FUNCTIONS ESSENTIALLY DEPENDING ON AT MOST ONE VARIABLE

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In this note we analyze the question: When does a function $f\colon \omega^d \to \omega$ essentially depend on at most one coordinate? (Here $\omega = \mathbb{N} \cup \{0\}$.) For example, the function

$$f(m,n) = \begin{cases} m, & \text{if } m \text{ is even,} \\ n, & \text{if } m \text{ is odd,} \end{cases}$$

depends on both variables. However, we can cover its domain by two rectangles, $2\omega \times \omega$ and $(2\omega+1) \times \omega$, such that f depends on at most one variable on each one of them.

Definition 1. A function $f\colon X^d \to X$ is *elementary* if its domain can be covered by finitely many rectangles such that f depends on at most one coordinate on each one of them. (By *rectangle* we mean a set of the form $A_0 \times A_1 \times \cdots \times A_{d-1}$.)

A function from a subset of X^d is elementary if it can be extended to an elementary function.

Define a map $\bar{g}$ from a subset of ω^2 into $\{0,1\}$ by

$$\bar{g}(m,n) = \begin{cases} 1, & \text{if } m < n, \\ 0, & \text{if } m = n, \\ \text{undefined}, & \text{if } m > n. \end{cases}$$

It is not difficult to prove that $\bar{g}$ is not elementary (see Lemma 4). In our main result, Theorem 3, we shall prove that $\bar{g}$ is the canonical non-elementary function. To make this statement more precise, we need a bit of nonstandard notation. If $I \cap J = \emptyset$, then for $\vec{x} \in X^I$ and $\vec{y} \in X^J$ we define $\vec{x}\hat{\ }\vec{y} \in X^{I \cup J}$ by

$$(\vec{x}\hat{\ }\vec{y})(\xi) = \begin{cases} \vec{x}(\xi), & \text{if } \xi \in I, \\ \vec{y}(\xi), & \text{if } \xi \in J. \end{cases}$$

Namely, in our notation $\vec{x}\hat{\ }\vec{y}$ is formed by concatenating $\vec{x}$ and $\vec{y}$, but $\vec{x}$ does not necessarily come before $\vec{y}$; the order depends on where the elements of $\vec{x}$ and $\vec{y}$ come from.

In the above situation we also define $f^{\vec{x}}\colon X^J \to X$ by

$$f^{\vec{x}}(\vec{y}) = f(\vec{x}\hat{\ }\vec{y}).$$

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Definition 2. Assume $f\colon X^d \to X$ and g is a map from a subset of Y^k into Y. We say that g is *reducible* to f if there is a disjoint partition

$$d = s_0 \cup s_1 \cup \cdots \cup s_{k-1}$$

of d into nonempty sets and maps $p_i\colon Y \to X^{s_i}$ for $i < k$ such that the map $p\colon Y^k \to X^d$ defined by

$$p(y_0, \dots, y_{k-1}) = p_0(y_0)^\frown p_1(y_1)^\frown \dots {}^\frown p_{k-1}(y_{k-1})$$

satisfies

$$g(\vec{y}) \neq g(\vec{z}) \qquad \text{implies} \qquad f(p(\vec{y})) \neq f(p(\vec{z})),$$

whenever $\vec{y}$ and $\vec{z}$ are in the domain of g.

We shall prove (Lemma 5) that if g is reducible to f and g is not elementary, then f is not elementary either. A question of dependence of functions on their variables has been previously studied from a different angle; see e.g., [1] and the references thereof. We can now state our main result (X stands for an arbitrary set).

Theorem 3. *Assume $f\colon X^d \to X$. Then exactly one of the following holds:*

(A) *f is elementary.*
(B) *The map $\bar{g}$ is reducible to f.*

The rest of this note is devoted to the proof of this result. Its application to confirm a conjecture of van Douwen is given in [2]. The special case $d = 2$ of Theorem 3 appears in [3, Theorem 4.2.1].

