ABSOLUTENESS FOR UNIVERSALLY BAIRE SETS AND THE UNCOUNTABLE I

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Cantor's Continuum Hypothesis was proved to be independent from the usual ZFC axioms of Set Theory by Gödel and Cohen. The method of forcing, developed by Cohen to this end, has lead to a profusion of independence results in the following decades. Many other statements about infinite sets, such as the Borel Conjecture, Whitehead's problem, and automatic continuity for Banach Algebras, were proved independent, perhaps leaving an impression that most nontrivial statements about infinite sets can be neither proved nor refuted in ZFC.[1] Moreover, some classical statements imply the consistency of ZFC and stronger theories, and by Gödel's incompleteness theorems the consistency of these statements with ZFC can be proved only by using strong axioms of infinity, so-called *large cardinal axioms*. A classical example is Banach's 'Lebesgue measure has a σ-additive extension to all sets of reals.' While it is fairly easy to find a model in which this is false and there is no known ZFC-proof of its negation, proving the consistency of this statement requires assuming the existence of a *measurable cardinal* ([32]).

A remarkable result was proved by Shoenfield ([31]): every statement of the form $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\phi(x, y)$, where all quantification in ϕ is over the natural numbers and all of its parameters are real numbers, is *absolute* between models of ZFC that are transitive and contain all countable ordinals. In this form Shoenfield's theorem is best possible, as it cannot even be improved by adding one more alternation of quantifiers ranging over $\mathbb{R}$. However, a corollary that the truth of any $\mathbf{\Sigma}^1_2$ statement (i.e., one of the above syntactical form) cannot be changed by forcing turned out to be susceptible to far-reaching generalizations. One of the more striking results in modern set theory is that the existence of suitable large cardinals implies that the theory of the inner model $L(\mathbb{R})$ (the smallest inner model of ZF, the usual axioms of Set Theory without the Axiom of Choice, containing all real numbers) cannot be changed by set forcing (see [13, 21]). In particular, a sentence with real parameters and any number of alterations of quantifiers ranging over $\mathbb{R}$, has a fixed truth value that cannot be changed by forcing. The impact of large cardinals on sets of reals goes well beyond $L(\mathbb{R})$ to imply absoluteness for certain canonical sets of reals, the *universally Baire* sets ([11]), as defined below. A remarkable consequence is that the existence of large cardinals outright implies that all sets of reals in $L(\mathbb{R})$, and indeed all universally Baire sets, share all the classical regularity properties of Borel sets such as Lebesgue measurability.

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[1]Shelah's pcf-theory witnesses that this impression is false even in the realm of cardinal arithmetic, a subject that was considered essentially closed after Easton's work in 1970.

The problem of precise calibration of the large cardinal axioms required to prove absoluteness for formulas of a given complexity is currently a very active field of research. In this paper we are taking a different line by proving forcing-absoluteness of sentences that are not statements of $L(\mathbb{R})$.

Forcing-absoluteness from large cardinals does not extend to $L(\mathcal{P}(\omega_1))$ (the smallest inner model of ZF containing all subsets of the first uncountable ordinal, ω_1) or to the closely related set-model $H(\aleph_2)$ (the set of all sets whose transitive closure is of cardinality not greater than $\aleph_1$), however. Theories of these models are highly susceptible to forcing. For example, the Axiom of Choice implies that the Continuum Hypothesis can be expressed as 'there exists a well-order of all reals all of whose proper initial segments are countable,' a statement of these structures. Large cardinals imply that this reformulation of CH is false in $L(\mathbb{R})$, since such a well-order provides a non-Lebesgue measurable subset of the plane, while Cantor's original reformulation, that every infinite set of reals is either countable or equinumerous with $\mathbb{R}$, holds in $L(\mathbb{R})$. Nevertheless, large cardinals do influence the theory of the uncountable, even in contexts free of metamathematics. Todorcevic ([37]) has shown that the existence of class many Woodin cardinals influences the structure of Banach spaces satisfying Namioka's 'generic continuity principle.' Todorcevic's conclusion was previously shown to fail in L by Namioka and Pol.

Also, a number of highly nontrivial objects in $H(\aleph_2)$ have been constructed without the aid of additional set-theoretic axioms: from gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ of Hausdorff and Luzin, special Aronszajn trees, to the more recent negative partition relations ([35]) and L-spaces ([26]). Statements asserting the existence of these objects are absolute simply because they are true. Lévy's absoluteness theorem ([23]) states that all Σ_1-statements, ones of the form $(\exists X)\phi(X)$ where all quantification in ϕ is over the members of X, are *absolute* between models of ZFC that are transitive, even allowing for parameters for real numbers. One consequence is that every Σ_1-statement true in the set-theoretic universe is already witnessed by a set in $H(\aleph_1)$ (the set of all sets whose transitive closure is at most countable), thus Σ_1-statements reflect to $L(\mathbb{R})$. An absoluteness result that is genuinely about ω_1 is a consequence of Keisler's completeness theorem for the logic $L_{\omega_1\omega}(Q)$ ([19]). It has been used to prove a couple of theorems for which no other proof, free of metamathematics, is known. Statements susceptible to Keisler's theorem are Σ_1-statements of $H(\aleph_2)$ expressed in an extension of the first-order logic that allows countably infinite conjunctions and has the quantifier 'there exists uncountably many.' These are not consequences of Lévy's theorem because the witness to the Σ_1 statement in question can be required to be uncountable. Describing the exact syntactical form of these statements is beyond this introduction (see §5). Keisler's theorem allows ϕ to contain predicates for arbitrary Borel sets, and a desire to add arbitrary universally Baire predicates to these absoluteness results was the starting point of our investigation. As an application, we prove that for every suitably definable partition of n-tuples of reals the statement 'there exists an uncountable homogeneous set' is forcing-absolute (Proposition 7.3) and therefore all suitably defined ccc forcings are productively ccc.

While the majority of absoluteness results depend on the syntactical form of the sentence in question, we also prove a forcing-absoluteness conditioned on a semantical assumption. Statements $(\exists X)\phi(X)$ of $H(\aleph_2)$ for which a witness $\bar{X}$ can be forced so that it is *indestructible*, in the sense that $\phi(\bar{X})$ holds in all further extensions in which ω_1 is not collapsed, are forcing absolute (Theorem 2.2).

Our results can be expressed in the following form: if a sufficiently large rank initial segment of the universe of sets can be forced to satisfy a sentence ϕ, then there exists a model of ϕ already which has certain correctness properties. The properties we consider in this paper are the following.

(1) Containing any given $\aleph_1$ many reals.
(2) Correctness about stationarity for subsets of ω_1.
(3) Correctness about any given universally Baire set of reals.

The corresponding result for (1) above is essentially Woodin's Σ^2_1-absoluteness result from CH ([40], see also [21]). After completing this work we have learned that the results for parts (2) and (3) were previously known to Woodin as well. In a sequel to this paper [10] we will give some additional properties.

