated either by left-multiplication by $p_{I(X,i)}$ or right-multiplication by $p_{I(X,j)}$. □

We note that if $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is such that the enumerating functions of elements of $\mathcal{U}$ form an eventually dominating family, then in Lemma 4.3 we may choose X to be an element of $\mathcal{U}$.

We now show how to define a continuous homomorphism from $\mathbb{T}^{\mathbb{N}}$ into the group of inner automorphisms of the special fragments $C_X(A)$ of the corona defined above. For this we will need the following key result which connects the construction in Section 2 with our present efforts.

Lemma 4.4. *Suppose that A_0 is a C^*-algebra which contains a sequence of orthogonal projections $q_0, \ldots, q_{n-1}$ satisfying*

- $\sum_{i<n} q_i = 1$, *and*
- $q_i\, A_0\, q_j \neq \{0\}$ *for all $i, j < n$.*

Letting $I = \{\, 0, \ldots, n-1 \,\}$, for each $\alpha \in \mathbb{T}^I$ we let $u_\alpha = \sum_{i<n} \alpha(i) q_i$. Then for all α, β we have

$$\Delta_I(\alpha, \beta) \leq \|\mathrm{Ad}\, u_\alpha - \mathrm{Ad}\, u_\beta\| \leq 2\Delta_I(\alpha, \beta)\ .$$

Proof. Since $\Delta_I(\alpha, \beta) = \Delta_I(\alpha\bar{\beta}, 1)$ and $\|\mathrm{Ad}\, u_\alpha - \mathrm{Ad}\, u_\beta\| = \left\|\mathrm{Ad}(u_\alpha u_\beta^*) - \mathrm{id}\right\|$ we may assume $\beta = 1$ (i.e., the constantly 1 function in $\mathbb{T}^n$). Now, to show that $\|\mathrm{Ad}\, u_\alpha - \mathrm{id}\| \leq 2\Delta_I(\alpha, 1)$, first note that we can multiply α by a fixed constant to assume that $\alpha(0) = 1$. This implies that for all $i < n$, $|\alpha(i) - 1| \leq \Delta(\alpha, 1)$, and so in particular

$$u_\alpha - 1 = \sum_{i<n} (\alpha(i) - 1) q_i$$

has norm $\leq \Delta(\alpha, 1)$. It follows that

$$\begin{aligned} \|u_\alpha\, a\, u_\alpha^* - a\| &\leq \|u_\alpha\, a\, u_\alpha^* - u_\alpha\, a\| + \|u_\alpha\, a - a\| \\ &\leq 2\Delta(\alpha, 1)\, \|a\| \end{aligned}$$

For the inequality $\Delta_I(\alpha, 1) \leq \|\mathrm{Ad}\, u_\alpha - \mathrm{id}\|$, fix i and j in I. For $a \in q_i\, A_0\, q_j$ we have

$$u_\alpha\, a\, u_\alpha^* - a = \alpha(i)\, q_i\, a\, q_j \overline{\alpha(j)} - q_i\, a\, q_j = (\alpha(i)\overline{\alpha(j)} - 1) a$$

and therefore

$$\|\mathrm{Ad}\, u_\alpha - \mathrm{id}\| \geq \max_{i,j \in I} \Delta_{\{\, i,j \,\}}(\alpha, 1) = \Delta_I(\alpha, 1)\ ,$$

as desired. □

Now, for $\alpha \in \mathbb{T}^{\mathbb{N}}$ define a unitary element of $M(A)$ by

$$u_\alpha = \sum_{i \in \mathbb{N}} \alpha(i) r_i\ ,$$

and let $\dot{u}_\alpha$ denote the image of u_α in $Q(A)$.

Lemma 4.5. *Suppose once again that A satisfies Hypothesis 4.1. Given an infinite subset $X \subset \mathbb{N}$, the map $\mathbb{T}^{\mathbb{N}} \to \mathrm{Aut}(C_X(A))$ defined by*

$$\alpha \mapsto \mathrm{Ad}\, \dot{u}_\alpha$$

has kernel precisely equal to $\mathbf{F}_X$.