Lemma 4. *The function $\bar{g}$ is not elementary.*

Proof. First note that $\bar{g}$ depends on more than one coordinate on any square $A \times A$ whose side has at least one element. This is because if $m < n$ then $\bar{g}(m,m) \neq \bar{g}(m,n)$ and $\bar{g}(m,n) \neq \bar{g}(n,n)$, therefore $\bar{g}$ depends on both coordinates on the rectangle $\{m,n\} \times \{m,n\}$.

If ω^2 is covered by finitely many rectangles, then one of them contains $\langle i,i\rangle$ and $\langle j,j\rangle$ for some $i < j$, and therefore contains the square $\{i,j\} \times \{i,j\}$. Hence $\bar{g}$ depends on more than one coordinate on this rectangle by the above. $\square$

Let $\bar{g}_n = \bar{g} \restriction \{1, \dots, n\}^2$, for a natural number n. Fred Galvin points out a curious consequence of Theorem 3, that $\bar{g}$ is reducible to f if and only if $\bar{g}_n$ is reducible to f for every n. Indeed, the direct implication is trivial while if $\bar{g}_n$ is reducible to f then X^d cannot be covered by $n-1$ many rectangles such that f depends on at most one variable on each one of them.

Lemma 5. *If g is reducible to f and f is elementary, then g is elementary as well.*

Proof. Assume g is a partial map from Y^k into Y which is reducible to an elementary map $f\colon X^d \to X$. Let $d = s_0 \cup \cdots \cup s_{k-1}$ and $p_i\colon Y \to X^{s_i}$ $(i < k)$ be as in Definition 2. Assume R is a rectangle on X^d such that f depends on at most one coordinate on R. Then (with a mild abuse of notation) we have $R = A_0 \times A_1 \times \cdots \times A_{k-1}$ for some $A_i \subseteq X^{s_i}$ $(i < k)$. Let B_i be the p_i-preimage of A_i. If s_i is such that the index of the only coordinate on which f depends is in s_i, then the restriction of g to $B_0 \times \cdots \times B_{k-1}$ depends only on i-th coordinate.

If X^d is covered by finitely many rectangles R_j $(j < m)$, then the rectangles constructed from R_j as above cover Y^k, and this completes the proof. $\square$

We shall prove the following "local" version of Theorem 3.

Theorem 6. *Assume $f\colon X^d \to X$ and that $\mathcal{U}_0, \dots, \mathcal{U}_{d-1}$ are ultrafilters on X. Then at least one of the following holds:*

(C) *There are sets $A_0 \in \mathcal{U}_0, \dots\ A_{d-1} \in \mathcal{U}_{d-1}$ such that the restriction of f to $\prod_{i=0}^{d-1} A_i$ depends on at most one coordinate.*
(D) *There is a disjoint partition $d = s\dot{\cup}t$ and tuples $\vec{x}_i \in X^s$, $\vec{y}_i \in X^t$ $(i \in \omega)$ such that for all i and all $j < k$ we have $f(\vec{x}_i{}^\frown\vec{y}_i) \neq f(\vec{x}_j{}^\frown\vec{y}_k)$.*

Before proving Theorem 6, let us show how it implies Theorem 3.

Proof of Theorem 3. By Lemma 4 and Lemma 5, (A) and (B) exclude each other.
Fix $f\colon X^d \to X$. Assume (D), and let $p_0\colon \omega \to X^s$ and $p_1\colon \omega \to X^t$ be

$$p_0(i) = \vec{x}_i, \qquad p_1(j) = \vec{y}_j.$$

These maps verify that $\bar{g}$ is reducible to f, and (B) follows.