Let us say a word about the proofs. A Σ^2_1-*statement* is the one of the form $(\exists A \subseteq \mathbb{R})\phi(A)$, where all quantifiers in ϕ range over real numbers and ϕ can have reals as parameters. Note that CH is an example of a Σ^2_1 statement. Assuming suitable large cardinals (stronger than the ones used for forcing-absoluteness of the theory of $L(\mathbb{R})$), all Σ^2_1 statements are absolute between forcing extensions satisfying CH. Moreover, every Σ^2_1 statement that holds in some forcing extension (even those in which CH is false) holds in every forcing extension satisfying CH. One of the methods used in this note is extracted from Woodin's proof; without assuming CH and applying Woodin's construction we show that certain Σ_1 statements true in the extension also hold in the ground model. We also use this method to prove an extension of Zapletal's results on isolating cardinal invariants [42], originally proved by using the Σ^2_1 absoluteness theorem. Woodin's work on the partial order $\mathbb{P}_{max}$ ([41]), and in particular the notion of an *iterable model* is used to show similar absoluteness results from a more modest large cardinal assumptions.

Organization of the paper. In §1 we introduce universally Baire sets and present a corollary to Woodin's construction of $\mathbb{P}_{\rm max}$. In §2 we use iterable structures to prove Σ_1-absoluteness for $H(\aleph_2)$. In §3 we present a proof of absoluteness for universally Baire sets between generic ultrapowers, due to John Steel. In §4 the proof of Σ^2_1 absoluteness is modified to give Σ_1 absoluteness for $H(\aleph_2)$ for models containing any $\aleph_1$ many reals. In §5 we enrich Keisler's theorem by adding predicates for universally Baire sets of reals and the nonstationary ideal. Zapletal's results on isolating cardinal invariants are extended in §6. Applications are given in §7.1 and in §7.2 we prove that some (rather weak) large cardinal assumptions are necessary for some of our results. A related result of John Steel is briefly discussed in §7.3.

Terminology. We are using the standard set-theoretic terminology. Definitions of more advanced notions such as Woodin cardinals, $\mathbb{P}_{\rm max}$, or the stationary towers $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$ can be found e.g., in [41] or [21]. For the general theory of large cardinals see also [17].

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1. Universally Baire sets and $\mathbb{P}_{\max}$

We consider the reals $\mathbb{R}$ to be ω^ω, the set of all ω-sequences of elements of ω equipped with the product topology. If γ is an ordinal, a tree on $\omega \times \gamma$ is a subset of $(\omega \times \gamma)^{<\omega}$ closed under initial segments. If T is a such a tree, the *projection* $p[T]$ of T is the set of $x \in \omega^\omega$ for which there exists a $z \in \gamma^\omega$ such that $(x{\restriction}n, z{\restriction}n)$ is in T for each $n \in \omega$.

For a given cardinal κ, a set of reals A is κ-*universally Baire* if there exist trees T and S such that $p[T] = A$, $p[S] = \omega^\omega \setminus A$ and T and S project to complements in every forcing extension by a partial order of cardinality κ (see [11]). A set is *universally Baire* if it is κ-universally Baire for each cardinal κ. A universally Baire set of reals then has a natural reinterpretation in every set forcing extension, the projection of T, for any T as above (it is a standard fact that this does not depend on the choice of T). This allows us to talk about properties of the set in these forcing extensions. If A is universally Baire, P is a partial order and $G \subset P$ is a V-generic filter, we denote the reinterpretation of A in $V[G]$ as A_G or A^{V^P}.

Universally Baire sets of reals, introduced in [11], tend to satisfy the standard regularity properties. For example, in the presence of a proper class of Woodin cardinals the model $L(A, \mathbb{R})$ satisfies AD^+ whenever A is universally Baire ([41]). More specifically, in the presence of a proper class of Woodin cardinals universally Baire sets are ∞-homogeneously Suslin (see [25, 34, 21]), a property from which the usual regularity properties are more easily derived. We let Γ_{uB} denote the set of universally Baire sets of reals, and let NS_{ω_1} denote the nonstationary ideal on ω_1.

In [41] Woodin introduces the partial order $\mathbb{P}_{\max}$ and proves the following theorem (see also [22]).

Theorem 1.1. *Suppose that there exist proper class many Woodin cardinals, and let A be a universally Baire set of reals. Let ϕ be a Π_2 formula in the language of set theory with additional unary predicate symbols for NS_{ω_1} and A. Then if in some set forcing extension the structure*

$$\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A^*, \in\rangle$$

satisfies ϕ, where A^* is the induced reinterpretation of A, then the structure

$$\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A, \in\rangle$$

satisfies ϕ in the $\mathbb{P}_{\max}$ extension of $L(A, \mathbb{R})$. □

The conclusion of Theorem 1.1 can be interpreted as saying that $L(A, \mathbb{R})$ satisfies the statement that the structure $\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A, \in\rangle$ of its $\mathbb{P}_{\max}$ extension satisfies ϕ. In particular, the conclusion is still meaningful if no $L(\mathbb{R})$-generic filter for $\mathbb{P}_{\max}$ exists. Since $\mathbb{P}_{\max}$ adds no reals, it is not necessary to reinterpret A in the conclusion of the theorem (i.e., we can write A instead of A^*).

Theorem 1.1 has the following immediate corollary.

Corollary 1.2. *Suppose that there exist proper class many Woodin cardinals, and let A be a universally Baire set of reals. Let ϕ be a sentence in the language of set theory with two additional unary predicate symbols for NS_{ω_1} and A, and suppose that ϕ and $\neg\phi$ are provably equivalent to Π_2 sentences over $H(\aleph_2)$ in the same expanded language. Then either ϕ holds in all set forcing extension or ϕ fails in all forcing extensions.*

Proof. Otherwise ϕ and $\neg\phi$ both hold in the $\mathbb{P}_{\max}$ extension of $L(A, \mathbb{R})$, by Theorem 1.1. □

The results mentioned here hold for the language with a symbol for each universally Baire set, since each sentence can have only finitely many such symbols, and finitely many (indeed, countably many) universally Baire sets are easily combined into one.

2. Σ_1-absoluteness for $H(\aleph_2)$

If ϕ is a Boolean combination of Σ_1 sentences, then ϕ and $\neg\phi$ are clearly both Π_2. It follows from Corollary 1.2 that in the presence of suitable large cardinals, the truth value of such ϕ cannot be changed by set forcing, even if ϕ contains additional predicates for NS_{ω_1} and any given universally Baire set. Absoluteness for Boolean combinations of Σ_1 sentences follows immediately from absoluteness for Σ_1 sentences, and, in fact, the only meaningful sentences ϕ we know of for $H(\aleph_2)$ with the property that ϕ and $\neg\phi$ are provably equivalent to Π_2 sentences are such Boolean combinations (note that while CH can be expressed as a Π_2 sentence in V, it cannot be so expressed in $H(\aleph_2)$). While the proof of Theorem 1.1 is too long to present here (and in any case is presented in full in [41, 22]), we can present a self-contained proof of Σ_1-absoluteness from a slightly weaker large cardinal assumption.

Theorem 2.1. *Suppose that there exist two Woodin cardinals $\delta < \lambda$, and let A be a λ^+-universally Baire set of reals. Let ϕ be a Σ_1 sentence in the expanded language with two additional unary predicates. Let P be a partial order in V_δ. Then ϕ holds in the structure*

$$\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A, \in\rangle$$

if and only if ϕ holds in

$$\langle H(\aleph_2)^{V^P}; \mathrm{NS}_{\omega_1}^{V^P}, A^{V^P}, \in\rangle,$$

where A^{V^P} is the induced reinterpretation of A in the P-extension.

