EXTREME AMENABILITY OF L_0, A RAMSEY THEOREM, AND LÉVY GROUPS

ILIJAS FARAH AND SŁAWOMIR SOLECKI

Abstract. We show that $L_0(\phi, H)$ is extremely amenable for any diffused submeasure ϕ and any solvable compact group H. This extends results of Herer–Christensen and of Glasner and Furstenberg–Weiss. Proofs of these earlier results used spectral theory or concentration of measure. Our argument is based on a new Ramsey theorem proved using ideas coming from combinatorial applications of algebraic topological methods. Using this work, we give an example of a group which is extremely amenable and contains an increasing sequence of compact subgroups with dense union, but which does not contain a Lévy sequence of compact subgroups with dense union. This answers a question of Pestov. We also show that many Lévy groups have non-Lévy sequences, answering another question of Pestov.

1. Introduction

A topological group is called *extremely amenable* if each of its continuous actions on a compact space has a fixed point. These types of groups and their connections with concentration of measure and with Ramsey theory have received considerable attention. For recent treatments of extreme amenability see [17] and [8]. The earliest examples of such groups were of the form $L_0(\phi, H)$ for certain locally compact second countable groups H and certain submeasures ϕ, see [7], [5]. While among very disconnected groups—those with a basis at the identity consisting of open subgroups—extreme amenability is well understood through its connection with structural Ramsey theory [8], this is not the case for many connected groups, such as $L_0(\phi, H)$. The present paper contributes to the clarification of this situation.

Now we explain the notions of submeasure and of $L_0(\phi, H)$. Let $\mathcal{B}$ be an algebra of subsets of a set X. A function ϕ mapping $\mathcal{B}$ to $\mathbb{R}$ is called a *submeasure* if $\phi(\emptyset) = 0$ and, for $U, V \in \mathcal{B}$, $\phi(U) \leq \phi(V)$, if $U \subseteq V$, and

Date: March 28, 2008.

2000 *Mathematics Subject Classification.* 05D10, 22F05, 43A07, 46A16, 54H25, 55M20.

Key words and phrases. extremely amenable groups, L_0, submeasures, Ramsey theory, Borsuk-Ulam theorem.

Research supported by NSERC (I.F.) and NSF grant DMS-0400931 (S.S.).

Filename: ea-jfa.tex.

$\phi(U \cup V) \le \phi(U) + \phi(V)$. A submeasure μ is called a *measure* if $\mu(U \cup V) = \mu(U) + \mu(V)$ for $U, V \in \mathcal{B}$ that are disjoint. Note that we do not require countable additivity. We say that a submeasure ϕ *has no measure below* if there is no non-zero measure μ such that $\mu \le \phi$. The generic submeasure turns out to have no measure below [7]. A submeasure ϕ is called *diffused* if for any $\epsilon > 0$ there is a covering of X by sets from $\mathcal{B}$ of ϕ-submeasure less than ϵ.

Let ϕ be a submeasure, and let H be a topological group. By *H-valued step functions* we understand functions on X with values in H, with finite ranges, and with preimages of points in $\mathcal{B}$. We make two step functions f, g equivalent if $\phi(\{x \in X : f(x) \neq g(x)\}) = 0$. Define $S(\phi, H)$ to be the space of all equivalence classes of H-valued step function with the topology of convergence in the submeasure ϕ. If H is locally compact, second countable (lcsc), we view $S(\phi, H)$ as equipped with the right invariant metric

$$d_\phi(f, g) = \inf\{\epsilon > 0 : \phi(\{x \in X : \rho(f(x), g(x)) > \epsilon\}) < \epsilon\}, \tag{1.1}$$

where ρ is a right invariant metric on H. (Such a metric ρ exists and is automatically complete as H is lcsc.) Of course, the value of $d_\phi(f, g)$ depends only on the equivalence classes of f and g in $S(\phi, H)$. We define $L_0(\phi, H)$ as the completion of $S(\phi, H)$ with respect to the metric d_ϕ. As usual, if $X = 2^{\mathbb{N}}$ and $\mathcal{B}$ is the algebra of all closed and open subsets of X, then elements of $L_0(\phi, H)$ can be identified with equivalence classes of certain Borel functions from $2^{\mathbb{N}}$ to H. It consists of all equivalence classes of Borel functions if and only if ϕ is *exhaustive*, i.e., if $\lim_n \phi(A_n) = 0$ for every disjoint family $A_n \in \mathcal{B}$. All measures are exhaustive. In an answer to an old question of Maharam, an exhaustive submeasure with no measure below was constructed recently by Talagrand [19].

To fix attention, we first consider the case of the circle group $H = \mathbb{T}$. We come back to the issue of more general groups H below.

1. The following observation goes back to Nikodým [13, pp.139–141]: ϕ is not diffused if and only if $L_0(\phi, \mathbb{T})$ has a nontrivial continuous homomorphism to $\mathbb{T}$. (In [13] this is proved only for ϕ a measure and with $\mathbb{R}$ in place of $\mathbb{T}$, but the method adapts with minor changes to the present situation.) In particular, if ϕ is not diffused, then $L_0(\phi, \mathbb{T})$ is not extremely amenable.

2. It follows from the results of Herer and Christensen [7] that if ϕ is a submeasure with no measure below, then $L_0(\phi, \mathbb{T})$ is extremely amenable. Note that such a ϕ is necessarily diffused. On the opposite end of the spectrum, it was proved by Glasner [5], and independently Furstenberg and Weiss, that if μ is a diffused measure, then $L_0(\mu, \mathbb{T})$ is also extremely amenable.

In light of 1 and 2 above, one is compelled to inquire whether $L_0(\phi, \mathbb{T})$ is extremely amenable if ϕ is only assumed to be diffused. We settle this question in the affirmative in the following result, whose sharper version is proved as Theorem 3.1.

Theorem 1.1. *Let ϕ be a diffused submeasure, and let H be a solvable compact second countable group. Then $L_0(\phi, H)$ is extremely amenable.*

It is worth pointing out that Herer and Christensen's method [7] is based on the spectral theorem while that of Glasner, Furstenberg, and Weiss [5] is based on concentration of measure. Our argument, presented in Sections 2 and 3, is different from both. It is combinatorial and uses a new Ramsey-type result for groups $\mathbb{Z}/p$ with p prime. This is Theorem 2.1, whose motivation and proof stem from some applications of algebraic topology to combinatorics. (For background see [2] and [11].) This seems to be the first result relating generalizations of the Borsuk–Ulam theorem to extreme amenability.

We now consider the following broader problem. So far there have been two main methods for proving extreme amenability: structural Ramsey theorems [12] and concentration of measure (Lévy property) [9]. The appropriate class of groups for application of structural Ramsey theory are automorphism groups of countable models or, in other words, closed subgroups of the group S_∞ of all permutations of $\mathbb{N}$, see [14], [8]. Such a group is extremely amenable if an appropriate structural Ramsey theorem holds. Furthermore, by [8], within this class of groups the two phenomena, extreme amenability of a group and existence of an appropriate structural Ramsey theorem, are essentially equivalent.

The appropriate class of groups for application of concentration of measure are groups that contain increasing sequences of compact subgroups with dense union, see [6], [5]: given such a group if a sequence of compact subgroups with dense union exhibits an appropriate concentration of measure, the group is extremely amenable. A question of Pestov, motivated by an older problem of Furstenberg, asks if this implication is an equivalence within the class of groups having increasing sequences of compact subgroups with dense unions (much like structural Ramsey theory/extreme amenability equivalence within closed subgroups of S_∞). See [17, §7.2] for a discussion of this problem. We give an example of a group, again of the form $L_0(\phi, H)$ for a certain submeasure ϕ and a compact group H, that answers this question in the negative.

We will now be more precise. The concentration of measure phenomenon is used in proving extreme amenability through the following notion (see e.g., [6], [5], [16]). Let G be a topological group. An increasing sequence (K_n) of compact subgroups of G, equipped with their Haar probability measures μ_n, is a *Lévy sequence* if for every open $V \ni 1$ and every sequence $A_n \subseteq K_n$ of measurable sets such that $\inf_n \mu_n(A_n) > 0$ we have that $\lim_n \mu_n(VA_n) = 1$.

A group is a *Lévy group* if it has a Lévy sequence of compact subgroups whose union is dense in G. Lévy groups were introduced by Gromov and Milman in [6], where it was shown that every Lévy group is extremely amenable. This theorem was applied in a number of situations to obtain various extreme amenability results. This prompted the question, which originated with Furstenberg in the late 1970s and was later reconsidered by Pestov, asking whether *each* group that is extremely amenable and has an increasing sequence of compact subgroups whose union is dense is a Lévy group ([17, p. 155]). For more background see [17, §7.2].

We consider Lévy groups in Section 4. Theorem 4.2 contains a slightly more precise formulation of the following result showing that Lévy sequences cannot be used to prove extreme amenbility of $L_0(\phi, H)$ if H is compact and not connected.

Theorem 1.2. *For every compact second countable group H that is not connected there exists a diffused submeasure ψ on the algebra of all closed and open subsets of $2^{\mathbb{N}}$ such that $L_0(\psi, H)$ is not Lévy.*

The Pestov question is now answered as follows. Let $H = \mathbb{Z}/2$, and let ψ be the submeasure from Theorem 1.2. Then $L_0(\psi, \mathbb{Z}/2)$ is extremely amenable by Theorem 1.1 as ψ is diffused. Each element of $L_0(\psi, \mathbb{Z}/2)$ has order 2, so there are increasing sequences (K_n) of compact, in fact finite, subgroups of $L_0(\psi, \mathbb{Z}/2)$ with $\bigcup_n K_n$ dense. Finally, $L_0(\psi, \mathbb{Z}/2)$ is not a Lévy group by Theorem 1.2.

One ingredient of the proof of Theorem 1.2 is Lemma 4.1, in which we show that in a Lévy group *each* increasing sequence of compact subgroups with dense union retains a certain amount of concentration.

Developing this theme, in Lemma 4.4, we give a simple method of constructing non-Lévy increasing sequences of compact subgroups with dense unions and apply it in Propositions 4.5 and 4.6 producing non-Lévy sequences in Lévy groups. This answers a question of Pestov [17, p.155]. In a different direction, we generalize the concentration of measure method from [5] and [16] to prove Proposition 4.7. On the one hand, this generalizes the Glasner–Furstenberg–Weiss–Pestov result that $L_0(\mu, H)$ is Lévy for a compact group H and a diffused measure μ. On the other hand, it gives examples of submeasures ϕ with no measure below, as in the Herer–Christensen result discussed earlier, such that $L_0(\phi, H)$ is Lévy. Such examples were not known according to the earlier version of [17].