We can therefore assume that (C) of Theorem 6 holds for every d-tuple of ultrafilters on X. Consider βX, the Čech–Stone compactification of the space (X is taken with the discrete topology). Recall that βX is the space whose points are the ultrafilters on X and the basic open sets are

$$V_A = \{\mathcal{U} \in \beta X : A \in \mathcal{U}\}$$

for $A \subseteq X$. Then (C) of Theorem 6 associates to every point $\langle \mathcal{U}_0, \dots, \mathcal{U}_{d-1}\rangle$ of the power $(\beta X)^d$ its open neighborhood, $V_{A_0} \times \dots \times V_{A_{d-1}}$. By the compactness, $(\beta X)^d$ can be covered by finitely many of these open sets. These sets give a covering of X^d by the rectangles such that f depends on at most one coordinate on each one of them, and (A) follows. □

Proof of Theorem 6. Let us first observe that we can assume that all the ultrafilters $\mathcal{U}_0, \dots, \mathcal{U}_{d-1}$ are nonprincipal. Namely, if say $\mathcal{U}_0$ is a principal ultrafilter generated by $\{\bar{x}_0\}$, then it suffices to prove the statement of the theorem for the function of $d-1$ many variables $g(\vec{x}) = f(\bar{x}_0{}^\frown\vec{x})$ and the ultrafilters $\mathcal{U}_1, \dots, \mathcal{U}_{d-1}$. In this manner we can eliminate all the principal ultrafilters from the d-tuple $\mathcal{U}_0, \dots, \mathcal{U}_{d-1}$.

Lemma 7. *If $d = s \cup t$ is a disjoint decomposition and $\vec{x}_i \in X^s$, $\vec{y}_i \in X^t$ are such that*

(1) *$f(\vec{x}_i{}^\frown\vec{y}_i) \neq f(\vec{x}_i{}^\frown\vec{y}_j)$ for all $i < j$ and*
(2) *$f(\vec{x}_i{}^\frown\vec{y}_j) = f(\vec{x}_i{}^\frown\vec{y}_k)$ for all $i < j < k$,*

then the alternative (D) of Theorem 6 holds.

Proof. Let $g\colon \omega^2 \to X$ be defined by $g(i,j) = f(\vec{x}_i{}^\frown\vec{y}_j)$. It clearly suffices to find an infinite $A \subseteq \omega$ such that $g(i,j) \neq g(k,k)$ for all $i < j$ and k in A. Define $x(i) = g(i,i)$ and $y(i) = g(i,i+1)$. Then $x(i) \neq y(i)$ for all i and $g(i,j) = y(i)$ for all $i < j$. If either x or y is constant on an infinite set, we have what we want. Otherwise, recursively find an infinite sequence $i_0 < i_1 < i_2 < \dots$ such that $x(i_n) \neq y(i_m)$ for all $n \neq m$ as follows. If i_j $(j \leq n)$ are chosen, there are at most finitely many j such that $x(j)$ or $y(j)$ is in $\{x(i_k), y(i_k) : 0 \leq k \leq n\}$, thus we can pick i_{n+1} as required. □

If $\mathcal{U}$ is an ultrafilter on a set X and h is a map whose domain includes X, then we say that $\lim_{x\to\mathcal{U}} h(x)$ *exists* if there is $A \in \mathcal{U}$ such that the restriction of h to A is constant. The value of the limit is the constant value of h on A.

Lemma 8. *Assume that (D) of Theorem 6 does not hold. Then for every $j < d$ there is $A_j \in \mathcal{U}_j$ such that for every $\vec{x} \in X^{d\setminus\{j\}}$ the following two conditions are equivalent:*

(1) *$f^{\vec{x}} \restriction A_j$ is constant,*
(2) *$\lim_{y\to\mathcal{U}_j} f^{\vec{x}}(y)$ exists.*