Proof. This will follow immediately from Theorem 2.8 proved below. □

Note the similarity with Levy's absoluteness theorem. The similar-looking assertion that $H(\aleph_2) \prec_{\Sigma_1} H(\aleph_2)^P$ is much stronger (it outright implies that P preserves $\aleph_1$). By [3] (also see [33]), familiar bounded forcing axioms such as MA or BPFA are equivalent to the assertion that $H(\aleph_2) \prec_{\Sigma_1} H(\aleph_2)^P$ for all P in the corresponding class of forcings. Reflection principles for $H(2^{\aleph_0})$ in relation to fragments of second order logic were studied already in the early 1980s, in a paper that has appeared only recently ([33]). Note that Theorem 2.1 cannot be extended to $H(\aleph_3)$ since the non-absolute '$2^{\aleph_0} \geq \aleph_2$' is a Σ_1 statement in this structure.

One way to state the idea behind the proof of Theorem 2.1 is that any sentence (with a parameter for some universally Baire set of reals) that can be forced to hold holds in some $\mathbb{P}_{\max}$ condition (Lemma 2.6) which can then be embedded elementarily into an uncountable structure which is correct about NS_{ω_1} and the universally Baire set A. Almost all of the details of the proof appear in [41, 22], in proving that certain types of $\mathbb{P}_{\max}$ conditions exist. This proof gives another result of somewhat different nature, in which the assumption on the absolute formula ϕ is semantical, rather than syntactical.

2.1. **Indestructible witnesses.** For certain formulas ϕ there exists a formula ψ (possibly ϕ itself) such that one can prove that whenever a set X satisfies ψ in a model N, X satisfies ϕ in any model $M \supseteq N$ such that $\mathrm{NS}_{\omega_1}^M \cap N = \mathrm{NS}_{\omega_1}^N$. In this case we say that X is an *indestructible witness* for ϕ. This property is at least as strong as continuing to be a witness in all NS_{ω_1}-preserving forcing extensions. Theorems proving that X is an indestructible witness essentially show that that the first-order properties of set X imply its (possibly second-order) property asserted by ϕ. Typical indestructible witnesses are Hausdorff/Luzin gaps and special Aronszajn trees. In the first case, 'zapping' between two orthogonal ω_1 towers in $\mathcal{P}(\omega)/\mathrm{Fin}$ assures that they cannot be separated by a single subset of ω (see [36, Theorem 8.6]). In the second case, an ω_1-tree is the union of countably many antichains, hence every cofinal branch defines a map from ω onto the height of the branch. Other standard constructions of objects of $H(\aleph_2)$ usually yield indestructible witnesses ([35], [26]).

Theorem 2.2. *If $(\exists x)\phi(x)$ is a formula of $H(\aleph_2)$ such that some forcing of size less than a Woodin cardinal forces an indestructible witness for it, then $(\exists x)\phi(x)$ holds in $H(\aleph_2)$.*

Proof. Readers interested only in the proof of this theorem can skip all references to universally Baire and weakly homogeneously Suslin sets and read only the beginning of §2.3 and the relevant part of the proof of Lemma 2.6. By Lemma 2.6 (with $A = \emptyset$) there is an iterable model $(N, \mathrm{NS}_{\omega_1}^N)$ and $X \in H(\aleph_2)^N$ such that $N \models \phi(X)$ and X is indestructible. Let $j\colon (N, \mathrm{NS}_{\omega_1}^N) \to (N^*, \mathrm{NS}_{\omega_1}^{N^*})$ be an iteration of length ω_1 such that $NS_{\omega_1} \cap N^* = \mathrm{NS}_{\omega_1}^{N^*}$. Then $Y = j(X)$ is an indestructible witness of ϕ in $H(\aleph_2)^{N^*} = H(\aleph_2) \cap N^*$ and therefore $H(\aleph_2) \models \phi(Y)$. □

Note that an anti-large cardinal assumption such as V=L implies the conclusion of Theorem 2.2 fails. One can strengthen Theorem 2.2 in an obvious way, by allowing parameters for universally Baire sets, adding an extra Woodin cardinal and appropriately weakening the indestructibility requirement. An axiom resembling the statement of Theorem 2.2 was studied by Hamkins in [15].

2.2. **Weakly homogeneously Suslin sets.** We refer the reader to [17, p. 453^+], [41, pages 22–23], [34, §1], or [21, Definitions 1.3.4 and 1.3.6] for the definition of "weakly homogeneously Suslin tree/set of reals." For every λ-weakly homogeneously Suslin tree T there exists a canonical tree S on $\omega \times (2^{|T|})^+$ such that $p[S] = \omega^\omega \setminus p[T]$ (see [21, p. 23]). This is the so-called Martin–Solovay tree (see [34, Theorem 1.20]). These trees S and T project to complements in every forcing extension by a partial order of cardinality less than λ, and therefore λ-weakly homogeneously Suslin sets are λ-universally Baire. Weakly homogeneously Suslin sets will figure prominently in §3. The following is due to Woodin.

Theorem 2.3. *Suppose that λ is a Woodin cardinal, and that A is a λ^+-universally Baire set of reals. Then A and $\omega^\omega \setminus A$ are both $< \lambda$-weakly homogeneously Suslin.*

Proof. This follows immediately from [21, Theorem 3.3.8]. □

The argument in the proof of the following is standard, and the theorem itself is a variation of standard facts. Since this exact statement does not seem to formally follow from the published results, we include the proof for the convenience of the reader.

Lemma 2.4. *Let θ be a regular cardinal, suppose that T is a weakly homogeneous tree on $\omega \times Z$ in $H(\theta)$, for some set Z. Let $\gamma \geq 2^\omega$ be an ordinal such that there exists a countable collection Σ of γ^+-complete measures witnessing the weak homogeneity of T. Assume that there is a measurable cardinal in the interval (γ, θ).*

Then for every elementary submodel X of $H(\theta)$ of cardinality less than γ with T, Σ, $\gamma \in X$, there is an elementary submodel Y of $H(\theta)$ containing X such that $Y \cap \theta$ is uncountable, $Y \cap \gamma = X \cap \gamma$ and $p[T \cap Y] = p[T]$.

Proof. Fixing θ, T, Σ and γ as in the statement of the theorem, the theorem follows from the following fact. Suppose that W is an elementary submodel of $H(\theta)$ of cardinality less than γ. Fix $x \in p[T]$ and a countably complete tower $\{\sigma_i : i < \omega\} \subset \Sigma$ such that for all $i < \omega$, $\{a \in Z^i : (x\restriction i, a) \in T\} \in \sigma_i$, and for each $i < \omega$, let $A_i = \bigcap(\sigma_i \cap W)$. Then since $\{\sigma_i : i < \omega\}$ is countably complete, there exists a $z \in Z^\omega$ such that for all $i < \omega$, $z\restriction i \in A_i$. Then the pair (x, z) forms a path through T, and, letting

$$W[z] = \{f(z\restriction i) \mid i < \omega \wedge f \colon Z^i \to H(\theta) \wedge f \in W\},$$

$W[z]$ is an elementary submodel of $H(\theta)$ containing W and $\{z\restriction i : i < \omega\}$, and, since each σ_i is γ^+-complete, $W \cap \gamma = W[z] \cap \gamma$.