Proof. Fix $\alpha \in \mathbb{T}^{\mathbb{N}}$. Given a multiplier $m \in DD_{X_e}(A)$, the proof of Lemma 4.3 shows that m can be written as $m_e + m_o$ where

$$m_e \in \sum p_{I(X,2i)\cup I(X,2i+1)}\, A\, p_{I(X,2i)\cup I(X,2i+1)} \ ,$$
$$m_o \in \sum p_{I(X,2i+1)\cup I(X,2i+2)}\, A\, p_{I(X,2i+1)\cup I(X,2i+2)} \ .$$

Dealing first with m_e, it is not hard to compute that

$$\|\dot{u}_\alpha\, \dot{m}_e\, \dot{u}_\alpha^* - \dot{m}_e\| = \limsup_j \left\|(u_\alpha\, m_e\, u_\alpha^* - m_e)p_{I(X,2j)\cup I(X,2j+1)}\right\| \ .$$

Using Lemma 4.4, whenever $\alpha \in \mathbf{F}_X$ we have that the right-hand side is zero. The same argument shows that we have $\|\dot{u}_\alpha\, \dot{m}_o\, \dot{u}_\alpha^* - \dot{m}_o\| = 0$, and it follows that α is in the kernel.

Conversely, if $\alpha \notin \mathbf{F}_X$ then there is $\epsilon > 0$ such that for infinitely many $i \in \mathbb{N}$ we have

$$\Delta_{I(X,i)\cup I(X,i+1)}(\alpha, 1) \geq \epsilon.$$

By Lemma 4.4, for such j we can find an element

$$a_i \in p_{I(X,i)\cup I(X,i+1)}\, A\, p_{I(X,i)\cup I(X,i+1)}$$

such that $\|u_\alpha\, a_i\, u_\alpha^* - a_i\| \geq \epsilon\, \|a_i\|$. We may assume either that all of the i's are even or that all of the i's are odd. Renormalizing so that $\|a_i\| = 1$ for all such i and letting $a = \sum_i a_i$, we have an element of $DD_X(A)$ (in fact a block-diagonal one) witnessing that $\mathrm{Ad}\, \dot{u}_\alpha$ is not the identity map. □

The final component of the argument involves patching together a coherent sequence of partial inner automorphisms to construct an outer automorphism. Suppose that $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is directed under $\supset^*$. For $X \in \mathcal{U}$ let

$$\mathbf{A}_X = \{\Phi \in \mathrm{Aut}(C_X(A)) \mid \Phi \text{ normalizes } C_Y(A) \text{ whenever } X \subset^* Y \in \mathcal{U}\} \ ,$$

and for $X \subset^* Y$ let $\pi_{YX} \colon \mathbf{A}_X \to \mathbf{A}_Y$ denote the map

$$\pi_{YX}(\Phi) = \Phi \restriction C_Y(A) \ .$$

Then $(\mathbf{A}_X)_{X\in\mathcal{U}}$ together with the maps π_Y^X forms an inverse system of groups. The following result shows that, assuming $\mathcal{U}$ is large enough, any thread through $(\mathbf{A}_X)_{X\in\mathcal{U}}$ gives rise to an element of $\mathrm{Aut}(Q(A))$. This is the evolution of [10, Lemma 1.3].

Lemma 4.6. *Suppose that A satisfies Hypothesis 4.1 and $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ satisfies Hypothesis 2.4. Then there is a group homomorphism from $\varprojlim \mathbf{A}_X \to \mathrm{Aut}(Q(A))$ of the form*

$$(\Phi_X)_{X\in\mathcal{U}} \mapsto \Phi$$

which satisfies $\Phi(\dot{m}) = \Phi_X(\dot{m})$ for $\dot{m} \in C_X(A)$.