Additional notation and conventions. For a set x, we write $|x|$ for the cardinality of x. The set of natural numbers, denoted by $\mathbb{N}$, includes 0. A natural number $n \in \mathbb{N}$ will be sometimes identified with the set of all smaller natural numbers, so $n = \{0, \ldots, n-1\}$. Given two sets x and y, by x^y we denote the set of all functions from y to x.

2. A Ramsey-type result

Our proof of extreme amenability will involve a Ramsey result for groups $\mathbb{Z}/p$ with p a prime, Theorem 2.1. In fact, we will only need a consequnce of this result which is stated as Corollary 2.2 at the end of this section.

Let A be a set. Let $n \geq l$ be positive natural numbers. We denote by $A^{n:l}$ the set of all partial functions from n to A whose domain has at least $n-l$ elements. If $A = \mathbb{Z}/p$, $h \in (\mathbb{Z}/p)^{n:l}$, and $r \in \mathbb{Z}/p$, we write $r+h$ for the element of $(\mathbb{Z}/p)^{n:l}$ whose domain is equal to that of h and that is such that

$$(r+h)(i) = r + h(i)$$

for any i in the domain of h. A subset L of $(\mathbb{Z}/p)^{n:l}$ is called *full* if there exists $h \in (\mathbb{Z}/p)^n$ and $a \subseteq n$ such that $|a| \geq n-l$ and for any $r \in \mathbb{Z}/p$,

$$(r+h) \restriction a_r \in L$$

for some $a_r \subseteq n$ with $a \subseteq a_r$.

Theorem 2.1. *Let $p_1, \ldots, p_k$ be prime numbers. Let $d \in \mathbb{N}$, $d > 0$. Then*

$$\exists l_1 \forall n_1 \geq l_1 \exists l_2 \forall n_2 \geq l_2 \cdots \exists l_k \forall n_k \geq l_k \text{ for any } d\text{-coloring of } \prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i:l_i}$$

$$\text{there exist full sets } L_1 \subseteq (\mathbb{Z}/p_1)^{n_1:l_1}, \ldots, L_k \subseteq (\mathbb{Z}/p_k)^{n_k:l_k}$$

$$\text{such that } L_1 \times \cdots \times L_k \text{ is monochromatic.}$$

It may be instructive for the reader to phrase and prove the above theorem for $k=1$, $p_1 = 2$, and $d = 2$. The proof in this special case is rather simple but, unlike in other Ramsey type results, the 2-color version is not representative of the general case. Needless to say it is the general case that is used in applications. See also the remarks following the proof of Corollary 2.2 below.

Let us first review terminology and results from algebraic topology that will be used in the proof. More details can be found in [11, p. 138]. We recall the basic notions around abstract complexes. Let X be a finite set. By a *complex with vertex set* X, we understand a closed under subsets family $\mathcal{F}$ of subsets of X. We call elements of $\mathcal{F}$ the *simplices* of the complex. Given two complexes $\mathcal{F}_1$ on X and $\mathcal{F}_2$ on Y, a function $f : X \to Y$ is called *simplicial* if $\{f(x) : x \in F\} \in \mathcal{F}_2$ for any $F \in \mathcal{F}_1$. For a group H, we say that a complex $\mathcal{F}$ with vertex set X is an *H-complex* if there is an action of H on X such that $\{hx : x \in F\} \in \mathcal{F}$ for any $F \in \mathcal{F}$ and $h \in H$. Given two H-complexes $\mathcal{F}_1$ on X and $\mathcal{F}_2$ on Y, we say that a simplicial map $f : X \to Y$ is *equivariant* if $f(hx) = hf(x)$ for any $x \in X$ and $h \in H$. If such an equivariant simplicial function exists, we write

$$\mathcal{F}_1 \xrightarrow{H} \mathcal{F}_2.$$

For a complex $\mathcal{F}$, define the complex $\mathrm{sd}(\mathcal{F})$ by declaring its set of vertices to be $\mathcal{F}$ and by letting

$$\mathrm{sd}(\mathcal{F}) = \{C \subseteq \mathcal{F} : C \neq \emptyset \text{ and } \forall F_1, F_2 \in C\ F_1 \subseteq F_2 \text{ or } F_2 \subseteq F_1\}.$$

We say that $\mathrm{sd}(\mathcal{F})$ is obtained from $\mathcal{F}$ by the *barycentric subdivision*. By sd^k, with $k \in \mathbb{N}$, we will indicate the k-fold iteration of the barycentric subdivision operation.

If $\mathcal{F}$ is an H-complex, then $\mathrm{sd}(\mathcal{F})$ with the natural induced action of H is an H-complex as well. If K is a simplicial complex, the polyhedron of its geometric representation is denoted by $\|K\|$. If K is an H-complex, the action of H on K naturally induces an action of H on $\|K\|$. A simplicial map f between simplicial complexes K and L induces a continuous function from $\|K\|$ to $\|L\|$ which sends geometric simplexes to simplexes. Moreover, if K and L are H-complexes and f is equivariant, then the induced map is also equivariant.

The following complex is a basic object. Let n be a natural number. Let $(\mathbb{Z}/p)^{*n}$ be the complex whose vertices are ordered pairs (i, r) with $i < n$ and $r \in \mathbb{Z}/p$. A set of vertices is a simplex of $(\mathbb{Z}/p)^{*n}$ if it forms a partial function from n to $\mathbb{Z}/p$. The complex $(\mathbb{Z}/p)^{*n}$ is a $\mathbb{Z}/p$-complex when considered with the following action on the vertices

$$(s, (i, r)) \to (i, s + r),$$

where $s \in \mathbb{Z}/p$ and (i, r) is a vertex of $(\mathbb{Z}/p)^{*n}$. We will use the following theorem which is an immediate consequence of Proposition 6.2.4 and Theorem 6.3.3 in [11]:

Let $k \in \mathbb{N}$. If $n \geq d \cdot (p - 1) + 1$, then for any continuous function from $\|\mathrm{sd}^k((\mathbb{Z}/p)^{*n})\|$ to $\mathbb{R}^d$ there exists an orbit of the $\mathbb{Z}/p$ action on $\|\mathrm{sd}^k((\mathbb{Z}/p)^{*n})\|$ that is mapped to a single point in $\mathbb{R}^d$.

Proof of Theorem 2.1. We start with a claim.

Claim. Let p be a prime number. For any $d \in \mathbb{N}$, $d > 0$, there exists $l \in \mathbb{N}$ such that for any $n \geq l$ any d-coloring of $(\mathbb{Z}_p)^{n:l}$ has a monochromatic full set.

Proof of Claim. We consider $(\mathbb{Z}/p)^{n:l}$ to be the set of vertices of the simplicial complex $(\mathbb{Z}/p)^{n:l}_c$ whose simplices are precisely those $C \subseteq (\mathbb{Z}/p)^{n:l}$ for which for any $h_1, h_2 \in C$ either $h_1 \subseteq h_2$ or $h_2 \subseteq h_1$. Consider it with the action of $\mathbb{Z}/p$ given by

$$\mathbb{Z}/p \times (\mathbb{Z}/p)^{n:l} \ni (r, h) \to r + h \in (\mathbb{Z}/p)^{n:l}.$$

Now we relate the complex $(\mathbb{Z}/p)^{n:l}_c$ to the complexes obtained by iterated application of the barycentric subdivision to $(\mathbb{Z}/p)^{*(l+1)}$. An immediate checking gives

$$\mathrm{sd}((\mathbb{Z}/p)^{*(l+1)}) = (\mathbb{Z}/p)^{l+1:l}_c. \tag{2.1}$$

We further show that for any $n > l$

$$\mathrm{sd}((\mathbb{Z}/p)^{n:l}_c) \xrightarrow{\mathbb{Z}/p} (\mathbb{Z}/p)^{n+1:l}_c. \tag{2.2}$$

To see this, let $h_1 \subseteq h_2 \subseteq \cdots \subseteq h_k$ be a simplex in $(\mathbb{Z}/p)^{n:l}_c$, that is, $v = \{h_1, \ldots, h_k\}$ is a vertex in $\mathrm{sd}((\mathbb{Z}/p)^{n:l}_c)$. If the domain of h_k has $> n - l$ elements, then h_k is a vertex in $(\mathbb{Z}/p)^{n+1:l}_c$, and in this case map v to h_k. If the domain of h_k has precisely $n - l$ points, in which case $h_k = h_1$ and $v = \{h_1\}$, map v to $h_k \cup \{(n, h_k(m))\}$, where m is the minimal element of the domain of h_k. Note that $h_k \cup \{(n, h_k(m))\}$ a vertex in $(\mathbb{Z}/p)^{n+1:l}_c$. One easily checks that this defines an equivariant simplicial map witnessing (2.2). Combining (2.1) and (2.2), we see that for $n > l$

$$\mathrm{sd}^{n-l}((\mathbb{Z}/p)^{*(l+1)}) \xrightarrow{\mathbb{Z}/p} (\mathbb{Z}/p)^{n:l}_c. \tag{2.3}$$