Repeated application of this fact (starting with X) for each real in the projection of T produces a Y containing X such that $Y \cap \gamma = X \cap \gamma$, and $p[T \cap Y] = p[T]$. Furthermore, Y will have cardinality less than γ.

If $Y \cap \theta$ happens to be countable, then fix a measurable cardinal κ in Y greater than γ and a normal measure μ on κ. Then letting η be any member of $\bigcap(\mu \cap Y)$,

$$Y[\eta] = \{f(\eta) \mid f \colon \gamma \to H(\theta) \wedge f \in Y\}$$

is an elementary submodel of $H(\theta)$ end-extending Y below κ. Iterating this construction ω_1 times produces an elementary submodel Y' end-extending Y below κ with $Y' \cap \theta$ uncountable. □

The last paragraph of the preceding proof is needed only when the projection of T is analytic. Note also that the theorem implies an ostensibly stronger version where X is expanded to a Y capturing the projection of two weakly homogeneous trees in X, which in any case would be proved directly in the same way.

2.3. **Iterable models.** The proof of Theorem 2.8 will also use the following notion. ZFC^0 is a fragment of ZFC that holds in $H(\theta)$ for a regular $\theta \geq \aleph_2$ (the precise definition will not be needed here; see [22, §1]). If N_0 is a transitive model of ZFC^0 and I_0 is a normal ideal on $\omega_1^{N_0}$ in N_0, then an *iteration of* (N_0, I_0) *of length* γ is $\langle (N_\eta, I_\eta), j_{\xi\eta}, G_\eta, \xi < \eta \leq \gamma \rangle$, where $j_{\xi\eta} \colon (N_\xi, I_\xi) \to (N_\eta, I_\eta)$ is a commuting family of elementary embeddings, $G_\eta \subseteq (\mathcal{P}(\omega_1)/I_\eta)^{N_\eta}$ is a generic filter, $j_{\eta\eta+1}$ is the corresponding generic ultrapower embedding, and for a limit η and $\xi < \eta$, $j_{\xi\eta}$ and N_η are the direct limit of $j_{\xi\zeta}$ and N_ζ for $\xi < \zeta < \eta$. An iteration is *well-founded* if all the models occurring in it are well-founded. A pair is *iterable* if all of its iterations are well-founded. If $A \in N_0$ is a universally Baire set then a pair (N_0, I_0) is *A-iterable* if it is iterable and its iterations compute A correctly. We shall follow the standard convention and identify an *iteration* of length γ with the final model together with the embedding $j_{0\gamma} \colon (N_0, I_0) \to (N_\gamma, I_\gamma)$. For more information we refer the reader to [41, 22]. The term 'iterable model' is also used to describe inner models of ZFC equipped with strategies for iterating extender-based ultrapowers. This notion will not be used in the present paper.

We will also use the following fact, due to Woodin (see [12, 14, 30, 41]). For our purposes, it is only important that presaturated ideals are *precipitous*: the corresponding generic ultrapower is forced to be well-founded.

Lemma 2.5. *If δ is a Woodin cardinal, then after forcing with* $\mathrm{Coll}(\omega_1, {<}\delta)$ *the nonstationary ideal on ω_1 is presaturated.* □

Note that the following lemma also applies in the case when ϕ has additional predicates for NS_{ω_1} and a universally Baire set. The phrase "ϕ is preserved by forcings of the form $\mathrm{Coll}(\omega_1, \gamma)$" in the statement of Lemma 2.6 means that every forcing extension of a model M of $ZFC + \phi$ by $\mathrm{Coll}(\omega_1, \gamma)^M$ for some ordinal $\gamma \in M$ satisfies ϕ.

Lemma 2.6. *Assume $\delta < \lambda$ are a Woodin and a measurable cardinal, A and $\omega^\omega \setminus A$ are δ^+-weakly homogeneously Suslin set of reals, and ϕ is a sentence whose truth is preserved by forcings of the form* $\mathrm{Coll}(\omega_1, \gamma)$*. If P is a partial order in V_δ that forces ϕ, then there exists an A-iterable model $(N, \mathrm{NS}^N_{\omega_1})$ that satisfies ϕ.*

Proof. Let θ be a regular cardinal greater than δ, and let X be a countable elementary submodel of $H(\theta)$ with δ, λ, A and P in X. For some fixed $\gamma \in X \setminus \delta$ there are countable sets of γ^+-complete measures in X witnessing that A and $\omega^\omega \setminus A$ are weakly homogeneously Suslin. By Lemma 2.4 there is an elementary submodel Y of $H(\theta)$ such that

- $X \subset Y$,
- $X \cap \gamma = Y \cap \gamma$,
- $Y \cap \theta$ is uncountable,
- there exist trees S, T on $\omega \times \eta$ in X, for some ordinal $\eta \in \lambda \cap X$, such that $p[S \cap Y] = A$ and $p[T \cap Y] = \omega^\omega \setminus A$.

Now let M be the transitive collapse of Y, and let $\bar{P}$, $\bar{A}$, $\bar{S}$, $\bar{T}$, $\bar{\gamma}$ be the respective images of P, A, S, T and γ under this collapse. Noting that $V^M_{\bar{\gamma}}$ is countable, let N be a forcing extension of M by the forcing $\bar{P} * \mathrm{Coll}(\omega_1, {<}\bar{\delta})$ as defined in M. Let $A_N = p[\bar{S}]$. Since $\bar{P}$ forces $\phi(A_N)$, N satisfies ϕ. By Lemma 2.5 the nonstationary ideal in N is precipitous. Furthermore, $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable. Since $N \cap Ord$ is uncountable, [22, Lemma 1.5] implies $(N, \mathrm{NS}^N_{\omega_1})$ is iterable. Let $j \colon (N, \mathrm{NS}^N_{\omega_1}) \to (N^*, \mathrm{NS}^{N^*}_{\omega_1})$ be an iteration. The trees $\bar{S}$ and $\bar{T}$ project to complements in N (in V, in fact), and, from the point of view of V, $p[\bar{S}] \subset p[j(\bar{S})]$ and $p[\bar{T}] \subset p[j(\bar{T})]$. Since $N^* \models p[j(\bar{S})] \cap p[j(\bar{T})] = \emptyset$, and this fact is absolute, it follows that $j(A_N) = p[j(\bar{S})] \cap N^* = A \cap N^*$ and $j((\omega^\omega)^N \setminus A_N) = p[j(\bar{T})] \cap N^* = (\omega^\omega \setminus A) \cap N^*$. Therefore, N^* is correct about A, and we have proved $(N, \mathrm{NS}^N_{\omega_1})$ is A-iterable. □

It only remains to prove that the iterable structures are robust under small forcing.

Lemma 2.7. *Let δ be a cardinal and suppose that A is δ-universally Baire. If a pair (N, I) is A-iterable then it is A-iterable in any forcing extension by a partial order P in V_δ.*

Proof. Iterability is a Π^1_2 property and it is therefore absolute (see [22, Remark 1.3]). Let S and T be trees witnessing the δ-universally Baireness of A, with $p[S] = A$.