Proof. Given Φ_X for $X \in \mathcal{U}$ and $b \in M(A)$ we define $\Phi(\dot{m})$ as follows. By Lemma 4.3 and the remark following its proof there exists $X \in \mathcal{U}$ such that $m = d + a$ where $d \in DD_X(A)$ and $a \in A$. Then we simply let:

$$\Phi(\dot{m}) = \Phi_X(\dot{d}) \;.$$

Then since $X \supset^* Y$ implies that $\Phi_Y \restriction \mathbf{A}_X$ agrees with Φ_X, we clearly have that $\Phi(\dot{m})$ does not depend on the choice of X. Moreover, Φ is invertible since its inverse corresponds to the thread $\left(\Phi_X^{-1}\right)_{X\in\mathcal{U}}$. □

Putting together the maps from Lemma 4.5 and Lemma 4.6 we now obtain the desired embedding.

Proof of Theorem 4.2. Let $(\alpha_X)_{X\in\mathcal{U}}$ be given representative of $\varprojlim_{X\in\mathcal{U}} \mathbf{G}_X$ (in other words, the residues $([\alpha_X]_{\mathbf{F}_X}])_{X\in\mathcal{U}}$ form a thread in the inverse system $(\mathbf{G}_X)_{X\in\mathcal{U}}$). Then it is easy to see that $(\operatorname{Ad}\dot{u}_{\alpha_X})_{X\in\mathcal{U}}$ form a thread in $(\mathbf{A}_X)_{X\in\mathcal{U}}$, and the map $\varprojlim_{X\in\mathcal{U}} \mathbf{G}_X \to \varprojlim_{X\in\mathcal{U}} \mathbf{A}_X$ defined by

$$(\alpha_X)_{X\in\mathcal{U}} \mapsto (\operatorname{Ad}\dot{u}_{\alpha_X})_{X\in\mathcal{U}}$$

is well-defined and one-to-one by Lemma 4.5. Hence we may let Φ be the corresponding element of $\operatorname{Aut}(Q(A))$ given by Lemma 4.6. □

5. A stronger result

In this section, we show it is possible to establish the conclusion of Theorem 4.2 using a hypothesis that is slightly weaker than Hypothesis 4.1. This will be done by establishing an analog of each of the lemmas from the previous section.

Hypothesis 5.1. We assume A has a sequence of positive elements r_i, for $i \in \mathbb{N}$, such that:

- for all i, j, k and all $\epsilon > 0$ there exists $a \in A$ such that $\|a\| = 1$ and $\left\| r_i^k \, a \, r_j^k \right\| \geq 1 - \epsilon$.
- the sequence of partial sums $p_n = \sum_{i<n} r_i$ form an increasing approximate unit for A with $p_{n+1}p_n = p_n$ for all n.

Using the property that $p_{n+1}p_n = p_n$, it is easy to see that $r_i r_j = 0$ whenever $|i - j| \geq 2$. Also, if we define

$$p_I = \sum_{i\in I} r_i$$

for I an interval of $\mathbb{N}$, then we similarly have (the definition of $I(X, i)$ is given before Definition 2.2) $p_{I(X,i)}p_{I(X,j)} = 0$ whenever $|i - j| \geq 2$.

Theorem 5.2. *Suppose that $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ satisfies Hypothesis 2.4 and that A satisfies Hypothesis 5.1. Then there is an embedding of $\varprojlim_{X\in\mathcal{U}} \mathbf{G}_X$ into the automorphism group of $Q(A)$.*

In the proof of Theorem 5.2, we will not use the first condition of Hypothesis 5.1 directly, but rather the following consequence of it.