Proof. Fix $j < d$. Note that: (i) for a given $\vec{x}$ the truth of (2) does not depend on the choice of A_j, (ii) for given $\vec{x}$ and A_j, (1) implies (2), and (iii) for every fixed $\vec{x}$ such that (2) holds one may find A_j such that (1) holds as well. Hence the conclusion of Lemma 8 is saying that A_j can be chosen so that for all $\vec{x}$ the clause (1) holds whenever (2) holds. We will assume that this fails and prove (D) of Theorem 6. Recursively find $\vec{x}_i \in X^{d\setminus\{j\}}$, $B_i \in \mathcal{U}_j$ and y_i $(i \in \omega)$ such that

(3) $B_1 \supset B_2 \supset B_3 \supset \dots$ are in $\mathcal{U}_j$,
(4) $\lim_{y\to\mathcal{U}_j} f^{\vec{x}_i}(y)$ exists, moreover $f^{\vec{x}_i} \restriction B_{i+1}$ is constant,
(5) $y_i \in B_i$ and $f^{\vec{x}_i}(y_i) \neq \lim_{y\to\mathcal{U}_j} f^{\vec{x}_i}$.

Let us describe the recursive construction. Assume that $\vec{x}_i, y_i, B_i$ $(i \leq n)$ are chosen to satisfy the above requirements. By our assumptions, (1) and (2) are not equivalent for the set $A_j = B_n$, and therefore there is $\vec{x}_{n+1}$ such that $\lim_{y\to\mathcal{U}_j} f^{\vec{x}_{n+1}}(y)$ exists yet $f^{\vec{x}_{n+1}} \restriction B_{n+1}$ is not constant. Let $y_{n+1} \in B_n$ be such that $f^{\vec{x}_{n+1}}(y_{n+1}) \neq \lim_{y\to\mathcal{U}_j} f^{\vec{x}_{n+1}}(y)$. Finally, let $B_{n+1} \in \mathcal{U}_j$ be such that $f^{\vec{x}_{n+1}} \restriction B_{n+1}$ is constant (and therefore equal to $\lim_{y\to\mathcal{U}_j} f^{\vec{x}_{n+1}}(y)$) and $B_{n+1} \subset B_n$. Then clearly the conditions are satisfied, and this describes the construction.

Note that for all $i < l < k$ we have $f^{\vec{x}_i}(y_l) = f^{\vec{x}_i}(y_k) = \lim_{y\to\mathcal{U}_j} f^{\vec{x}_i}(y)$. Therefore the conditions of Lemma 7 are satisfied, and (D) of Theorem 6 holds. □

We shall now prove Theorem 6 by induction on d, starting from the first nontrivial case, when $d = 2$. We assume that (D) does not hold.

Case 1. If for some $A_0 \in \mathcal{U}_0$ and all $x_0 \in A_0$ the limit $\lim_{x_1\to\mathcal{U}_1} f(x_0, x_1)$ exists, then let $A_1 \in \mathcal{U}_1$ be a set as guaranteed by Lemma 8, so that for every $x_0 \in A_0$ function $f^{x_0} \restriction A_1$ is constant. Then $f \restriction A_0 \times A_1$ depends only on x_0, and (C) is satisfied.

Case 2. If for some $A_1 \in \mathcal{U}_1$ and all $x_1 \in A_1$ the limit $\lim_{x_0\to\mathcal{U}_0} f(x_0, x_1)$ exists, and $A_0 \in \mathcal{U}_0$ is as guaranteed by Lemma 8, then $f \restriction A_0 \times A_1$ depends only on x_1.

Case 3. For every $A_0 \in \mathcal{U}_0$ there is $x_0 \in A_0$ such that the limit $\lim_{x_1\to\mathcal{U}_1} f(x_0, x_1)$ does not exist and for every $A_1 \in \mathcal{U}_1$ there is $x_1 \in A_1$ such that the limit $\lim_{x_0\to\mathcal{U}_0} f(x_0, x_1)$ does not exist. Let

$$B = \Big\{x \in X : \lim_{y\to\mathcal{U}_1} f(x, y) \text{ does not exist}\Big\},$$

$$C = \Big\{y \in X : \lim_{x\to\mathcal{U}_0} f(x, y) \text{ does not exist}\Big\}.$$