Since every real appearing in any iteration of length ω_1 appears in some countable initial segment of the iteration, it suffices to consider iterations which have countable length in some extension by a forcing in V_δ. Fix a well-founded iteration $j\colon (N, I) \to (N^*, \mathrm{NS}_{\omega_1}^{N^*})$. We need to check that $p[S] \cap j(N \cap (\omega^\omega \setminus A)) = p[T] \cap j(N \cap A) = \emptyset$. The existence of a real $z \in p[S] \cap j(N \cap (\omega^\omega \setminus A))$ is witnessed by a path x through S with first coordinate projection equal to z, a real y coding an iteration of (N, I) (of possibly illfounded ordertype) and an embedding π from the ordertype of this iteration into δ (i.e., into the ω_1 of the extension). There is a natural tree (with S as its first two coordinates and functions from ω into δ as its last coordinate) such that any such triple z, y, π corresponds to a path through the tree. Since the wellfoundedness of this tree is absolute between wellfounded models, we conclude $p[S] \cap j(N \cap (\omega^\omega \setminus A)) = \emptyset$. The argument for the nonexistence of a real $z \in p[T] \cap j(N \cap A)$, for j some wellfounded iteration of (N, I), is identical. □

Theorem 2.8. *Suppose that there exist two Woodin cardinals $\delta < \lambda$, and let A be a λ^+-universally Baire set of reals. Let ϕ be a Σ_1 sentence in the expanded language with two additional unary predicates. Then the following are equivalent.*

(1) ϕ holds in the structure $\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A, \in\rangle$.
(2) There is an A-iterable pair $(N, \mathrm{NS}_{\omega_1}^N)$ such that $N \models \phi$.
(3) ϕ holds in $\langle H(\aleph_2)^{V^P}; \mathrm{NS}_{\omega_1}^{V^P}, A^{V^P}, \in\rangle$ for some partial order P in V_δ.
(4) ϕ holds in $\langle H(\aleph_2)^{V^P}; \mathrm{NS}_{\omega_1}^{V^P}, A^{V^P}, \in\rangle$ for every partial order P in V_δ.

Proof. Clearly, (4) implies (1) and (1) implies (3), using the trivial forcing. Assume that (3) holds, and let $P \in V_\delta$ be such that every condition in P forces ϕ to hold. By Theorem 2.3, A and $\omega^\omega \setminus A$ are $<\lambda$-weakly homogeneous, and Lemma 2.6 implies (2).

Assume (2) holds, and let $(N, \mathrm{NS}_{\omega_1}^N)$ be an A-iterable model for ϕ. There is an iteration j of $(N, \mathrm{NS}_{\omega_1}^N)$ with final model N^* such that $\mathrm{NS}_{\omega_1}^{N^*} = N^* \cap \mathrm{NS}_{\omega_1}$ (this can be assured by a straightforward bookkeeping argument, see [41] or [22]). Since the iteration is correct about A, ω_1, and NS_{ω_1}, the witness for ϕ in N^* is a witness for ϕ in V, and therefore (1) holds.

Finally, assume (2). By Lemma 2.7, an A-iterable model in which ϕ holds remains A-iterable after forcing with any $P \in V_\delta$ and therefore we may apply the implication from (2) to (1) to conclude the proof of (4). □

The following weakening of (2): 'There is an A-iterable pair (N, I) such that $N \models \phi$' does not in general imply (1). Since in every ω_1-iteration $j\colon (N, I) \to (N^*, I^*)$ every set in I^* is disjoint from the critical sequence we have $\mathrm{NS}_{\omega_1} \cap N^* \supseteq I^*$. Therefore if ϕ states that $\mathrm{NS}_{\omega_1} \neq I$ then no iteration of (N, I) can be correct about NS_{ω_1}.

Remark 2.9. Theorem 2.8 (and other results in this paper) have natural reinterpretations in terms of Woodin's Ω-logic (see [39]). Briefly, given suitable large cardinals, there is a universally Baire set of reals A such that if M is an A-closed model (in the sense of [39]) then in M every set is a member of an inner model containing a Woodin cardinal below a measurable cardinal κ (of M), such that this inner model is iterable (in V) by some measure on κ in M. Given this fact, the arguments in this section show that if ϕ is a Σ_1 statement for $\langle H(\aleph_2), \mathrm{NS}_{\omega_1}, B, \in\rangle$ for a universally Baire set B then there is a universally Baire set A such that ϕ

holds in V if and only if it holds in some A-closed model. Such an A is a *proof* of ϕ in Ω-logic. Similar arguments are discussed in [22].

3. Weakly homogeneously Suslin sets and stationary tower embeddings

Reader's familiarity with stationary tower forcing $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$ as presented in [21] is assumed in this section. We want to extend the arguments in the previous section to arguments using the stationary tower. In this case, handling the predicates for universally Baire sets requires Theorem 3.1 below (see [21, Theorem 1.7.1 and Theorem 1.7.2] for more on this result, and §2.2 for the definitions).

Theorem 3.1 (Steel)**.** *Assume that T is a λ-weakly homogeneously Suslin tree for some strongly inaccessible cardinal λ, and that P is $\mathbb{Q}_{<\delta}$ or $\mathbb{P}_{<\delta}$ for some strongly inaccessible $\delta < \lambda$. If S is the Martin–Solovay tree such that $p[S] = \omega^\omega \setminus p[T]$, then the generic embedding $k\colon V \to M$ derived from P satisfies $k(S) = S$.*

This is [21, Theorem 3.3.17] and the case when T is λ-homogeneous is proved in [34, Corollary 4.6(1)]. Since the proof is only outlined in [21], we include some more details for the convenience of the reader. The proof relies on two lemmas, in which we assume λ, P, δ and k are as in Theorem 3.1.

Lemma 3.2. *Assume μ is a $|P|^+$-complete ultrafilter on $\kappa^{<\omega}$. Then P forces*

$$j_{k(\mu)} \restriction \mathrm{Ord} = j_\mu \restriction \mathrm{Ord}\,.$$

Proof. The proof closely follows the proof of [21, Lemma 1.7.3]. Let $M \subseteq V[G]$ be an inner model such that $k\colon V \to M$. Let

$$X = \{f\colon k(\kappa^{<\omega}) \to \mathrm{Ord}\,|f \in M\}$$
$$Y = \{g\colon \kappa^{<\omega} \to \mathrm{Ord}\,|g \in V\}.$$

Define $\pi\colon X \to Y$ via $(\pi(f))(s) = f(k(s))$. We claim the following.

(1) For each $f \in \mathrm{dom}(\pi)$ there exists $g\colon \kappa^{<\omega} \to \mathrm{Ord}$ in V and $A \in \mu$ such that $(\pi(f))(s) = g(s)$ for all $s \in A$.

(2) For each $g\colon \kappa^{<\omega} \to \mathrm{Ord}$ in V there exists an $A \in \mu$ and a $f \in \mathrm{dom}(\pi)$ such that $(\pi(f))(s) = g(s)$ for all $s \in A$.

Once proved, (1) and (2) will imply that $\pi^*\colon X/k(\mu) \to Y/\mu$ defined by $\pi^*([f]_{k(\mu)}) = [\pi(f)]_\mu$ is a bijection. We will then show that it is an order-isomorphism.