Lemma 5.3. *Suppose that the sequence r_i satisfies the first condition in Hypothesis 5.1. Then for every i, j and $\epsilon > 0$ there exists $a \in A$ such that $\|a\| = 1$, $\|r_i\, a\, r_j\| \geq 1 - \epsilon$, and $\|r_i\, a\, r_j - a\| < \epsilon$.*

Proof. By the continuous functional calculus, for every contraction r we have $r^{k+1} - r^k \to 0$. It follows that for every $\delta > 0$ there exists k large enough so that $\left\| r_i^{k+1} - r_i^k \right\| \leq \delta$ and $\left\| r_j^{k+1} - r_j^k \right\| \leq \delta$. Now, choose a_0 such that $\|a_0\| = 1$ and $\left\| r_i^{k+1}\, a_0\, r_j^{k+1} \right\| \geq 1 - \delta$. Then it is easy to see that the element $a = r_i^k\, a_0\, r_j^k$ satisfies $\|r_i\, a\, r_j - a\| \leq 2\delta$ and $\|r_i\, a\, r_j\| \geq 1 - \delta$. Since the r_i are norm-decreasing, we of course have $\|a\| \geq 1 - \delta$ as well. It follows that we can renormalize a and choose δ small enough to obtain the desired inequality. □

We begin the proof of Theorem 5.2 by again defining the system of multipliers

$$DD_X(A) = \left\{ m \in M(A) \mid p_{I(X,i)}\, m\, p_{I(X,j)} = 0 \text{ whenever } |i - j| \geq 2 \right\} ,$$

and let $C_X(A)$ denote the image of $DD_X(A)$ in $Q(A)$. The next result again shows that the $C_X(A)$ stratify all of $Q(A)$. The proof is formally identical to that of Lemma 4.3.

Lemma 5.4. *For all $m \in M(A)$ there exists a subset $X \subset \mathbb{N}$ such that $m \in DD_X(A) + A$.*

The next result gives the slight strengthening of Lemma 4.4 necessary for our situation. Note that the difficulty stems from the fact that this time, u_α is not necessarily a unitary element of A_0.

Lemma 5.5. *Suppose that A_0 is a C^*-algebra containing a sequence of positive elements $q_0, \ldots, q_{n-1}$ such that*

- *for all i, j, k and all $\epsilon > 0$ there exists $a_0 \in A_0$ such that $\|a_0\| = 1$ and $\left\| r_i^k\, a_0\, r_j^k \right\| \geq 1 - \epsilon$, and*
- $\sum_{i<n} q_i = 1$.

Letting $I = \{ 0, \ldots, n-1 \}$, for each $\alpha \in \mathbb{T}^I$ we set $u_\alpha = \sum_{i \in I} \alpha(i) q_i$. Then we have

$$\Delta_I(\alpha, 1) \leq \|\mathrm{Ad}\, u_\alpha - \mathrm{id}\| \leq 2\Delta_I(\alpha, 1) .$$

Proof. The proof used in Lemma 4.4 again shows that $\|\mathrm{Ad}\, u_\alpha - \mathrm{id}\| \leq 2\Delta_I(\alpha, 1)$.

For the inequality $\Delta_I(\alpha,1) \leq \|\operatorname{Ad} u_\alpha - \mathrm{id}\|$, we first fix i_0, j_0 and for any $a \in A_0$ we write:

$$\begin{aligned}
u_\alpha \, a \, u_\alpha^* - a &= \sum \alpha(i)\overline{\alpha(j)} q_i \, a \, q_j - \sum q_i q_j \\
&= \sum \Big(\alpha(i)\overline{\alpha(j)} - 1\Big) q_i \, a \, q_j \\
&= \Big(\alpha(i_0)\overline{\alpha(j_0)} - 1\Big) q_{i_0} \, a \, q_{j_0} + \sum_{\substack{i \neq i_0 \\ j \neq j_0}} \Big(\alpha(i)\overline{\alpha(j)} - 1\Big) q_i \, a \, q_j \; . \qquad (9)
\end{aligned}$$

Given ϵ, we apply Lemma 5.3 to choose a such that $\|a\| = 1$, $\|q_{i_0} \, a \, q_{j_0} - a\| < \epsilon$ and $\|q_{i_0} \, a \, q_{j_0}\| \geq 1 - \epsilon$. Note that the latter inequality implies that the whole right-hand term of Equation (9) is very small. Indeed, it follows that $\|q_i \, a \, q_j\| < \epsilon$ whenever $i \neq i_0, j \neq j_0$, and hence that this last term is bounded by $n\epsilon$.