Let now $c : (\mathbb{Z}/p)^{n:l} \to d$ be a d-coloring. Let e_j, $j = 0, \ldots, d-1$, be the standard basic vectors in $\mathbb{R}^d$. Let Δ be the simplex whose vertex set is $\{e_j : j = 0, \ldots, d-1\}$ (and whose simplices are all non-empty subsets of the vertex set). Consider the simplicial map f from $(\mathbb{Z}/p)^{n:l}_c$ to Δ induced by the function

$$(\mathbb{Z}/p)^{n:l} \ni h \to e_{c(h)}.$$

By composing the function given by (2.3) with f we get a simplicial map

$$\mathrm{sd}^{n-l}((\mathbb{Z}/p)^{*(l+1)}) \to \Delta$$

which induces a continuous map from $\|\mathrm{sd}^{n-l}((\mathbb{Z}/p)^{*(l+1)})\|$ to the geometric simplex $\|\Delta\| \subseteq \mathbb{R}^d$ spanned by the vectors $e_0, \ldots, e_{d-1}$. Now if we take $l + 1 \geq d \cdot (p-1) + 1$, by the theorem stated as the final remark preceding this proof, there exists a point in $\|\mathrm{sd}^{n-l}((\mathbb{Z}/p)^{*(l+1)})\|$ whose whole $\mathbb{Z}/p$ orbit is mapped by the above function to a singe point in $\|\Delta\|$. Since the function in (2.3) preserves the action, we obtain a point $x_0 \in \|(\mathbb{Z}/p)^{n:l}_c\|$ such that

$$F(x_0) = F(r + x_0) \quad \text{for each } r \in \mathbb{Z}/p, \tag{2.4}$$

where $F : \|(\mathbb{Z}/p)^{n:l}_c\| \to \|\Delta\|$ is the continuous induced by f. Since $F(x_0)$ is a point in $\|\Delta\|$, at least one of its coordinates, say i_0, is not equal to 0. Point x_0 lies in the geometric realization of a maximal simplex of $(\mathbb{Z}/p)^{n:l}_c$, that is, of a simplex of the form

$$h_l \subseteq h_{l-1} \subseteq \cdots \subseteq h_0$$

where $h_i \in (\mathbb{Z}/p)^{n:l}$ has $n-i$ elements in its domain. Fix $r \in \mathbb{Z}/p$. Point $r+x_0$ lies in the geometric realization of the maximal simplex

$$r+h_l \subseteq r+h_{l-1} \subseteq \cdots \subseteq r+h_0.$$

By (2.4), the i_0-th coordinate of $F(r+x_0)$ is non-zero. Thus, there is $0 \le j_r \le l$ such that the point of $\|(\mathbb{Z}/p)^{n:l}_c\|$ corresponding to $r+h_{j_r}$ is mapped by F to a point with non-zero i_0-th coordinate. By the definition of F, this implies that $e_{c(r+h_{j_r})}$ has non-zero i_0-th coordinate, which means that $c(r+h_{j_r}) = i_0$. Since

$$\{r+h_{j_r} : r \in \mathbb{Z}/p\}$$

is a full set, the claim is established. □

We now prove the theorem by induction on k. For $k=1$ it is simply Claim. Assume our theorem holds for k. In order to prove it for $k+1$, we pick l_i for $i \le k$ given $l_1, n_1, \ldots, l_{i-1}, n_{i-1}$ using our inductive assumption. Here is how to pick l_{k+1} given $l_1, n_1, \ldots, l_k, n_k$. Apply Claim with the number of colors equal to

$$d' = d \cdot \left| \prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i:l_i} \right|$$

to obtain l. Let l_{k+1} be equal to this l. Now given a d-coloring c of the product $\prod_{i=1}^{k+1} (\mathbb{Z}/p_i)^{n_i:l_i}$ consider the following d'-coloring c' of $(\mathbb{Z}/p_{k+1})^{n_{k+1}:l_{k+1}}$:

$$c'(h) = (c(h_1, \ldots, h_k, h) : (h_1, \ldots, h_k) \in \prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i:l_i}).$$

Let $L_{k+1} \subseteq (\mathbb{Z}/p_{k+1})^{n_{k+1}:l_{k+1}}$ be a c'-monochromatic full set. For an $h \in L_{k+1}$ let

$$c''(h_1, \ldots, h_k) = c(h_1, \ldots, h_k, h)$$

be a d-coloring of $\prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i:l_i}$. Note that since L_{k+1} is c'-monochromatic, the definition of c'' does not depend on the choice of $h \in L_{k+1}$. Find full sets

$$L_1 \subseteq (\mathbb{Z}/p_1)^{n_1:l_1}, \ldots, L_k \subseteq (\mathbb{Z}/p_k)^{n_k:l_k}$$

such that $L_1 \times \cdots \times L_k$ is c''-monochromatic. Then it is clear that $L_1 \times \cdots \times L_k \times L_{k+1}$ is c-monochromatic. □

J. Matoušek has pointed out to us that an isomorphic version of the simplicial complex $(\mathbb{Z}/p)^{n:l}_c$ used in the proof above occurs in [10] (it is denoted there by K), where the lower bound for its $\mathbb{Z}/p$-index is derived by Sarkaria's trick.

In the sequel, we will need only the following consequence of Theorem 2.1. In its statement, for $X \subseteq \prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i}$ we write $X - X = \{h_1 - h_2 : h_1, h_2 \in X\}$.

Corollary 2.2. *Let $p_1, \dots, p_k$ be prime numbers. Let $d \in \mathbb{N}$, $d > 0$. Then*

$$\exists l_1 \forall n_1 \geq l_1 \exists l_2 \forall n_2 \geq l_2 \cdots \exists l_k \forall n_k \geq l_k \text{ for any } d\text{-coloring of } \prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i}$$
$$\text{there exist } v_1 \subseteq n_1, \dots, v_k \subseteq n_k \text{ and a color } X \text{ such that}$$

$|v_1| = l_1, \dots, |v_k| = l_k$ and for every $f \in \prod_{i=1}^{k} \mathbb{Z}/p_i$ there is $h \in X - X$ satisfying $h(j) = f(i)$ for all $i \leq k$ and all $j \in n_i \setminus v_i$.

Proof. We claim that, given $p_1, \dots, p_k$ and d, any sequence $l_1, n_1, \dots, l_k, n_k$ obtained by using Theorem 2.1 is as required. Given a coloring c of $\prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i}$ define a coloring c' of $\prod_{i=1}^{k} (\mathbb{Z}/p_i)^{n_i \cdot l_i}$ by letting $c'(h) = c(\check{h})$, where $\check{h} \supseteq h$ is an extension defined by $\check{h}(j) = 1$ for $j \notin \text{dom}(h)$. An application of Theorem 2.1 to c' completes the proof. $\square$

Note that Theorem 2.1 becomes false even when $k = 1$ if $\mathbb{Z}/p$ is replaced by $\mathbb{Z}$. In fact, for every $l \geq 1$, for $n = 2l + 1$, we have a partition $\mathbb{Z}^n = X_1 \cup X_2$ such that for every $h \in \mathbb{Z}^n$ and color $X \in \{X_1, X_2\}$ there is $a \in \mathbb{Z}$ such that $|\{j : f(j) \neq a + h(j)\}| \geq l + 1$ for every $f \in X$. To see it, let $f \in X_1$ if $|\{j : f(j) > 0\}| \geq l + 1$ and $f \in X_2$ otherwise. Since for every $h \in \mathbb{Z}^{2l+1}$ we can find a and a' such that $a + h(j) > 0$ for all j and $a' + h(j) < 0$ for all j, the conclusion follows.

On the other hand, it is not difficult to see that Corollary 2.2 with $k = 1$ and $d = 2$ and with $\mathbb{Z}/p$ replaced by $\mathbb{Z}$ (or any other group) remains true already with $l = 1$. However, the argument does not seem to generalize to the higher values of d.

3. Extreme amenability of L_0

In this section, we will prove the following theorem of which Theorem 1.1 is an immediate consequence.

Theorem 3.1. *Let ϕ be a diffused submeasure. Let H be a second countable group such that the closure of each finitely generated subgroup of H is compact and solvable. Then $L_0(\phi, H)$ is extremely amenable.*

We will be using the following old characterization of extreme amenability due to Pestov [15] (see also [17, Theorem 3.4.9])

> G is extremely amenable if and only if SS^{-1} is dense for every
> $S \subseteq G$ such that $FS = G$ for some finite F.

In the following lemma, we register some well-known preservation properties of extremely amenable groups.

Lemma 3.2. *Let G be a topological group.*

(i) *Let H be a closed normal subgroup of G. If both H and G/H are extremely amenable, then so is G.*
(ii) *If G is the union of a family of extremely amenable subgroups which is directed under inclusion, then G is extremely amenable.*
(iii) *If G contains a dense extremely amenable subgroup subgroup, then G is extremely amenable.*
(iv) *Let (H_n) be a sequence of topological groups, and let $\pi_n : H_{n+1} \to H_n$ be surjective continuous homomorphisms such that $G = \varprojlim(H_n, \pi_n)$. If each H_n is extremely amenable, then so is G.*
(v) *The product of an arbitrary family of extremely amenable groups is extremely amenable.*
(vi) *Continuous homomorphic images of extremely amenable groups are extremely amenable.*

Proof. These are all well-known. Points (iii) and (vi) are immediate from the definition of extreme amenability, as are (i) and (ii) if one only realizes that the set of fixed points of a continuous action of a topological group on a compact space is compact. Point (v) is a consequence of (i), (ii) and (iii).

Point (iv) follows from Pestov's characterization of extreme amenability stated in the beginning of the present section. Indeed, let $G = FS$ for some finite $F \subseteq G$. Let $\pi_n^\infty : G \to H_n$ be the projection. It follows from our assumptions that each π_n^∞ is surjective. Thus, since each H_n is extremely amenable, $\pi_n^\infty(S)\pi_n^\infty(S)^{-1}$ is dense in H_n for each n. This immediately implies that SS^{-1} is dense in G since the topology on G is generated by sets of the form $(\pi_n^\infty)^{-1}(U)$ for $U \subseteq H_n$ open in H_n. □

Now we have a couple of lemmas giving an analysis of groups $S(\phi, H)$.