Since both $\mathcal{U}_0$ and $\mathcal{U}_1$ are ultrafilters, we have $B \in \mathcal{U}_0$ and $C \in \mathcal{U}_1$. Now we construct $x_i \in B$ and $y_i \in C$ for $i \in \omega$ such that for all $i < j$ we have

(6) $f(x_i, y_j) \notin \{f(x_k, y_l) : \max(k, l) < \max(i, j)\}$, and
(7) $f(x_i, y_i) \notin \{f(x_j, y_k) : j < i, k \leq i\}$.

Let us describe the recursive construction. If x_i, y_i $(i \leq n)$ have been defined, choose $y_{n+1} \in C$ satisfying (6) for all $i < j = n + 1$ (note that the set of such y is nonempty, as it belongs to $\mathcal{U}_1$). Then choose $x_{n+1} \in B$ satisfying (6) for all

$j < i = n + 1$ and (7) for $i = n + 1$ (the set of such x is nonempty, as it belongs to $\mathcal{U}_0$). The sequences x_i, y_i ($i \in \omega$) chosen in this way satisfies the requirements, as well as (D) of Theorem 6.

This completes the proof of the case $d = 2$.

Let us now assume that the statement is true for $d - 1 \geq 2$ and prove it for d. Fix $f\colon X^d \to X$, and for every $j < d$ let A_j be a set as guaranteed by Lemma 8.

Fix $\bar{n} < d$. By using the inductive assumption, for $x \in X$ we can find $A^x_k \in \mathcal{U}_k$ ($k < d$, $k \neq \bar{n}$) such that if we let $A^x_{\bar{n}} = \{x\}$ then the map

$$h^x = f \restriction \prod_{k=0}^{d-1} A^x_k$$

depends on at most one variable. (Note that h^x and A^x_k tacitly depend on $\bar{n}$.) We may assume $A^x_k \subseteq A_k$ for all $k \neq \bar{n}$. Let $B_{\bar{n}} \in \mathcal{U}_{\bar{n}}$ and $m(\bar{n}) \in d \setminus \{\bar{n}\}$ be such that for every $x \in B_{\bar{n}}$, h^x depends at most on $m(\bar{n})$-th coordinate. (If h^x is constant on this set, we let $m(\bar{n})$ be arbitrary.)

Claim 9. *For every $x \in B_{\bar{n}}$, we can assume that $A^x_k = A_k$, (where A_k is as provided by Lemma 8) for every $k \neq m(\bar{n})$.*

Proof. For convenience we assume that $\bar{n} = 0$ and $m(\bar{n}) = d - 1$.

We need to prove that whenever $y_i \in A^x_i$ ($1 \leq i < d$), $x_i \in A_i$ ($1 \leq i < d - 1$), and $x_{d-1} = y_{d-1}$, then

$$f(x, y_1, \ldots, y_{d-1}) = f(x, x_1, \ldots, x_{d-1}).$$

We prove this by the induction on the size of the set

$$\{1 \leq i < d - 1 : x_i \notin A^x_i\}$$

which we denote by l. If $l = 0$, then both $(x, y_1, \ldots, y_{d-1})$ and $(x, x_0, \ldots, x_{d-1})$ belong to $\prod_{k=0}^{d-1} A^x_k$, and the statement follows from our assumption on h^x and $x_{d-1} = y_{d-1}$. Let us assume the statement is proved for l and prove it for $l + 1$. Without a loss of generality, we may assume $x_i \notin A^x_i$ for $1 \leq i \leq l + 1$. By the inductive assumption, we have

(8) $f(x, y_1, \ldots, y_l, y_{l+1}, x_{l+2}, \ldots, x_{d-1}) = f(x, x_1, \ldots, x_l, y_{l+1}, x_{l+2}, \ldots, x_{d-1})$.