We work in V below some $p \in P$ throughout. To prove (1), fix an $f\colon k(\kappa^{<\omega}) \to \mathrm{Ord}$ and a P-name τ for f. For each $s \in \kappa^{<\omega}$ fix $p_s \leq p$ and γ_s such that $p_s \Vdash (\pi(\tau))(\check{s}) = \check{\gamma}_s$. Since μ is $|P|^+$-complete there exist $A \in \mu$ and $q \leq p$ such that for all $s \in A$ we have $q = p_a$. Define $g\colon \kappa^{<\omega} \to \mathrm{Ord}$ via $g(s) = \gamma_s$ for all $s \in \kappa^{<\omega}$. Then $q \Vdash (\pi(\tau))(\check{s}) = g(\check{s})$ for all $s \in A$.

To prove (2), fix a $g\colon \kappa^{<\omega} \to \mathrm{Ord}$ in V. For each $s \in \kappa^{<\omega}$ find $p_s \leq p$, $a_s \in P$ and $h_s\colon a_s \to \mathrm{Ord}$ such that $p_s \Vdash [\check{h}_s]_G = \check{g}(\check{s})$ (see [21, §2.2]). Since μ is $|P|^+$-complete there exist $A \in \mu$, $q \leq p$ and $a \in V_\delta$ such that for all $s \in A$ we have $q = p_s$ and $a = a_s$. Define $F\colon a \to \mathrm{Ord}^{\kappa^{<\omega}}$ by letting $(F(X))(s)$ be $h_s(X)$ when $s \in A$ and 0 otherwise. We wish to see that for each $s \in A$, q forces that $\pi([F]_G)(s)$ (which is $[F]_G(k(s))$ by definition) is equal to $g(s)$. This follows from the fact that q forces that h_s represents $g(s)$ in the embedding k, the constant function $X \mapsto s$ represents $k(s)$, and $F(X)(s) = h_s(X)$ for all $X \in a$.

In order to check π^* is order-preserving, we closely follow the proof of [21, Claim 3 on page 11] to prove the following.

(3) For all $A \subseteq \kappa^{<\omega}$ in V and $C \subseteq k(\kappa^{<\omega})$ in M such that $A \in \mu$ and $C \in k(\mu)$ we have $k[A] \cap C \neq \emptyset$.

Fix A and C and $p \in P$. By extending p if necessary we may find a p' below p such that $C = [h]_G$ for some $h\colon a \to \mu$ with $p' \leq a$. Since μ is $|P|^+$-complete, there is an $A \in \mu$ which is contained in $h(X)$ for all $X \in a$. Then $k(s) \in C$ for all $s \in A$.

Since (3) immediately implies π^* is order-preserving, we conclude π^* is an isomorphism. Fix an ordinal γ. Then $j_\mu(\gamma) = [g]_\mu$ for a function $g\colon \kappa^{<\omega} \to \mathrm{Ord}$ in V that is constantly equal to γ. Similarly, $j_{k(\mu)}(\gamma) = [f]_{k(\mu)}$ for $f\colon \kappa^{<\omega} \to \mathrm{Ord}$ in M that is constantly equal to γ. Since $\pi(f) = g$, the conclusion follows. □

Lemma 3.3. *Assume μ_0 is a λ-complete ultrafilter on κ^{i_0} and μ_1 is a λ-complete ultrafilter on κ^{i_1} that projects to μ_0. Then P forces*

$$j_{k(\mu_0)k(\mu_1)} \restriction \mathrm{Ord} = j_{\mu_0\mu_1} \restriction \mathrm{Ord}\,.$$

Proof. The proof is very similar to the proof of Lemma 3.2 and we omit it. □

Proof of Theorem 3.1. This is analogous to the proof of [21, Theorem 1.3.14], but using Lemma 3.3 instead of [21, Lemma 1.1.25]. As in [21, p. 23], S is the increasing union of trees S_γ ($\gamma < (2^{|T|})^+$) and each S_γ is defined using only $j_{\mu_i\mu_j} \restriction \gamma$, where μ_i ($i < \omega$) is an enumeration of λ-complete measures witnessing T is weakly homogeneous. By Lemma 3.3, $j_{j_E(\mu_i)j_E(\mu_j)} \restriction \mathrm{Ord} = j_{\mu_i\mu_j} \restriction \mathrm{Ord}$ for all i, j. Furthermore, S is definable from $|T|^+$ in $L(\{j_{\mu_i\mu_j} \restriction |T|^+ : i, j < \omega\})$, and so, since $j(|T|^+) = |T|^+$, $j(S) = S$. □

Theorem 3.4. *Assume there are class many Woodin cardinals. Let A be a universally Baire set in V and let $V[G]$ be a set forcing extension of V in which $j\colon V \to M$ is a V-generic elementary embedding obtained via forcing with $\mathbb{P}_{<\delta}$ or $\mathbb{Q}_{<\delta}$. Then in $V[G]$, $j(A) = A_G$.*

Proof. Fix an inaccessible λ greater than δ and the rank of the partial order for which G is a generic filter. By Theorem 2.3, A and its complement are respectively the projections of λ-weakly homogeneous trees T_0 and T_1 which project to complements in $V[G]$. By Theorem 3.1, we have trees S_0 and S_1 in V such that

- $p[S_0] = A$,
- $p[S_1] = \omega^\omega \setminus A$,
- the pairs (S_0, T_1) and (S_1, T_0) project to complements in $V[G]$,
- $j(S_0) = S_0$,
- $j(S_1) = S_1$.

By the absoluteness of well-foundedness of trees, $j(A) = (p[S_0])^M = A_G \cap M$. □

Corollary 3.5. *Assume there are class many Woodin cardinals and let A be a universally Baire set. Suppose $V[h]$ is a forcing extension of V, $j_0\colon V \to M_0$ and $j_1\colon V[h] \to M_1$ are stationary tower embeddings existing in some forcing extension of $V[h]$. Then $j_0(A) \cap M_1 = j_1(A) \cap M_0$.*

Proof. Let δ_i, G_i be a Woodin cardinal an generic corresponding to j_i for $i < 2$. Pick an inaccessible $\lambda > \max(\delta_0, \delta_1)$ and a λ-homogeneously Suslin tree S such that $A = p[S]$. By Theorem 3.4 the conclusion follows. □

Remark 3.6. Note that Theorem 3.4 (as well as corresponding variants of Corollary 3.5) remain true if $\delta < \zeta$ for an inaccessible ζ and $j \colon V_\zeta \to M$. This remark will be used in the proof of Theorem 4.1.

Remark 3.7. It is well-known that analogues of Theorem 3.4 and Corollary 3.5 hold for generic ultrapowers obtained from precipitous ideals. This is a consequence of an appropriate analogue of Theorem 3.1, also due to Steel (see [21, Theorem 1.7.1, also Theorem 1.7.3]).

4. Σ^2_1-absoluteness with predicates for universally Baire sets and NS_{ω_1}

Theorem 4.1 below subsumes Woodin's Σ^2_1-absoluteness theorem, and is proved in this section by a generalization of his proof. An 'elementary' proof (one without using the machinery of stationary tower forcing) can be obtained by modifying Todorcevic's proof of the Σ^2_1-absoluteness theorem (see [38], [8]). This proof uses the Lévy collapse followed by $\mathcal{P}(\omega_1)/\mathcal{I}$, for a saturated ideal $\mathcal{I}$ defined in the collapse. As Matt Foreman pointed out, an iteration of Lévy collapse and $\mathcal{P}(\omega_1)/\mathcal{I}$ for a saturated I is regularly embedded into the stationary tower by [12]. It is not difficult to see that Theorem 4.1 subsumes another well-known fact, that under large cardinal assumptions for every universally Baire set A the theory of $L(A, \mathbb{R})$ cannot be changed by forcing.