These computations imply that the expression in Equation (9) can be made arbitrarily close to $(\alpha(i_0)\overline{\alpha(j_0)} - 1)a$, and it follows that $\|\operatorname{Ad} u_\alpha - \mathrm{id}\| \geq \Big|\alpha(i_0)\overline{\alpha(j_0)} - 1\Big|$. Since this is true for all $i_0, j_0 \in I$, we can conclude that $\|\operatorname{Ad} u_\alpha - \mathrm{id}\| \geq \Delta_I(\alpha, 1)$, as desired. $\square$

Now for each $\alpha \in \mathbb{T}^{\mathbb{N}}$, we again define the corresponding elements u_α of $M(A)$ by

$$u_\alpha = \sum_{i \in \mathbb{N}} \alpha(i) r_i \; .$$

These need not be unitaries, but the following result shows that for plenty of α, the image $\dot{u}_\alpha$ in $Q(A)$ will in fact be unitary.

Lemma 5.6. *If $\alpha \in \mathbb{T}^{\mathbb{N}}$ satisfies $\alpha(i+1) - \alpha(i) \to 0$ then $\dot{u}_\alpha$ is a unitary in $Q(A)$.*

Proof. Recall that $r_i r_j = 0$ for $|i - j| \geq 2$. Hence we have:

$$\begin{aligned}
u_\alpha u_\alpha^* - 1 &= \sum \alpha(i) r_i \sum \overline{\alpha(i)} r_i - \sum r_i \sum r_i \\
&= \sum \Big[2\Re(\alpha(i)\overline{\alpha(i+1)}) r_i r_{i+1} + r_i^2\Big] - \sum \big[2 r_i r_{i+1} + r_i^2\big] \\
&= \sum 2 \Big[\Re(\alpha(i)\overline{\alpha(i+1)}) - 1\Big] r_i r_{i+1} \; .
\end{aligned}$$

Since every partial sum lies in A, it is enough to show that the tails of this last series converge to zero in norm. By our hypothesis, given ϵ, we can find N such that $i > N$ implies

$$2 \Big|\Re(\alpha(i)\overline{\alpha(i+1)}) - 1\Big| < \epsilon \; .$$

Since the r_i commute, we can regard them as complex-valued functions on the Gelfand space X of $C^*(\{ r_i \})$. Since $r_j r_k = 0$ whenever $|j - k| \geq 2$, each $x \in X$ can only lie in the support of at most three of the terms $r_i r_{i+1}$. Hence we can bound the tail

$$\left\| \sum_{n>N} 2 \left[\Re(\alpha(i)\overline{\alpha(i+1)}) - 1 \right] r_i r_{i+1} \right\| < 3\epsilon \,,$$

as desired. □

Thus, if we let $Z \subset \mathbb{T}^{\mathbb{N}}$ be the set of α such that $\alpha(i+1) - \alpha(i) \to 0$, we have that for all $\alpha \in Z$ the conjugation $\operatorname{Ad} \dot{u}_\alpha$ defines an element of $\operatorname{Aut}(Q(A))$. In order to proceed, we must argue that Z is large enough that there are still $2^{\aleph_1}$ many elements of $\varprojlim \mathbf{G}_X$ consisting just of elements of Z.