Lemma 3.3. *Let H be a compact, second countable, Abelian group. Then $H = \varprojlim(H_n, \pi_n)$, where each H_n is a compact, second countable, Abelian group in which torsion elements are dense, and $\pi_n : H_{n+1} \to H_n$ is a continuous surjective homomorphism.*

Proof. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since each compact second countable Abelian group is isomorphic to a closed subgroup of $(\mathbb{T})^{\mathbb{N}}$, it is the inverse limit, with continuous surjective homomorphisms as bonding maps, of closed subgroups of $(\mathbb{T})^n$ with $n \in \mathbb{N}$. The lemma follows, therefore, from the fact that the torsion elements are dense in any closed subgroup H of $(\mathbb{T})^n$ with $n \in \mathbb{N}$. This can be seen as follows. There is a surjective homomorphism from the dual group of $(\mathbb{T})^n$, which is $\mathbb{Z}^n$, onto the dual group of H. Thus, this last group is finitely generated and, therefore, it is the product of a finite Abelian group and $\mathbb{Z}^k$, for some $k \in \mathbb{N}$. Now H being the dual of such a group is isomorphic to $H' \times (\mathbb{T})^k$ with H' finite. □

Lemma 3.4. *Let ϕ be a submeasure.*

(i) *Let (H_n) be a sequence of topological groups, and let $\pi_n : H_{n+1} \to H_n$ be continuous surjective homomorphisms. Then $\pi_n^* : S(\phi, H_{n+1}) \to S(\phi, H_n)$ given by $\pi_n^*(f) = \pi_n \circ f$ is a surjective continuous homomorphism and*

$$S(\phi, \varprojlim(H_n, \pi_n)) = \varprojlim(S(\phi, H_n), \pi_n^*).$$

(ii) *If H' is a dense subgroup of a topological group H, then $S(\phi, H')$ is a dense subgroup of $S(\phi, H)$.*

(iii) *If H is the union of a directed by inclusion family of subgroups $\mathcal{H}$, then $S(\phi, H) = \bigcup\{S(\phi, H') : H' \in \mathcal{H}\}$.*

(iv) *Let H' be a closed normal subgroup of a topological group H. Then*

$$S(\phi, H/H') = S(\phi, H)/S(\phi, H').$$

Proof. (i) Define $r : \varprojlim(S(\phi, H_n), \pi_n^*) \to S(\phi, \varprojlim(H_n, \pi_n))$ by letting

$$r((f_n))(x) = (f_n(x)).$$

We leave it to the reader to check that r is a bijective homomorphism which is a homeomorphism.

(ii) is immediate as is (iii) which, however, depends on the fact that each function in $S(\phi, H)$ attains finitely many values.

(iv) Define $r : S(\phi, H) \to S(\phi, H/H')$ be $r(f)(x) = f(x)/H'$. One checks that r is a continuous surjective homomorphism whose kernel is $S(\phi, H')$. Thus, it induces a bijective continuous homomorphism from $S(\phi, H)/S(\phi, H')$ onto $S(\phi, H/H')$. We leave it to the reader to check that this homomorphism is open. □

Extreme amenability will be proved using the lemma below.

Lemma 3.5. *Assume G is a topological group such that for every partition $G = X_1 \cup \cdots \cup X_d$, every finite $F \subseteq G$ and every open neighborhood W of the identity there is $t \leq d$ such that $F \subseteq WX_t(X_t)^{-1}$. Then G is extremely amenable.*

Proof. We will use Pestov's characterization of extreme amenability stated at the beginning of this section. Let $S \subseteq G$ be such that $G = \bigcup_{t \leq d} g_t S$. Put $X_t = g_t S$. By our assumption, for every W and finite F there is $t = t(W, F) \leq d$ such that $F \subseteq WX_t(X_t)^{-1}$. We claim there is $t = t(W) \leq d$ such that $G = WX_t(X_t)^{-1}$. Otherwise, for every $t \leq d$ there is $h_t \in G \setminus WX_t(X_t)^{-1}$. Then $F = \{h_1, \ldots, h_d\}$ is not included n $WX_t(X_t)^{-1}$ for any $t \leq d$, contradicting our assumption. A similar argument shows that there is $t_0 \leq d$ such that $G = UX_{t_0}(X_{t_0})^{-1}$ for every open neighborhood U of the identity. Therefore,

$X_{t_0}(X_{t_0})^{-1} = g_{t_0}SS^{-1}g_{t_0}^{-1}$ is dense in G, which makes SS^{-1} dense, and the extreme amenability follows. □

A condition easily equivalent to the one in Lemma 3.5 was considered by Christensen already in [3, p.104]. In fact, the condition from the lemma is equivalent to extreme amenability as proved in [17, Theorem 3.4.9].

Proof of Theorem 3.1. By Lemma 3.2(iii), Lemma 3.4(iii), and Lemma 3.2(ii), it suffices to prove the theorem for $S(\phi, H)$, where H second countable, compact and solvable. Now, by Lemma 3.4(iv) and Lemma 3.2(i), it is enough to prove it for H second countable compact and Abelian. By Lemma 3.3, Lemma 3.4(i), and Lemma 3.2(iv), we can assume that H is second countable, compact, Abelian, and that the torsion elements are dense in H. Therefore, by applying Lemma 3.4(ii) and (iii) and Lemma 3.2(ii) and (iii), we reduce proving the theorem to the case when H is finite. One more application of Lemma 3.4(iv) and Lemma 3.2(i) shows that we can restrict ourselves to $H = \mathbb{Z}/p$ for a prime number p. We consider this case in what follows.

By Lemma 3.5 it suffices to prove that $S(\phi, \mathbb{Z}/p)$ has the following property:

Let $X_1 \cup \cdots \cup X_d = S(\phi, \mathbb{Z}/p)$, let $F \subseteq S(\phi, \mathbb{Z}/p)$ be finite, and let $W \subseteq S(\phi, \mathbb{Z}/p)$ be an open neighborhood of 0. Then there exists $t_0 \leq d$ with $F \subseteq W + (X_{t_0} - X_{t_0})$.

Our intention is to use Corollary 2.2. We can assume that F consists of all elements of $S(\phi, \mathbb{Z}/p)$ which are constant on sets in a fixed partition $b_1, \ldots, b_k$ of X into sets from $\mathcal{B}$. This is because each finite subset of $S(\phi, \mathbb{Z}/p)$ is contained in a set of this form. Since ϕ is diffused, we can also assume that for some $\epsilon > 0$,

$$W = \{g \in S(\phi, \mathbb{Z}/p) : \phi(\{x \in X : g(x) \neq 0\}) < \epsilon\}.$$

At this point we are given ϵ, the number k of elements in the partition of X, and the number d of elements in the partition of $S(\phi, \mathbb{Z}/p)$. With these k and d, use Corollary 2.2 to find l_1. Partition b_1 into finitely many sets in $\mathcal{B}$ of ϕ-submeasure $< \delta_1$ such that $0 < \delta_1 \cdot l_1 < \epsilon/k$. (We use diffusion of ϕ here.) Let n_1 be the number of elements in this partition of b_1. We can assume that $n_1 \geq l_1$. Now given l_1, n_1 use Corollary 2.2 to find l_2. Repeat what was done above with b_2 replacing b_1. That is, partition b_2 into finitely many sets in $\mathcal{B}$ of ϕ-submeasure $< \delta_2$ such that $0 < \delta_2 \cdot l_2 < \epsilon/k$. Let n_2, which we can assume to be $\geq l_2$, be the number of elements in this partition of b_2. Continue in this fashion using Corollary 2.2 until all sets b_i are taken care of. For each $1 \leq i \leq k$, let

$$a_0^i, a_1^i, \ldots, a_{n_i-1}^i$$

be the partition of b_i constructed above.

Now we describe a coloring of $\prod_{i=1}^k (\mathbb{Z}/p)^{n_i}$ into d colors. Given

$$\bar{c} = (c_1, \ldots, c_k) \in \prod_{i=1}^k (\mathbb{Z}/p)^{n_i}$$

consider $g_{\bar{c}} \in S(\phi, \mathbb{Z}/p)$ defined by

$$g_{\bar{c}} \restriction a^i_j = c_i(j).$$

The color assigned to $\bar{c} = (c_1, \ldots, c_k)$ is equal to the subscript of the element of the partition $X_0, \ldots, X_{d-1}$ of $S(\phi, \mathbb{Z}/p)$ to which $g_{\bar{c}}$ belongs.

By our choice of the numbers l_i and n_i there are

$$h_1 \in (\mathbb{Z}/p)^{n_1}, \ldots, h_k \in (\mathbb{Z}/p)^{n_k},$$

$t_0 < d$, and $v_1 \subseteq n_1, \ldots v_k \subseteq n_k$ such that $|v_1| = l_1, \ldots |v_k| = l_k$, and for every $\bar{d} = (d_1, \ldots, d_k) \in (\mathbb{Z}/p)^k$ there are $\bar{f}$ and $\bar{f}'$, both colored by t_0, such that $d_i = \bar{f}(j) - \bar{f}'(j)$ for all i and all $j \in n_i \setminus v_i$.

Every $g \in F$ is constant on each b_i, hence by the last observation there are $g_{\bar{f}}$ and $g_{\bar{f}'}$ satisfying

$$\begin{aligned} \phi(\{x : g(x) \neq (g_{\bar{f}'} - g_{\bar{f}})(x)\}) &\leq \phi(\textstyle\bigcup_{i=1}^k \bigcup \{a^i_j : j \in v_i\}) \\ &\leq \sum_{i=1}^k \sum_{j \in v_i} \phi(a^i_j) < k \cdot \epsilon / k = \epsilon. \end{aligned}$$

It follows from it that $g \in W + (g_{\bar{f}'} - g_{\bar{f}})$. Since $g \in F$ was arbitrary and since $g_{\bar{f}'}, g_{\bar{f}} \in X_{t_0}$, we see that $F \subseteq W + X_{t_0} - X_{t_0}$, and we are done. □

4. Extreme amenability and Lévy groups

In this section, we study connections between extreme amenability and being Lévy and between being Lévy and having non-Lévy sequences of compact subgroups. This is done principally for groups of the form $L_0(\phi, H)$, but also for the unitary group of the separable Hilbert space and the group of measure preserving transformations.

We start with a lemma proving a property of general Lévy groups. This property may be of some independent interest and we make some additional comments on it in Section 5. It says that in a Lévy group all increasing sequences of compact subgroups with dense unions exhibit some degree of concentration as in (4.1). This degree of concentration has to be carefully calibrated since, as shown by Propositions 4.5 and 4.6, in general Lévy groups arbitrary increasing sequences of compact subgroups with dense unions are not Lévy, that is, full concentration fails on them.

The immediate usefulness of the lemma below is in proving that a given group is not Lévy. The lemma allows us to replace the necessity of showing that no increasing sequence of compact subgroups with dense union is Lévy by constructing one increasing sequence of compact subgroups with dense union on which concentration fails in a uniform way, that is, on which condition (4.1) is violated.

Lemma 4.1. *Let a topological group G be Lévy. Then for any increasing sequence (K_n) of compact subgroups of G with dense union, an open neighborhood V of 1, a compact set $F \subseteq G$, and Borel sets $A_n \subseteq K_n$, we have*

$$\liminf_n \nu_n(VA_n) - \nu_n(FA_n) \geq 0, \tag{4.1}$$

where ν_n is the probability Haar measure on K_n, considered as a measure on G with support K_n.