Fix a cardinal κ. An $N \in \mathcal{P}_{\aleph_2}(H(\kappa))$ is *internally approachable* if there is a continuous chain $(N_\alpha)_{\alpha<\omega_1}$ such that $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and each proper initial segment of this chain belongs to N. This is a special case of [12, Definition 2.1] when the length is equal to ω_1. We shall denote the set of all internally approachable structures by IA. Internally approachable structures form a rather large stationary subset of $\mathcal{P}_{\aleph_2}(H(\kappa))$, roughly corresponding to $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega_1\}$. In all of our applications, κ can be taken to be any large enough cardinal.

If a sentence ϕ of the language of Set Theory has an additional unary predicate for a set $A \subseteq \mathbb{R}$ then a model M of ϕ is *A-correct* if it is transitive and $A \cap M = A^M$. Similarly (assuming $\omega_1 \subseteq M$), M is *NS_{ω_1}-correct* if $\mathrm{NS}_{\omega_1} \cap M = \mathrm{NS}^M_{\omega_1}$.

Theorem 4.1. *Suppose that κ is a measurable Woodin cardinal. Let A be a κ-universally Baire set of reals and let ϕ be a sentence in the language of set theory with one additional unary predicate. Then the following statement is absolute between forcing extensions by posets in V_κ:*

> *The set $\mathfrak{M} = \{M \cap H(\aleph_1) \mid M$ is an A-correct and NS_{ω_1}-correct model of $\phi\}$ contains a club in $\mathcal{P}_{\aleph_2}(H(\aleph_1))$ relative to IA.*

The strength of the large cardinal assumption in Theorem 4.1 can be weakened to 'κ is a full Woodin cardinal and there is a Woodin cardinal $\lambda > \kappa$' (see §4.1 below for definition). The case of Theorem 4.1 when ϕ is a Σ_1 formula of $H(\aleph_2)$ implies the conclusion of Theorem 2.1. Since under CH the stationarity of $\mathfrak{M}$ is equivalent to the existence of an A-correct and NS_{ω_1}-correct model of ϕ containing all the reals, the case when ϕ is a Σ^2_1 formula implies Woodin's theorem.

We emphasize that in Theorem 4.1 ϕ is not assumed to be a statement of $H(\aleph_2)$, and this result may be considered as an extension of the fact that every sentence that can be forced is true in a transitive model. The version of Theorem 4.1 in which $\mathbb{R}$ is replaced with $\mathcal{P}(\omega_1)$ is false: CH implies there is a subset of ω_1 that does not belong to any correct model of $\neg$CH.

4.1. **Robust and full Woodin cardinals.** If D is a collection of stationary sets (i.e., sets in $\mathbb{P}_\infty$), then we say that a set X *captures* D if there exists a $d \in D \cap X$ such that $X \cap (\cup d) \in d$. Say that a strongly inaccessible cardinal κ is *robust* if the following holds in every forcing extension by a partial order in V_κ: for every $b \in \mathbb{Q}_{<\kappa}$ and every $f\colon b \to \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ the set of countable $X \prec V_{\kappa+1}$ such that

- X captures every predense $D \subset \mathbb{Q}_{<\kappa}$ in X,
- $\mathrm{o.t.}(X \cap \kappa) \in f(X \cap (\cup b))$ (hence in particular $X \cap \cup b \in b$)

is stationary and (as an element of $\mathbb{P}_{<\lambda}$ or $\mathbb{Q}_{<\lambda}$ for some $\lambda > \kappa$) compatible with every member of $\mathbb{Q}_{<\kappa}$.

In the following lemma G is a canonical name for a $\mathbb{Q}_{<\lambda}$ generic filter.

Lemma 4.2. *Assume $\kappa < \lambda$ are Woodin cardinals and κ is moreover robust. Then $\mathbb{Q}_{<\kappa}$ forces that for every name $\dot{S}$ for a stationary subset of ω_1 in the generic ultrapower there is $a \in \mathbb{Q}_{<\lambda}$ that forces $\kappa \in \dot{S}$ and $G \cap \mathbb{Q}_{<\kappa}$ is $\mathbb{Q}_{<\kappa}$-generic. The same conclusion holds when $\mathbb{Q}_{<\lambda}$ is replaced with $\mathbb{P}_{<\lambda}$.*

Proof. We shall give an argument for $\mathbb{Q}_{<\lambda}$ since the proof for $\mathbb{P}_{<\lambda}$ is identical. Fix $b \in \mathbb{Q}_{<\kappa}$ and $f\colon b \to \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ that represents $\dot{S}$. Let a be the stationary subset of $[V_{\kappa+1}]^{\aleph_0}$ as in the definition of robust Woodin cardinals. Since $\bigcup a \supseteq V_\kappa$, for every $c \leq a$ in $\mathbb{Q}_{<\lambda}$ we have $c \cap V_\kappa \subseteq a$ and every $Y \in c$ still captures all predense subsets of $\mathbb{Q}_{<\kappa}$. Therefore if $D \subseteq \mathbb{Q}_{<\kappa}$ is predense and $d \in D$ is such that $Y \cap (\cup d) \in D$ for all $Y \in c$ then d is compatible with c in $\mathbb{Q}_{<\lambda}$. Hence a forces $G \cap \mathbb{Q}_{<\kappa}$ is generic.

The function $X \mapsto \mathrm{o.t.}(X \cap \alpha)$ represents α in any $\mathbb{Q}_{<\lambda}$ or $\mathbb{P}_{<\lambda}$ embedding for $\lambda > \alpha$ (for $\mathbb{P}_{<\kappa}$ this is [21, Fact 2.2.10] and the proof for $\mathbb{Q}_{<\kappa}$ is similar). Therefore, the requirement that $\mathrm{o.t.}(X \cap \kappa) \in f(X \cap (\cup b))$ ensures that κ will be in the set represented by f in any stationary tower embedding derived from $\mathbb{Q}_{<\lambda}$ or $\mathbb{P}_{<\lambda}$ for some $\lambda > \kappa$ by a filter containing the set above. □

Lemma 4.3. *If κ is a measurable Woodin cardinal, then the set of robust Woodin cardinals is in every normal measure on κ.*

Proof. Aside from the condition that $\mathrm{o.t.}(X \cap \kappa) \in f(X \cap (\cup b))$, this follows from Lemma [21, Lemma 2.7.14]. The remaining part follows from the fact that if κ is measurable, then every countable elementary submodel of $V_{\kappa+2}$ can be end-extended (as in the last paragraph of the proof of Lemma 2.4), and so the stationarity of the desired set reflects to a member of every normal measure on κ. □

For a strongly inaccessible cardinal λ, let $a(\lambda)$ be the set of $X \prec V_{\lambda+1}$ such that

1. $\mathrm{o.t.}(X \cap \lambda) = \omega_1$,
2. X captures every predense $D \subset \mathbb{Q}_{<\lambda}$ in X,
3. letting $\bar{X}$ be the transitive collapse of X, there exists a closed unbounded set $C \subset \bar{X} \cap \omega_1$ such that
   - each member of C is a robust Woodin cardinal in $\bar{X}$,
   - for each limit point γ of C, if $E \in \mathcal{P}(\gamma)^{\bar{X}}$ is a club subset of γ, then E contains a tail of $C \cap \gamma$.