Lemma 5.7. *Under the hypotheses of Theorem 2.5, there are more than continuum many threads through $\varprojlim \mathbf{G}_X$ of the form $(\alpha_X)_{X \in \mathcal{U}}$ where $\alpha_X \in Z$.*

Proof. We must simply inspect the construction given in the proof of Theorem 2.5, and check that it can be carried out with the additional condition:

- $\alpha_s(i) - \alpha_s(i+1) \to 0$

For the successor step, this is essentially immediate. Indeed, using the notation of Theorem 2.5, recall that the witness $\alpha \in \mathbf{F}_\xi \smallsetminus \mathbf{F}_{\xi+1}$ which we produced satisfies $|\alpha(n) - \alpha(n+1)| \leq \pi/m$ for $n \in I(X_{\xi+1}, n(m))$ and $|\alpha(n) - \alpha(n+1)| = 0$ elsewhere.

For the inductive step, recall that given α_n we constructed a set X_∞, an element α_s, and constants γ_n such that $\alpha_s(i) = \gamma_n \alpha_n(i)$ for all $i \in I(X_\infty, n)$. Assuming additionally that the α_n satisfy $\alpha_n(i) - \alpha_n(i+1) \to 0$, we can achieve the same for α_s by simply thinning out X_∞ in advance so that $|\alpha_n(i) - \alpha_n(i+1)| < 1/n$ for all $i \in I(X_\infty, n)$. □

Finally, it is easy to see that the proofs of Lemma 4.5, Lemma 4.6, and the conclusion of the proof of Theorem 4.2 together yield an injection from the set of threads through $\varprojlim \mathbf{G}_X$ consisting of elements of Z into $\operatorname{Aut} Q(A)$. This concludes the proof of Theorem 5.2.

6. Algebras satisfying our hypotheses

In this section, we give a series of conditions on a C*-algebra A which are sufficient to guarantee that A satisfies either Hypothesis 4.1 or Hypothesis 5.1. In particular, we complete the proof of the main theorem (Theorem 1.4) by showing that each of its hypotheses (1)–(4) is sufficient as well. We also prove that there are at most $2^{\aleph_0}$ trivial automorphisms of the corona of any separable C*-algebra.

In this section, we will always assume that A is σ-unital.

Proposition 6.1. *If A has a σ-unital, non-unital quotient with a faithful irreducible representation, then A satisfies Hypothesis 5.1.*

By Theorems 2.5 and 5.2, this completes the proof of the main theorem 1.4 in the case that condition (4) holds. And clearly, condition (1) is a special case of condition (4).

Proof of Proposotion 6.1. Let $\pi\colon A \to \mathcal{B}(H)$ be an irreducible representation such that $\pi[A]$ is non-unital. Let r_j, for $j \in \mathbb{N}$, be contractions of norm 1 such that $p_n = \sum_{j<n} r_j$ for $n \in \mathbb{N}$ form an approximate unit for A.

To verify Hypothesis 5.1, fix $\epsilon > 0$, $i < j$ and k, and choose δ small enough that $(1-\delta)^2/(1+\delta) > 1-\epsilon$. Fix unit vectors ξ_i and ξ_j in H such that $\|\xi_i - \pi(r_i^k)\xi_i\| < \delta$ and $\|\xi_j - \pi(r_j^k)\xi_j\| < \delta$. By Kadison's Transitivity Theorem (see for instance [2, II.6.1.12]) we can find $a \in A$ of norm $\leq 1+\delta$ such that $\pi(ar_i^k)\xi_i = (1-\delta)\xi_j$. Then

$$\left\| r_j^k\, a\, r_i^k \right\| \geq \left\| \pi(r_j^k\, a\, r_i^k)\xi_i \right\| \geq (1-\delta)^2 > 1-\epsilon$$

and it follows that $\frac{1}{1+\delta}a$ is as required. □

Proposition 6.2. *If A is simple and has an approximate unit consisting of projections, then A satisfies Hypothesis 4.1.*

Proof. Since A is simple it has a faithful irreducible representation. Let r_i, for $i \in \mathbb{N}$, be projections such that their partial sums form an approximate unit for A. Since $r_i^k = r_i$ for all i and all $k \geq 1$, applying the proof of Proposition 6.1 to projections r_i, for $i \in \mathbb{N}$, we verify that they satisfy Hypothesis 4.1. □

Proposition 6.3. *If A satisfies Hypothesis 5.1 and B is a σ-unital C*-algebra then $A \otimes B$ satisfies Hypothesis 5.1 for any product norm on $A \otimes B$.*

This completes the proof of the main theorem 1.4 in the case that condition (3) holds. And again, since Proposition 6.1 implies that $\mathcal{K}$ satisfies Hypothesis 5.1, condition (2) is a special case of condition (3).