Proof. Assume the conclusion fails and pick (K_n), V, F, and A_n witnessing the failure of (4.1). By going to a subsequence of (K_n), we can suppose that there are δ_1, δ_2 so that

$$\nu_n(FA_n) > \delta_2 > \delta_1 > \nu_n(VA_n). \tag{4.2}$$

Let (H_m) be an arbitrary increasing sequence of compact subgroups of G with dense union. We will show that the sequence is not Lévy, thereby proving that G is not a Lévy group. Let μ_m be the probability Haar measure on H_m. Let $W \ni 1$ be an open neighborhood of 1 with $W^2 \subseteq V$. Using compactness of F, find m_0 such that $F \subseteq H_{m_0}W$ and $h_1, \dots, h_p \in H_m$ so that

$$F \subseteq \bigcup_{i=1}^{p} h_i W. \tag{4.3}$$

We claim that there are constants $\beta_1, \beta_2 > 0$ such that for $m \geq m_0$ we can find Borel $C \subseteq H_m$ with

$$\mu_m(C) > \frac{\beta_2\delta_2}{p} \quad \text{and} \quad \mu_m(H_m \setminus WC) > \beta_1(1-\delta_1), \tag{4.4}$$

thereby proving that (H_m) is not a Lévy sequence. Let

$$W' = \text{the interior of } \bigcap_{f\in F} fWf^{-1}. \tag{4.5}$$

The open set W' is a neighborhood of 1 since F is compact. Let $m \geq m_0$ be fixed. Fix n large enough so that

$$H_m \subseteq W'K_n. \tag{4.6}$$

Claim. Let $B \subseteq K_n$ be a measurable set and let $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Then

$$\nu_n(\{g \in K_n : \mu_m(W'Bg^{-1}) \geq \beta\nu_n(B)\}) \geq \alpha\nu_n(B).$$

Proof. Consider the compact group $H_m \times K_n$ with the product measure $\mu_m \times \nu_n$. By (4.6) all vertical sections of the set $\{(h,g) \in H_m \times K_n : hg \in W'\}$ are nonempty. By the translation invariance of ν_n each vertical section of

$$X = \{(h,g) \in H_m \times K_n : hg \in W'B\}$$

is of ν_n-measure at least $\nu_n(B)$. By Fubini's theorem, $\{g \in K_n :$ the horizontal section of X at g has μ_m-measure $\geq \beta\nu_n(B)\}$ has ν_n-measure at least $\alpha\nu_n(B)$. But the horizontal section of X at g is $H_m \cap W'Bg^{-1}$. □

Using (4.2), we can fix $0 < \alpha_1, \alpha_2 < 1$ such that

$$\alpha_1(1-\delta_1) + \alpha_2\delta_2 > 1. \tag{4.7}$$

Set $\beta_1 = 1 - \alpha_1$ and $\beta_2 = 1 - \alpha_2$. We show that for this choice of β_1 and β_2 (4.4) holds.

Applying the claim to the set $K_n \setminus VA_n$ with α_1 and β_1 and using (4.2), we get a set of $g \in K_n$ of ν_n-measure $\geq \alpha_1\nu_n(K_n \setminus VA_n) > \alpha_1(1-\delta_1)$ such that

$$\mu_m(W'(K_n \setminus VA_n)g^{-1}) \geq \beta_1\nu_n(K_n \setminus VA_n) > \beta_1(1-\delta_1).$$

For each such g we have

$$\mu_m(H_m \setminus W'VA_ng^{-1}) > \beta_1(1-\delta_1). \tag{4.8}$$

Now apply the claim to the set FA_n with α_2 and β_2 in combination with (4.2) to obtain a set of $g \in K_n$ of ν_n-measure $\geq \alpha_2\nu_n(FA_n) > \alpha_2\delta_2$ such that

$$\mu_m(W'FA_ng^{-1}) > \beta_2\nu_n(FA_n) > \beta_2\delta_2. \tag{4.9}$$

Therefore, using (4.5) and (4.3), we obtain

$$\begin{aligned} W'FA_n = W'FA_n &\subseteq FWA_n \subseteq h_1W^2A_n \cup \cdots \cup h_pW^2A_n \\ &\subseteq h_1VA_n \cup \cdots \cup h_pVA_n. \end{aligned}$$

In view of this, (4.9) implies

$$\mu_m(h_1VA_ng^{-1} \cup \cdots \cup h_pVA_ng^{-1}) > \beta_2\delta_2.$$

Since μ_m is invariant under translations by $h_1, \ldots, h_p$ as they are elements of H_m, we see that for a set of $g \in K_n$ of ν_n-measure $> \alpha_2\delta_2$ we have

$$\mu_m(VA_ng^{-1}) > \frac{\beta_2\delta_2}{p}. \tag{4.10}$$

Taking into account (4.7), we get $g_0 \in K_n$ for which both (4.8) and (4.10) hold. Thus, (4.4) is fulfilled by $C = VA_ng_0^{-1} \cap H_m$. □

Theorem 4.2 below is a more precise version of Theorem 1.2. We need some definitions. Let M_i, $i \in \mathbb{N}$ be a sequence of non-zero natural numbers. Put $\Sigma = \bigcup_{i=0}^{\infty} \prod_{j<i} M_j$. For $\sigma \in \Sigma$, if $\sigma \in \prod_{j<i} M_j$, let $|\sigma| = i$ and let

$[\sigma] = \{x \in \prod_i M_i : x \restriction |\sigma| = \sigma\}$. Define a submeasure on all subsets of $\prod_i M_i$ by letting

$$\psi_0(A) = \inf\{\textstyle\sum_{\sigma\in I} 2^{-|\sigma|} : I \subseteq \Sigma,\ A \subseteq \bigcup_{\sigma\in I}[\sigma]\}.$$

The submeasure ψ_0 coincides with the first element of the sequence of the so-called auxiliary measures in the definition of the one-dimensional Hausdorff measure on $\prod_i M_i$ taken with the metric $d(x, y) = 2^{-\min\{i:x_i\neq y_i\}}$ [18, p.50].

Theorem 4.2. *For every compact group H that is not connected there is a sequence of natural numbers M_i, $i \in \mathbb{N}$, such that the submeasure ψ_0 on the algebra of all clopen subsets of $\prod_i M_i$ has the following properties:*

(i) *it is diffused;*
(ii) *$L_0(\psi_0, H)$ is not a Lévy group.*

In order to state the next lemma, we need some definitions. Let M and d be natural numbers. For $g : M \to d$ and $j_0 < d$, let

$$\mu(g) = j_0$$

if

$$|\{l < M : g(l) = j_0\}| = \max_{j<d} |\{l < M : g(l) = j\}| \tag{4.11}$$

and the value of $\min\{l < M : g(l) = j\}$ for $j = j_0$ is the smallest of all such numbers evaluated for j realizing the maximum on the right side of (4.11). Assume nonzero natural numbers $M_0, \dots, M_k$ and d are given. Let

$$\mathcal{X}_k = \{(x_\sigma : \sigma \in \textstyle\bigcup_{i<k} M_0 \times \cdots \times M_i) : \forall\sigma\ x_\sigma \subseteq M_{|\sigma|} \text{ and } |x_\sigma| < 2^{|\sigma|}\}.$$

Let $f \in d^{M_0\times\cdots\times M_k}$, and let $\bar{x} \in \mathcal{X}_k$. For $i \le k$ define

$$f^i_{\bar{x}} : M_0 \times \cdots \times M_i \to d$$

as follows. Put $f^k_{\bar{x}} = f$. Assume $f^{i+1}_{\bar{x}}$ is defined. Let $\sigma \in M_0 \times \cdots \times M_i$. Put $f^i_{\bar{x}}(\sigma) = \mu(g)$, where $g : M_{i+1} \to d$ is given by

$$g(l) = \begin{cases} f^{i+1}_{\bar{x}}(\sigma^\frown l), & \text{if } l \in M_{i+1} \setminus x_\sigma; \\ 0, & \text{if } l \in x_\sigma. \end{cases}$$

This defines $f^i_{\bar{x}}$ for all $i \le k$ and all $\bar{x} \in \mathcal{X}_k$.

Lemma 4.3. *Let $\epsilon > 0$ and $d \in \mathbb{N}$ be given. There exists a sequence of natural numbers $M_0, M_1, \dots$ such that for each k*

$$|\{f \in d^{M_0\times\cdots\times M_k} : \exists\bar{x} \in \mathcal{X}_k\ f^0_{\bar{x}}(\emptyset) = 0\}| < (\frac{1}{d} + \epsilon) \cdot d^{M_0\cdots M_k}.$$

Proof. First we specify how fast the sequence M_i, $i \ge 0$, grows. We will use a probability argument involving the claim below, which is a consequence of the central limit theorem.

Claim. Let $d \geq 1$ be a natural number. Let $m \in \mathbb{N}$ and $\Delta > 0$ be given. There exists $M \in \mathbb{N}$ and $\delta > 0$ such that if $Y_0, \ldots, Y_{M-1}$ are independent random variables taking values $0, \ldots, d-1$ so that for any $i < M$ and $k < d$

$$|\Pr(Y_i = k) - \frac{1}{d}| < \delta, \tag{4.12}$$

then

$$\Pr\big(|\{i < M : Y_i = 0\}| + 2m > \max_{0 \leq k < d} |\{i < M : Y_i = k\}|\big) < \frac{1}{d} + \Delta. \tag{4.13}$$

Proof of Claim. Let $X_0, X_1, \ldots$ be a sequence of independent random variables with

$$\Pr(X_i = k) = \frac{1}{d} \tag{4.14}$$

for $k = 0, \ldots, d-1$. For each i define a random variable Z_i with values in $\mathbb{R}^d$ by letting $Z_i = e_{X_i}$, where $e_0, \ldots, e_{d-1}$ is the standard orthonormal basis in $\mathbb{R}^d$. Put $S_M = \sum_{i<M} Z_i$. Note that

$$\begin{aligned} &\Pr\big(|\{i < M : X_i = 0\}| + 2m > \max_{0 \leq k < d} |\{i < M : X_i = k\}|\big) \\ &= \Pr\big(\frac{S_M}{M} \cdot e_0 + \frac{2m}{M} > \max_{0 \leq k < d} \frac{S_M}{M} \cdot e_k\big), \end{aligned} \tag{4.15}$$

where $\cdot$ is the ordinary inner product. For a fixed m, $2m/M$ goes to 0 faster than $1/\sqrt{M}$ and the distribution of each Z_i is invariant under the orthonormal transformations of $\mathbb{R}^d$ induced by any permutation of the vectors e_i; thus, by the central limit theorem [1, Theorem 29.5], for large M not depending on the particular sequence $X_0, X_1, \ldots$, the second term in (4.15) is $< \frac{1}{d} + \frac{\Delta}{2}$. Therefore,

$$\Pr\big(|\{i < M : X_i = 0\}| + 2m > \max_{0 \leq k < d} |\{i < M : X_i = k\}|\big) < \frac{1}{d} + \frac{\Delta}{2}. \tag{4.16}$$