Let $\vec{x} = \langle x, x_1, \ldots, x_l, \cdot, x_{l+2}, \ldots, x_{d-1}\rangle \in X^{d\setminus\{l+1\}}$. Note that $\lim_{y\to\mathcal{U}_{l+1}} f^{\vec{x}}(y)$ exists (say, $\lim_{y\to\mathcal{U}_{l+1}} f^{\vec{x}}(y) = c$), since $f^{\vec{x}}$ is constant on A^x_{l+1}. Therefore by the choice of A_{l+1} we have $f^{\vec{x}}(y_{l+1}) = c$. Now let $\vec{x}' = \langle x, y_1, \ldots, y_l, \cdot, x_{l+2}, \ldots, y_{d-1}\rangle \in X^{d\setminus\{l+1\}}$. Thus (8) can be restated as $f^{\vec{x}}(y_{l+1}) = f^{\vec{x}'}(y_{l+1}) = c$, moreover by Lemma 8 this formula remains true when y_{l+1} is replaced with an arbitrary $y \in A_{l+1}$. In particular, $\lim_{y\to\mathcal{U}_{l+1}} f^{\vec{x}'}(y) = \lim_{y\to\mathcal{U}_{l+1}} f^{\vec{x}}(y) = c$. By the choice of A_{l+1} and $x_{l+1} \in A_{l+1}$, we have $f^{\vec{x}}(x_{l+1}) = c$ and

$$f(x, x_1, \ldots, x_{d-1}) = f^{\vec{x}}(x_{l+1}) = c = f^{\vec{x}'}(y_{l+1}) = f(x, y_1, \ldots, y_{l+1}, x_{l+2}, \ldots, x_{d-1}),$$

concluding the proof. □

Recall that h^x and A^x_k tacitly depend on $\bar{n}$. In the following argument the expressions h^{x_i} and $A^{x_i}_k$ should be interpreted by taking $\bar{n} = i$.

Claim 10. *Assume that i, j, k are distinct and less than d. If h^{x_i} does not depend on x_j for $\mathcal{U}_i$-many x_i then h^{x_k} does not depend on x_j for $\mathcal{U}_k$-many x_k either.*

Proof. Assume the contrary, that there is $B_i \in \mathcal{U}_i$ such that h^{x_i} does not depend on x_j for all $x_i \in B_i$, and that there is $B_k \in \mathcal{U}_k$ such that h^{x_k} does depend on x_j for all $x_k \in B_k$.

Since $j \neq i$, h^{x_k} does not depend on x_i for $x_k \in B_k$.

Pick $\bar{x}_i \in A_i \cap B_i$ and $\bar{x}_k \in A_k^{\bar{x}_i} \cap B_k$. We need to prove that $h^{\bar{x}_k}$ does not depend on x_j. Since $\bar{x}_k \in B_k$, $h^{\bar{x}_k}$ does not depend on x_i.

Let $y_l \in A_l^{\bar{x}_k}$ (for $l < d$) and $y'_j \neq y_j$ in $A_j^{\bar{x}_k}$ be arbitrary. We want to use the fact that $h^{\bar{x}_i}$ does not depend on j, and it would therefore be useful to have

(9) $y_l \in A_l^{\bar{x}_i}$ for all $l < d$ and $y'_j \in A_j^{\bar{x}_i}$.

By Claim 9 we have $y'_j \in A_j^{\bar{x}_i}$ and $y_l \in A_l^{\bar{x}_i}$ if $l \neq i$ and $h^{\bar{x}_i}$ does not depend on x_l. If $h^{\bar{x}_i}$ depends on x_l but $l = k$, we have $y_l = \bar{x}_k \in A_k^{\bar{x}_k}$ by the definition of $A_k^{\bar{x}_k} = \{\bar{x}_k\}$. Finally, if $k \neq l$ and $h^{\bar{x}_i}$ does depend on x_l, then we have $y_l \in A_l^{\bar{x}_k} = A_l$. Pick $y'_l \in A_l \cap A_l^{\bar{x}_i}$, and note that (we assume $j < l$ in order to simplify the notation) $f(y_0, \dots, y_j, \dots, y_l, \dots, y_{d-1}) = f(y_0, \dots, y_j, \dots, y'_l, \dots, y_{d-1})$, and similarly for y'_j instead of y_j. Thus we may assume (9). Therefore (now assume $i < j$ to simplify the notation):