Proof of Proposition 6.3. Let $r_i \in A$ witness that A satisfies Hypothesis 5.1 and let $p_n = \sum_{i<n} r_i$. Let q_n be an increasing approximate unit for B. We claim that

$$s_i = p_{i+1} \otimes q_{i+1} \; - \; p_i \otimes q_i$$

witness that $A \otimes B$ satisfies Hypothesis 5.1. Indeed, fix $i < j$, k and $\epsilon > 0$. Then there is $a \in A$ such that $\|a\| = 1$ and $\left\| r_i^k\, a\, r_j^k \right\| \geq 1-\epsilon$. We will show that $a \otimes q_j$ satisfies $\left\| s_i^k (a \otimes q_j)\, s_j^k \right\| \geq 1$.

To see this, fix a pure state ϕ of B such that $|\phi(q_i)| = 1$. Since $q_i \leq q_j$, this implies $|\phi(q_j)| = 1$. Then $\mathrm{id} \otimes \phi$ is a completely positive, contraction mapping from $A \otimes B$ into A. A straightforward computation shows that

$$(\mathrm{id} \otimes \phi)(s_i^k (a \otimes q_j) s_j^k) = (p_{i+1} - p_i)^k a (p_{j+1} - p_j)^k$$

and since $\mathrm{id} \otimes \phi$ is a contraction we conclude that $\|s_i^k (a \otimes q_j) s_j^k\| \geq 1-\epsilon$. □

We close this section with the following additional special case. Here, an element h of a C*-algebra A is *strictly positive* if $ah \neq 0$ for all nonzero $a \in A$.

Proposition 6.4. *Assume A has a subalgebra B that satisfies Hypothesis 5.1 and that B contains an element which is strictly positive in A. Then A satisfies Hypothesis 5.1.*

Proof. Let p_i, for $i \in \mathbb{N}$, be an approximate unit for B such that $r_i = p_i - p_{i-1}$ (with $p_{-1} = 0$) witness that B satisfies Hypothesis 5.1. Since B contains an element which is strictly positive in A, p_i is an approximate unit for A. Also, r_i for $i \in \mathbb{N}$ clearly witness that A satisfies Hypothesis 5.1. □

7. Trivial automorphisms

In this section, we discuss the claim in the introduction that Definition 1.1 is the most comprehensive definition of a trivial automorphism of a corona of a separable non-unital C*-algebra. Several arguments in this section require some standard results from descriptive set theory (see e.g., [16]).

Recall that if Φ is an automorphism of $Q(A)$ we write

$$\Gamma_\Phi = \{(a, b) : \Phi(a/A) = b/A\} ,$$

and that Φ is said to be trivial if and only if Γ_Φ is Borel.

Lemma 7.1. *Assume A is a separable, non-unital, C*-algebra. Then the trivial automorphisms of $Q(A)$ form a group.*

Proof. It is clear that Φ is trivial if and only if Φ^{-1} is trivial, so we only need to check that the composition of two trivial automorphisms Φ and Ψ is trivial. So suppose that Γ_Φ and Γ_Ψ are Borel; we need to check that $\Gamma = \Gamma_{\Phi\circ\Psi}$ is Borel. We shall use the fact that a subset of a Polish space is Borel if and only if it has a $\mathbf{\Pi}^1_1$ definition and a $\mathbf{\Sigma}^1_1$ definition ([16, Theorem 14.11]). Clearly, a Σ^1_1 definition is given by

$$\Gamma = \{(a, c) : (\exists b \in M(A))\ (a, b) \in \Gamma_\Psi \text{ and } (b, c) \in \Gamma_\Phi\} .$$