Fix M as above. Let $\delta = \frac{\Delta}{2M}$. Let $Y_0, \ldots, Y_{M-1}$ be independent random variables fulfilling (4.12). It is easy to see that given $X_0, \ldots, X_{i-1}$ each fulfilling (4.14) and such that $X_0, \ldots, X_{i-1}, Y_i, \ldots, Y_M$ are independent, there is X_i fulfilling (4.14) such that $X_1, \ldots, X_{i-1}, X_i, Y_{i+1}, \ldots, Y_{M-1}$ are independent and $\Pr(|X_i - Y_i| \neq 0) < \delta$. Note now that the probability that one of the events $|\{i < M : X_i = k\}| \neq |\{i < M : Y_i = k\}|$ happens with $0 \leq k < d$ is $< \delta \cdot M < \Delta/2$. Thus, it follows from (4.16) that (4.13) holds. □

Now we are ready to finish proving Lemma 4.3. Let $\epsilon > 0$ be given. Let $\delta_{-1} = \epsilon$. If $\delta_i > 0$ is defined, using Claim with $m = 2^i$ and $\Delta = \delta_i$, find $M_{i+1} \in \mathbb{N}$ and $\delta_{i+1} > 0$. This describes a sequence $(M_i)_{i=0}^{\infty}$ along with an auxiliary sequence of positive real numbers $(\delta_i)_{i=-1}^{\infty}$.

Fix k. We show by backward induction on $i_0 \leq k$ that for any $\sigma_0 \in \bigcup_{i<k} M_0 \times \cdots \times M_i$ with $|\sigma_0| = i_0$

$$|\{f \in d^{M_0\times\cdots\times M_k} : \exists \bar{x} \in \mathcal{X}_k\ f^{i_0}_{\bar{x}}(\sigma_0) = 0\}| < (\frac{1}{d} + \delta_{i_0-1}) d^{M_0\cdots M_k}.$$

For $i_0 = 0$ this gives the conclusion of the lemma. The estimate clearly holds for $i_0 = k$. Assume this is true for all $\sigma_0^\frown l$ with $l \in M_{i_0}$. Define Y_l, $l < M_{i_0}$, to be independent random variables such that $Y_l = 0$ with probability

$$\frac{1}{d^{M_0\cdots M_k}} |\{f \in d^{M_0\times\cdots\times M_k} : \exists \bar{y} \in \mathcal{X}_k\ f^{i_0+1}_{\bar{y}}(\sigma_0^\frown l) = 0\}|,$$

which by inductive assumption is $< \frac{1}{d} + \delta_{i_0}$. Obviously, it is $\geq \frac{1}{d}$. Further, let $\Pr(Y_l = i) = \Pr(Y_l = i')$ for $1 \leq i, i' < d$. It follows that for all $i < d$

$$\frac{1}{d} - \delta_{i_0} \leq \Pr(Y_l = i) < \frac{1}{d} + \delta_{i_0}. \tag{4.17}$$

Fix $\bar{x} \in \mathcal{X}_k$ for a moment. Note now that from the definition of f^{i_0} in the following list each condition implies the next one:

$$f^{i_0}_{\bar{x}}(\sigma_0) = 0 \tag{4.18a}$$

$$|\{l : f^{i_0+1}_{\bar{x}}(\sigma_0^\frown l) = 0\}| + 2^{i_0} > \max_{i>0} |\{l : f^{i_0+1}_{\bar{x}}(\sigma_0^\frown l) = i\}| - 2^{i_0} \tag{4.18b}$$

$$\begin{aligned} &|\{l : \exists \bar{y} \in \mathcal{X}_k\ f^{i_0+1}_{\bar{y}}(\sigma_0^\frown l) = 0\}| + 2 \cdot 2^{i_0} \\ &\quad > \max_{0<i<d} |\{l : \forall \bar{y} \in \mathcal{X}_k\ f^{i_0+1}_{\bar{y}}(\sigma_0^\frown l) > 0 \text{ and } f^{i_0+1}_{\bar{x}}(\sigma_0^\frown l) = i\}|. \end{aligned} \tag{4.18c}$$

The left side of (4.18c) does not depend on $\bar{x}$. We analyze now the right side of (4.18c). The set of all $f \in d^{M_0\times\cdots\times M_k}$ with $\forall \bar{y} \in \mathcal{X}_k\ f^{i_0+1}_{\bar{y}}(\sigma_0^\frown l) > 0$ does not depend on $\bar{x}$. Furthermore, in this set, for a given $\bar{x}$, the fraction of functions f with $f^{i_0+1}_{\bar{x}}(\sigma_0^\frown l) = i$ is the same for all $0 < i < d$, and this fraction does not depend on $\bar{x}$. Thus, the implications of (4.18) give the first inequality in the formula below; the second inequality follows from (4.17) and from the choice of δ_{i_0} and M_{i_0}.

$$\begin{aligned} &\frac{1}{d^{M_0\cdots M_k}} |\{f \in d^{M_0\times\cdots\times M_k} : \exists \bar{x} \in \mathcal{X}_k\ f^{i_0}_{\bar{x}}(\sigma_0) = 0\}| \\ &\qquad \leq \Pr(|\{l < M_{i_0} : Y_l = 0\}| + 2 \cdot 2^{i_0} > \max_{i<d} |\{l < M_{i_0} : Y_l = i\}|) \\ &\qquad < \frac{1}{d} + \delta_{i_0-1}, \end{aligned}$$

Thus, we obtain our inductive conclusion. □

Proof of Theorem 4.2. By our assumption H is a compact group that is not connected, hence it contains a proper open subgroup V. The index of V in H,

$$d = [H : V],$$

is finite. Pick (M_i) so that the conclusion of Lemma 4.3 holds with d and some $0 < \epsilon_0 < 1/2$. The submeasure ψ_0 is as in the paragraph preceding the statement of Theorem 4.2.

Point (i) is immediate since for each $n \in \mathbb{N}$, $\prod_i M_i$ is covered by $\{[\sigma] : |\sigma| = n\}$, and $\psi_0([\sigma]) \leq 2^{-n}$ if $|\sigma| = n$.

We prove (ii). We aim to apply Lemma 4.1. Let K_n consist of all elements of $S(\psi_0, H)$ that are constant on all $[\sigma]$ for $\sigma \in \Sigma$ with $|\sigma| = n+1$. Clearly K_n is a compact group, $K_n \subseteq K_{n+1}$, and $\bigcup_n K_n = S(\psi_0, H)$ is dense in $L_0(\psi_0, H)$.

Let F be a finite subset of H intersecting each left coset of V. Let F' be the finite set of constant functions in $S(\psi_0, H)$ with values in F. Let $V' \subseteq L_0(\psi_0, H)$ be an open set whose intersection with $S(\psi_0, H)$ gives the relatively open in $S(\psi_0, H)$ set

$$\{f \in S(\psi_0, H) : \psi_0(\{x : f(x) \notin V\}) < 1/2\}.$$

Using Lemma 4.3, we will find Borel sets $A_n \subseteq K_n$ such that

$$F' A_n = K_n \ \text{ and } \ \nu_n(V' A_n) \leq \frac{1}{d} + \epsilon_0, \tag{4.19}$$

where ν_n is the Haar probability measure on K_n. This clearly implies that (4.1) fails since $\epsilon_0 < 1 - \frac{1}{d}$, and therefore $L_0(\psi_0, H)$ is not Lévy. We identify H/V with d. Since each $f \in K_n$ is constant on each $[\sigma]$ for $\sigma \in \Sigma$ with $|\sigma| = n+1$, each such f induces a function $\hat{f} : M_0 \times \cdots \times M_n \to d$ by $\hat{f}(\sigma) = f(x)/V$ for any $x \in [\sigma]$, $\sigma \in M_0 \times \cdots \times M_n$. Recall the notation from Lemma 4.3. Let $\bar{0}$ be the element of $\mathcal{X}_n$ with all its entries equal to the empty set. The set

$$A_n = \{f \in K_n : \hat{f}^0_{\bar{0}}(\emptyset) = 0\}$$

is clearly Borel and $F' A_n = K_n$. It remains to check the second part of (4.19). Let $h \in V' A_n \cap K_n$. Pick $f \in A_n$ such that $h \in V' f$. We have

$$\psi_0\big(\bigcup\{[\sigma] : \sigma \in \Sigma,\, |\sigma| = n+1, \text{ and } \hat{f}(\sigma) \neq \hat{h}(\sigma)\}\big) < \frac{1}{2}.$$

Let $I \subseteq \Sigma$ be such that

$$\begin{aligned} \bigcup\{[\sigma] : \sigma \in \Sigma,\, |\sigma| = n+1 \text{ and } \hat{f}(\sigma) \neq \hat{h}(\sigma)\} \subseteq \bigcup\{[\tau] : \tau \in I\} \\ \text{and } \sum_{\tau \in I} 2^{-|\tau|} < \frac{1}{2}. \end{aligned} \tag{4.20}$$

Note that the condition in Lemma 4.3 guarantees that $M_k \geq 2^k$ for each k. With this in mind, we see that the second part of (4.20) gives that if $|\sigma| = n+1$, then $[\sigma] \neq \bigcup\{[\tau] : \tau \in I,\, \sigma \subseteq \tau,\, \sigma \neq \tau\}$ allowing us to remove from I any τ with $|\tau| > n+1$. Thus, we assume that $|\tau| \leq n+1$ for each $\tau \in I$. For $\sigma \in \bigcup_{i<n} M_0 \times \cdots \times M_i$ let

$$x_\sigma = \{l < M_{|\sigma|} : \sigma^\frown l \in I\}.$$

Note that by the second part of (4.20) we have $|x_\sigma| < 2^{|\sigma|}$. So $\bar{x} = (x_\sigma : \sigma \in \bigcup_{i<n} M_0 \times \cdots \times M_i) \in \mathcal{X}_n$. By the first part of (4.20), we have

$$\hat{h}^0_{\bar{x}}(\emptyset) \leq \hat{f}^0_0(\emptyset) = 0.$$

Thus,

$$V' A_n \cap K_n \subseteq \{h \in K_n : \exists \bar{x} \in \mathcal{X}_n \; \hat{h}^0_{\bar{x}}(\emptyset) = 0\}. \tag{4.21}$$

Since the map $K_n \ni h \to \hat{h} \in d^{M_0 \times \cdots \times M_n}$ is such that for any $D \subseteq d^{M_0 \times \cdots \times M_n}$ we have

$$\nu_n(\{h \in K_n : \hat{h} \in D\}) = \frac{|D|}{d^{M_0 \cdots M_n}},$$

it follows from (4.21) that

$$\nu_n(V' A_n) \leq \frac{1}{d^{M_0 \cdots M_n}} |\{g \in d^{M_0 \times \cdots \times M_n} : \exists \bar{x} \in \mathcal{X}_n \; g^0_{\bar{x}}(\emptyset) = 0\}|,$$

which is $< \frac{1}{d} + \epsilon_0$ by Lemma 4.3. □

Propositions 4.5, 4.6 and 4.7 below complement Theorem 4.2.