$$\begin{aligned} &f(y_0, \dots, y_i, \dots, y_j, \dots, y_{d-1}) \\ &\qquad = f(y_0, \dots, \bar{x}_i, \dots, y_j, \dots y_{d-1}), \text{ since } h^{\bar{x}_k} \text{ does not depend on } x_i, \\ &\qquad = f(y_0, \dots, \bar{x}_i, \dots, y'_j, \dots, y_{d-1}), \text{ since } h^{\bar{x}_i} \text{ does not depend on } x_j, \\ &\qquad = f(y_0, \dots, y_i, \dots, y'_j, \dots, y_{d-1}), \text{ since } h^{\bar{x}_k} \text{ does not depend on } x_i. \end{aligned}$$

Since y_i ($i < d$) and y'_j were arbitrary members of $A_j^{\bar{x}_k}$, this concludes the proof. □

Pick an arbitrary $m < d$. If for $\mathcal{U}_m$ many x_m the function h^{x_m} is constant, then we are done, since on the set $\prod_{j=0}^{d-1} A_j$ (where $A_m \in \mathcal{U}_m$ is chosen so that h^{x_m} is constant for all $x_m \in A_m$, while $A_j \in \mathcal{U}_j$ for $j \neq m$ are chosen by Lemma 8) the function f depends on at most one coordinate, x_m. Otherwise, let n be such that for some $B_m \in \mathcal{U}_m$ and all $x_m \in B_m$, h^{x_m} depends on x_n only. Let $l < d$ be distinct from both m and n (recall that $d \geq 3$). Then by Claim 10 applied with $i = l$, $j = n$ and $k = m$, for $\mathcal{U}_l$ many x_l the map h^{x_l} depends on x_n, and therefore it does not depend on x_m. Applying Claim 10 again ($i = l$, $j = m$, $k = n$), we have that h^{x_n} does not depend on x_m for $\mathcal{U}_n$ many x_n. Applying it again ($i = m$, $j = l$, $k = n$), we conclude that h^{x_n} does not depend on x_l for $\mathcal{U}_n$ many x_n. Since l was an arbitrary index in $\{0, \dots, d-1\} \setminus \{m, n\}$, the function h^{x_n} is constant for $\mathcal{U}_n$ many x_n; let A_n be the set of such x_n. Then the restriction of f to $\prod_{i=0}^{d-1} A_i$ depends on x_n only, and this completes the proof of Theorem 6. □

The above proof of Theorem 3 uses the assumption that every discrete space has the Čech–Stone compactification. The use of this substantial fragment of the Axiom of Choice can be avoided at least in the case when $X = \omega$. As pointed out by the referee of [2], the above proof of this case can be modified to use compactness of the Cantor set instead of the existence of the Čech–Stone compactification of ω. This is because this proof takes place in a countable subalgebra of the power-set of ω, consisting of sets definable by first-order formulas with f as a parameter. Some care is needed only in Case 3 of the proof of Lemma 8, where $B \in \mathcal{U}_0$ need not hold. This clause has to be replaced by a slightly weaker clause: for every $A \in \mathcal{U}_0$ there is an $m \in A$ such that $\lim_{n \to \mathcal{U}_1} f(m, n)$ does not exist. The clause that $C \in \mathcal{U}_1$ has to be modified in an analogous manner. It is easily checked that this is all that is needed in the proof.

Question 11. *Is there a finitary version of Theorem 3?*

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