Furthermore, since $\Phi \circ \Psi$ is an automorphism of $Q(A)$ we can write

$$\Gamma = \{(a, c) : (\forall b \in M(A))\ (a, b) \notin \Gamma_\Psi \text{ or } (b, c) \in \Gamma_\Phi\}$$

which gives a Π^1_1 definition. □

It is easy to see that every inner automorphism Φ of $Q(A)$ is trivial. Indeed, if $\pi\colon M(A) \to Q(A)$ denotes the quotient map, let $v \in M(A)$ be such that Φ is conjugation by $\pi(v)$. Then $\Gamma_\Phi = \{(a, b) : b - vav^* \in A\}$ is Borel. We now show that in the case of the Calkin algebra, an automorphism is inner if and only if it is trivial (cf. [10, Theorem 2.6]).

Lemma 7.2. *An automorphism of the Calkin algebra is trivial if and only if it is inner.*

Proof. We need only to prove the direct implication. Consider $\mathcal{B}(H)^2$ with respect to the product of strict topology. Let Φ be an automorphism of the Calkin algebra such that Γ_Φ is Borel. A straightforward computation shows that for a Borel subset Γ of $\mathcal{B}(H)^2$ the complexity of the assertion that $\Gamma = \Gamma_\Phi$ for some automorphism Φ of the Calkin algebra is at most Π^1_2, with a code for Γ as a parameter. Hence by Shoenfield's absoluteness theorem ([15, Theorem 13.15]), in every forcing extension one can use the Borel code for Γ_Φ to define an automorphism of $Q(H)$. This automorphism is an extension of Φ and we denote it by $\tilde{\Phi}$. The assertion that $\tilde{\Phi}$ is inner is Σ^1_2 with a code for Γ as a parameter, and again by Shoenfield's absoluteness theorem Φ is inner if and only if $\tilde{\Phi}$ is inner in all forcing extensions. By [10, Theorem 1] there exists a forcing extension in which all automorphisms of the Calkin algebra are inner. Therefore Φ is inner. □

The situation with coronas of abelian C*-algebras is similar. In [7, §4] the second author considered trivial homeomorphisms of Čech–Stone remainders of locally compact Polish spaces. Such $F\colon \beta X \setminus X \to \beta X \setminus X$ is *trivial* if there are compact subsets K and L of X and a homeomorphism $f\colon X \setminus K \to X \setminus L$ such that the continuous extension of f to βX agrees with F on $\beta X \setminus X$. In [7, §4] it was proved that the assertion "all homeomorphisms of $\beta X \setminus X$ are trivial" is relatively consistent with ZFC for all countable locally compact spaces X. Forcing axioms conjecturally imply all homeomorphisms of Čech–Stone remainders of locally compact Polish spaces are trivial ([7, §4]). The absoluteness argument of Lemma 7.2 shows that for such X and $A = C_0(X)$ an automorphism of $Q(A)$ is trivial if and only if the corresponding homeomorphism of $\beta X \setminus X$ is trivial.

It would be desirable to define trivial automorphisms of $Q(A)$ as those that have a representation with certain algebraic properties. However, even in the case of inner automorphisms of the Calkin algebra one cannot expect to have a representation that is an automorphism of $M(A)$, or even a *-homomorphism of $M(A)$ into itself. This situation is analogous to the problem of whether "topologically trivial" automorphisms of quotient Boolean algebras $\mathcal{P}(\mathbb{N})/I$ are necessarily "algebraically trivial" (see [9]).

Unlike [9] or [7] where the main theme was the existence of isomorphisms between quotient structures, we have considered only automorphisms of corona algebras. One reason for this is that we are unable to answer the following question.

Question 7.3. Are there separable non-unital C*-algebras A and B whose coronas are isomorphic, but there is no trivial isomorphism between them?

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