A *non-Lévy sequence* for G is an increasing sequence of compact subgroups with dense union that is not a Lévy sequence. Below we prove two propositions on the existence of non-Lévy sequences. In the first one, we show that groups $L_0(\phi, H)$ with H compact disconnected and ϕ an arbitrary diffused submeasure contain non-Lévy sequences despite the fact that some of them are Lévy groups as proved in [16] or in Lemma 4.7.

We start with a lemma on constructing non-Lévy sequences. For $\epsilon > 0$ and a subset A of a group G with a right invariant metric d, we write $(A)_\epsilon = \{g \in G : d(g,h) < \epsilon \text{ for some } h \in A\}$.

Lemma 4.4. *Assume a metric group G with a right invariant metric d is the increasing union of compact subgroups K_n, $n \in \mathbb{N}$. Assume there are $\varepsilon > 0$ and compact groups L_n, $n \in \mathbb{N}$ such that $K_n < L_n$, $\sup_n [L_n : K_n] < \infty$, and $L_n \not\subseteq (K_n)_\epsilon$. Then G can be written as an increasing union of a non-Lévy sequence.*

Proof. Our assumption $\sup_n [L_n : K_n] < \infty$ guarantees that for each n there is n' with $L_n \subseteq K_{n'}$. By going to a subsequence, we can ensure two things, first that $L_n \subseteq K_{n+1}$, and so $L_n \subseteq L_{n+1}$, and second that for all n, $[L_n : K_n] = m$ for some fixed $m \in \mathbb{N}$.

If ν_n is the Haar measure on L_n, we have $\nu_n(K_n) = 1/m$. Further, since d is right invariant, if $L_n \not\subseteq (K_n)_\epsilon$, then a left coset of K_n in L_n is disjoint from

$(K_n)_\epsilon$. Thus, $\nu_n((K_n)_\epsilon) \leq 1 - (1/m)$. It follows that (L_n) is an increasing sequence of compact subgroups whose union is G and which is not Lévy. □

Proposition 4.5. *The group $L_0(\phi, H)$ for a diffused submeasure ϕ and a compact disconnected Abelian group H contains a non-Lévy sequence.*

Proof. Fix a diffused ϕ on an algebra $\mathcal{B}$ and a compact disconnected group H with an invariant metric d_H. There is $\alpha > 0$ such that for every finite subalgebra $\mathcal{A}$ of $\mathcal{B}$ there is $b \in \mathcal{B}$ satisfying $\phi(a \Delta b) > \alpha$ for all $a \in \mathcal{A}$. Otherwise, the group $S(\phi, \mathbb{Z}/2)$ is totally bounded with respect to d_ϕ, hence its completion $L_0(\phi, \mathbb{Z}/2)$ is compact, contradicting its extreme amenability. Since H is disconnected, it has a closed normal subgroup G of finite index m. By the compactness of H there is $0 < \varepsilon < \alpha$ such that $d_H(g, h) > \varepsilon$ for all g, h belonging to different cosets of G. Let $\mathcal{B}_n$ be an increasing sequence of finite algebras with union $\mathcal{B}$ and let K_n be the subgroup of all $\mathcal{B}_n$-measurable maps. By Lemma 4.4 applied to $S(\phi, H)$, it will suffice to find $\varepsilon > 0$ such that for every n there is a compact group $L_n \supseteq K_n$ with $[L_n : K_n] = m$ and $(K_n)_\varepsilon \cap L_n = K_n$.

For each n find b_n such that $\phi(b_n \Delta a) > \varepsilon$ for all $a \in \mathcal{B}_n$. Let $\mathcal{B}'_n$ be the subalgebra generated by $\mathcal{B}_n$ and b_n. Then with $\pi\colon H \to H/G$ being the natural projection, $L_n = \{f \in S(\phi, H) : f$ is $\mathcal{B}'_n$-measurable and $f = f_1 + f_2$ with $f_1 \in K_n$ and $\pi \circ f_2$ measurable with respect to the subalgebra generated by $b_n\}$ is a compact subgroup of $S(\phi, H)$ including K_n. For $h \in H$ let $\tilde{h} \in L_0(\phi, H)$ be equal to h on b_n and equal to 1 on $b_n^{\complement}$. Then $\tilde{h} \in L_n$ and for $F \subseteq H$ such that $FG = H$ we have $\tilde{F} K_n = L_n$ with $\tilde{F} = \{\tilde{f} : f \in F\}$. Therefore $[L_n : K_n] = m$. If f and g in L_n belong to different cosets of K_n, then $\{x : d_H(f(x), g(x)) > \varepsilon\}$ is of the form $a \Delta b_n$ or $a \Delta b_n^{\complement}$ for some $a \in \mathcal{B}_n$, hence $d_\phi(f, g) > \varepsilon$. Hence K_n is a subset of L_n of Haar measure $1/m$ such that $(K_n)_\varepsilon \cap L_n = K_n$. □

The next proposition gives an application of Lemma 4.4 to groups other than $L_0(\phi, H)$. Both groups in this proposition are Lévy; see [17, §4].

Proposition 4.6. *The unitary group $U(\mathcal{H})$ of a separable infinite-dimensional Hilbert space (with the strong topology) and the group* $\mathrm{Aut}(X, \mu)$ *of measure preserving transformations of a standard measure space (with the weak topology) have non-Lévy sequences.*

Proof. Fix an orthonormal basis (e_n) such that $U(n)$ is the group of all operators that fix all vectors in the closed linear span of $\{e_i \mid i > n\}$. Let g_n be an involution defined by the permutation of the basis $\{e_i\}_{i=1}^\infty$ that swaps e_{n+1} with e_{n+2} and sends e_i to itself for $i \notin \{n, n+1\}$. Then $g_n \in U(n+2)$ and it clearly commutes with all elements of $U(n)$. Also, for a, b in $U(n)$ the vectors $g_n a(e_n) = e_{n+1}$ and $b(e_n) = e_n$ are orthogonal, and the conclusion follows by Lemma 4.4 applied to the dense subgroup $\bigcup_n U(n)$ with $K_n = U(n)$ and $L_n = K_n \cup g_n K_n$.

For $\mathrm{Aut}(X, \mu)$ it suffices to consider the case when $X = [0, 1]$ and μ is the restriction of the Lebesgue measure. Let K_n be the finite group of all transformations interchanging the intervals $I_{jn} = [j2^{-n}, (j+1)2^{-n})$ for $0 \leq j < 2^n$ by translations. Since the weak topology is the weakest topology making all functions $f \mapsto \mu(A \cap f[A])$, for $A \subseteq X$ measurable, continuous, by Lemma 4.4 applied to $\bigcup_n K_n$ it suffices to find $g_n \in K_{n+1}$ of order 2 which commutes with elements of K_n and is such that for some measurable J_n of measure 1/2 and all a, b in K_n we have $a[J_n] \cap g_n b[J_n] = \emptyset$. (Then $L_n = K_n \cup g_n K_n$ will do the job.) Let $\tau \in S_{2^{n+1}}$ be the permutation swapping $2j$ with $2j+1$ for all $j < 2^n$. Let $g_n \in K_{n+1}$ be the transformation moving each $I_{j,n+1}$ to $I_{\tau(j),n+1}$ by translations. Then $g_n \in K_{n+1}$ has order 2 and it commutes with all elements of K_n. If $J_n = \bigcup_{j<2^n} I_{2j,n+1}$ then $\mu(J_n) = 1/2$, $a[J_n] = J_n$ for all $a \in K_n$ and $g_n[J_n] = [0, 1] \setminus J_n$. □

In contrast with Theorem 4.2, the next result shows that there is a large class of diffused submeasures ϕ such that $L_0(\phi, H)$ is Lévy for any compact H.

A submeasure ϕ on an algebra $\mathcal{B}$ of subsets of X is called *strongly diffused* if for any $\epsilon > 0$ there exists $\delta > 0$ for which there exist partitions $\mathcal{P}$ of X into arbitrarily large number of sets from $\mathcal{B}$ such that for any $\mathcal{A} \subseteq \mathcal{P}$ if $|\mathcal{A}| < \delta \cdot |\mathcal{P}|$, then $\phi(\bigcup \mathcal{A}) < \epsilon$. Note that all diffused measures are strongly diffused. Many submeasures with no measures below are also strongly diffused, see the examples on page 98 of [20] with $\sigma_n \leq Cn$ for some finite constant $C \geq 2$.

Proposition 4.7. *Let ϕ be a strongly diffused submeasure.*

(i) *If H compact, then $L_0(\phi, H)$ is a Lévy group.*
(ii) *If H is an amenable locally compact second countable group, then $L_0(\phi, H)$ is extremely amenable.*

Proof. (i) Fix a metric d_ϕ on $S(\phi, H)$ as in (1.1). Let $\mathcal{P}$ be a partition of X into non-empty sets from $\mathcal{B}$. We will refer to such $\mathcal{P}$ simply as a partition. We write $H(\mathcal{P})$ for the subgroup of $S(\phi, H)$ consisting of all the functions constant on each set in $\mathcal{P}$. This group is isomorphic to H^k where k is the number of elements in $\mathcal{P}$ and by λ_k we will denote the probability Haar measure on it. Further, by d_k we will denote the Hamming metric on $H(\mathcal{P})$ given by

$$d_k(f, g) = \frac{1}{k} |\{a \in \mathcal{P} : f \restriction a \neq g \restriction a\}|.$$

We produce a Lévy sequence consisting of groups of the form $H(\mathcal{P})$ where $\mathcal{P}$ is a partition. If a partition $\mathcal{P}$ refines another partition $\mathcal{Q}$, then $H(\mathcal{Q}) < H(\mathcal{P})$. Thus, it will suffice to show that given a partition $\mathcal{Q}$ and $\epsilon > 0$, there is a partition $\mathcal{P}$ refining $\mathcal{Q}$ such that for any measurable $A \subseteq H(\mathcal{P})$ with $\lambda(A) > \epsilon$, we have $\lambda_m(\{f \in H(\mathcal{P}) : d_\phi(f, A) < \epsilon\}) > 1 - \epsilon$. Let $k = |\mathcal{Q}|$. Let $\delta > 0$ be as in the definition of strong diffusion chosen for the given $\epsilon > 0$. By concentration of measure (e.g., [9]), find $n_0 \in \mathbb{N}$ so that for any partition $\mathcal{P}$ with $|\mathcal{P}| = m \geq n_0$ and any $A \subseteq H(\mathcal{P})$ with $\lambda_m(A) \geq \epsilon$ we have

$$\lambda_m(\{f \in H(\mathcal{P}) : d_m(f, A) < \frac{\delta}{k}\}) > 1 - \epsilon. \tag{4.22}$$

This is possible since $H(\mathcal{P})$ is isomorphic to $H^{|\mathcal{P}|}$. Find a partition $\mathcal{P}'$ with at least n_0 elements such that for any $\mathcal{A} \subseteq \mathcal{P}'$ with $|\mathcal{A}| < \delta \cdot |\mathcal{P}'|$ we have $\phi(\bigcup \mathcal{A}) < \epsilon$. Let $\mathcal{P}$ be the partition induced by the elements of $\mathcal{P}' \cup \mathcal{Q}$. Put $m_0 = |\mathcal{P}|$. Note that $n_0 \leq m_0 \leq |\mathcal{Q}| \cdot |\mathcal{P}'|$.

Let $A \subseteq H(\mathcal{P})$ be measurable and such that $\lambda_{m_0}(A) > \epsilon$. Since (4.22) holds for $m = m_0$, it suffices to show that for $f \in H(\mathcal{P})$, $d_{m_0}(f, A) < \delta/k$ implies $d_\phi(f, A) < \epsilon$. Let $g \in A$ be such that $d_{m_0}(f, g) < \delta/k$. Then

$$|\{a \in \mathcal{P} : f \restriction a \neq g \restriction a\}| < \frac{\delta}{k} \cdot m_0 \leq \frac{\delta}{k} \cdot |\mathcal{Q}| \cdot |\mathcal{P}'| = \delta \cdot |\mathcal{P}'|.$$

It follows that $\bigcup\{a \in \mathcal{P} : f \restriction a \neq g \restriction a\}$ can be covered by at most $\delta \cdot |\mathcal{P}'|$ elements of the partition $\mathcal{P}'$. Thus, $\phi(\bigcup\{a \in \mathcal{P} : f \restriction a \neq g \restriction a\}) < \epsilon$, and therefore $d_\phi(f, A) < \epsilon$.

(ii) We obtain this point by combining the argument above with a use of Følner sequences as in [16, Theorem 2.2]. □

5. Remaining questions

Herer and Christensen [7] proved that $L_0(\phi, \mathbb{R})$ is extremely amenable if ϕ has no measure below. Pestov [16, Theorem 2.2] showed that $L_0(\mu, H)$ is extremely amenable if μ is a diffused measure and H is an amenable lcsc group. Neither of these results is implied by our Theorem 3.1. In fact, no group of the form $L_0(\phi, H)$ with H Abelian lcsc containing an element which generates a subgroup whose closure is not compact is covered by Theorem 3.1. Therefore, it is natural to ask for generalizations of this theorem as in questions 1 and 2 below.

1. Is every group of the form $L_0(\phi, H)$, for ϕ diffused and H compact, extremely amenable?

2. For what diffused ϕ is $L_0(\phi, \mathbb{Z})$, or $L_0(\phi, \mathbb{R})$, extremely amenable?

Question 1 is related to proving an analogue of Theorem 2.1, or at least Corollary 2.2, for an arbitrary finite group in place of $\mathbb{Z}/p$. In fact, it would suffice to prove it for simple finite groups. Similarly for Question 2. Hence the next question, the answer to which might involve extending the methods of [4].

3. Is there a Ramsey-type result along the lines of Theorem 2.1 or Corollary 2.2 for arbitrary finite (or simple finite) groups? Is there a Ramsey-type result analogous to Corollary 2.2 with $\mathbb{Z}$ replacing $\mathbb{Z}/p$?

A rudimentary argument shows that Corollary 2.2 with $k = 1$ and $d = 2$ and $\mathbb{Z}/p$ replaced by an arbitrary group remains true already with $l = 1$. However, the argument does not seem to generalize to the higher values of d.

Now, a question related to Theorem 4.2.

4. Let ϕ be a diffused submeasure, and let H be a compact connected second countable group. Is $L_0(\phi, H)$ Lévy?

In Lemma 4.1 we introduced the following property of Polish groups G.

(∗) For any increasing sequence (K_n) of compact subgroups of G with dense union, an open neighborhood V of 1, a compact set $F \subseteq G$, and Borel sets $A_n \subseteq K_n$, we have

$$\liminf_n \nu_n(VA_n) - \nu_n(FA_n) \geq 0,$$

where ν_n is the probability Haar measure on K_n, considered as a measure on G with support K_n.

In connection with the lemma, Pestov suggested the following question.

5. (Pestov) Let G be a Polish group containing an increasing sequence of compact groups with dense union. Assume G satisfies (∗) above. Is G Lévy?

The following proposition, communicated to us by the referee, shows that (∗) implies extreme amenability.

Proposition 5.1. *Let G be a Polish group that contains an increasing sequence of compact groups with dense union and satisfies* (∗). *Then G is extremely amenable.*

Proof. Fix an increasing sequence (K_n) of compact subgroups of G with dense union, and let ν_n be the normalized Haar measure on K_n. Fix $A \subseteq G$ such that $FA = G$ for some finite F, and let V be a symmetric neighborhood of the identity. Let $\overline{A}$ be the closure of A. (We consider the closure of A only because of questions of measurability.) Then (∗) implies

$$\limsup_n \nu_n(V(\overline{A} \cap K_n)) \geq \limsup_n \nu_n(F(\overline{A} \cap K_n)) = 1$$

We claim $V\overline{A}(\overline{A})^{-1}V$ is dense in G. Otherwise, there is $g \in \bigcup_n K_n$ for which $V\overline{A}$ and $gV\overline{A}$ are disjoint, and consequently $\limsup_n \nu_n(V(\overline{A} \cap K_n)) \leq \frac{1}{2}$.

Since V was arbitrary, $\overline{A}(\overline{A})^{-1}$ is dense in G and, therefore, so is AA^{-1}. It follows that G is extremely amenable by Pestov's characterization stated at the beginning of Section 3. □

Acknowledgement. We thank Jiří Matoušek and Vladimir Pestov for valuable comments.

References

[1] P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third edition, 1995.

[2] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[3] J. P. R. Christensen. *Topology and Borel structure*. North-Holland Mathematics Studies, 10. North-Holland Publishing Co., Amsterdam, 1974.

[4] M. de Longueville. Notes on the topological Tverberg theorem. *Discrete Math.*, 247(1-3):271–297, 2002.

[5] E. Glasner. On minimal actions of Polish groups. *Topology Appl.*, 85(1-3):119–125, 1998.

[6] M. Gromov and V. D. Milman. A topological application of the isoperimetric inequality. *Amer. J. Math.*, 105(4):843–854, 1983.

[7] W. Herer and J.P.R. Christensen. On the existence of pathological submeasures and the construction of exotic topological groups. *Math. Ann.*, 213:203–210, 1975.

[8] A. S. Kechris, V. G. Pestov, and S. Todorcevic. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geom. Funct. Anal.*, 15(1):106–189, 2005.

[9] M. Ledoux. *The concentration of measure phenomenon*. Mathematical Surveys and Monographs, 89. American Mathematical Society, Providence, RI, 2001.

[10] J. Matoušek. On the chromatic number of Kneser hypergraphs. *Proc. Amer. Math. Soc.*, 130(9):2509–2514, 2002.

[11] J. Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003.

[12] J. Nešetřil. Ramsey theory. In *Handbook of combinatorics*, pages 1331–1403. Elsevier, Amsterdam, 1995.

[13] O. Nikodým. Contribution à la théorie des fonctionnelles linéaires en connexion avec la théorie de la mesure des ensembles abstraits. *Mathematica (Cluj)*, 5:130–141, 1931.

[14] V. Pestov. On free actions, minimal flows, and a problem by Ellis. *Trans. Amer. Math. Soc.*, 350(10):4149–4165, 1998.

[15] V. Pestov. Some universal constructions in abstract topological dynamics. In *Topological dynamics and applications (Minneapolis, MN, 1995)*, Contemp. Math., 215, pages 83–99. Amer. Math. Soc., 1998.

[16] V. Pestov. Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups. *Israel J. Math.*, 127:317–357, 2002.

[17] V. Pestov. *Dynamics of infinite-dimensional groups: the Ramsey-Dvoretzky-Milman phenomenon*. University Lecture Series, 40. Amer. Math. Soc., 2006.

[18] C. A. Rogers. *Hausdorff measures*. Cambridge University Press, London, 1970.

[19] M. Talagrand. Maharam's problem. *Ann. of Math.* to appear.

[20] M. Talagrand. A simple example of a pathological submeasure. *Math. Ann.*, 252:97–102, 1980.

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 35, Belgrade

E-mail address: `ifarah@mathstat.yorku.ca`

Department of Mathematics, 1409 W. Green St., University of Illinois, Urbana, IL 61801, USA

E-mail address: ssolecki@math.uiuc.edu