Let us say that λ is *full* if it is a Woodin cardinal and $a(\lambda)$ is stationary.

Lemma 4.4. *Every measurable Woodin cardinal κ is full. Moreover, for any normal uniform measure on κ full Woodin cardinals form a set of measure one.*

Proof. Except for the robustness requirement, the proof adapted from Woodin's proof of Σ^2_1 absoluteness ([40]) was given in [21, Claim on page 94]. The additional requirement of robustness does not change the proof, though, as the proof used only the fact that the set of Woodin cardinals below a measurable Woodin is in every normal measure, which holds for robust Woodin cardinals by Lemma 4.3. □

Lemma 4.5. *If λ is full Woodin and $\delta \geq \lambda$ then in $\mathbb{P}_{<\delta}$ the condition $a(\lambda)$ forces $j(\omega_1) = \lambda$ and $G \cap V_\lambda$ is V-generic for $\mathbb{Q}_{<\lambda}$. Also, it forces that there is a club $C \subset \lambda$ in M such that*

(a) each member of C is a robust Woodin cardinal in V_λ,
(b) for each limit member γ of C and each club subset E of γ in V_λ, E contains a tail of $C \cap \gamma$.

Proof. We have $j(\omega_1) = \lambda$ by (1) and [21, Corollary 2.2.11]. Condition (2) implies the $\mathbb{Q}_{<\lambda}$ genericity of G as in the proof of Lemma 4.2. Since $a(\lambda) \in G$ and $\bigcup a(\lambda) = V_{\lambda+1}$, by [21, Fact 2.2.8] we have $j[V_{\lambda+1}] \in j(a(\lambda))$. Now V_λ, being the transitive collapse of $j[V_\lambda]$, is an element of M and (a) and (b) follow by (3). □

Lemma 4.6. *Assume λ is full Woodin, $\delta > \lambda$ is Woodin, and $G \subseteq \mathbb{P}_{<\delta}$ is V-generic below $a(\lambda)$ with $j\colon V \to M$ the corresponding generic embedding. Then in M for every forcing $P \in V_\lambda$ we can find a V-generic $h * H \subseteq P * \mathbb{Q}_{<\lambda}$. Furthermore, for a fixed h the set $\{Y \cap H(\aleph_1) \mid Y$ is a $\mathbb{Q}_{<\delta}$-generic ultrapower of $V[h]$ which is correct for $\mathrm{NS}_{\omega_1}\}$ contains a club in $\mathcal{P}_{\aleph_2}(H(\aleph_1))^M$ relative to IA.*

Proof. We shall need only the following property of internally approachable structures.
Claim. *If $N \prec H((2^{\mathfrak{c}})^+)$ is internally approachable then there is a well-ordering of the reals of N in type ω_1 all of whose proper initial belong to N.*

Proof. Let $(N_\alpha)_{\alpha<\omega_1}$ be the continuous chain of models with N as its union such that each initial segment belongs to N. By going to a club, we may assume this is an elementary chain. For each $X \in \mathcal{P}_{\aleph_2}(H(\aleph_1))$ fix a well-ordering $<_X$ of X in order type $\leq \omega_1$ and fix a bijection $g\colon \omega_1 \to \omega_1 \times \omega_1$. By elementarity, such g and a map $X \mapsto <_X$ both exist in N_0. For $\xi < \omega_1$ let $y_\xi \in N_{g_0(\xi)}$ be the $g_1(\xi)$-th real in $<_{N_{g_0(\xi)}}$. Then $\langle y_\xi : \xi < \omega_1\rangle$ enumerates all reals of N (with repetitions) and each proper initial segment of this ordering is in N. □

Let ζ be a strongly inaccessible cardinal in V between λ and δ. Since M is closed under sequences of length less than δ in $V[G]$ ([21, Theorem 2.5.8]), V_ζ and $G \cap V_\lambda$ are in M. Note that V is not included in M, but all the dense subsets of $P * \mathbb{Q}_{<\lambda}$ in V belong to V_ζ. Hence it will suffice to build a V_ζ generic $h * H \subseteq P * \mathbb{Q}_{<\lambda}$.

Since $\mathcal{P}(\mathbb{P})^V$ is countable in $V[G \cap V_\lambda]$, by [21, Theorem A.0.9] we may fix a V-generic filter $h \subset \mathbb{P}$ in $M \cap V[G \cap V_\lambda]$. Let C be a club as in Lemma 4.5. Now, $\mathbb{P}$ is ρ-c.c. for some $\rho < \lambda$, so for every regular $\xi \geq \rho$, every club subset of ξ in $V[h]$ contains a club set in V, which shows that members of $C \setminus \rho$ (which we may assume is the same as C) retain properties (a) and (b) in $V[h]$.

In M fix a well-ordering $<_w$ of $H((2^\delta)^+)$ and an internally approachable $(Z, <_w) \prec (H((2^\delta)^+), <_w)$ of size $\aleph_1$ containing everything relevant. In order to verify that the family of sets of reals of generic ultrapowers of $V[h]$ correct about NS_{ω_1} contains a club, we shall construct a generic ultrapower of $V[h]$ whose reals are the reals of Z. In M fix an enumeration $\{x_\alpha : \alpha < \omega_1\}$ of $\mathbb{R}^Z$ in type ω_1 all of whose proper initial segments are in Z.

Fix a partition in M of ω_1^M into stationary sets A_α ($\alpha < \omega_1$). Working in M we will produce a continuous increasing sequence of ordinals $\langle \eta_\alpha : \alpha < \omega_1 \rangle \subset C$ and a sequence $\langle H_\alpha : \alpha < \omega_1 \rangle$ such that (writing $(V[h])_\eta$ for $V_\eta^{V[h]}$):

(c) each H_α is a $V[h]$-generic filter contained in $\mathbb{Q}_{<\eta_\alpha}^{V[h]}$,
(d) each $x_\alpha \in (V[h][H_{\alpha+1}])_\zeta$,
(e) $\alpha' < \alpha$ implies $H_\alpha \cap (V[h])_{\eta_{\alpha'}} = H_{\alpha'}$.
(f) The pair (η_α, H_α) is the $<_w$-minimal pair satisfying (c)–(e).

Note that (d) is equivalent to $x_\alpha \in V[h][H_{\alpha+1}]$ but the latter is not a statement of Z (or of M). Note also that (f) implies that each H_α belongs to Z. Since each of $\mathbb{Q}_{<\eta_\alpha}^{V[h]}$, $(V[h])_{\eta_\alpha}$, and H_α is countable and belongs to Z, each of these sets is included in Z. Our $V[h]$-generic filter $H \subset \mathbb{Q}_{<\lambda}$ will be $\bigcup_{\eta<\omega_1} H_\eta$.

As we choose η_α and H_α, for each $b \in \mathbb{Q}_{<\lambda}^{V[h]}$ and each $\mathbb{Q}_{<\lambda}$-name $\dot{S}$ for a stationary subset of ω_1 in the generic ultrapower we will associate some A_β and ensure that $A_\beta \cap \{\eta_\alpha : \alpha < \omega_1\}$ is included in the interpretation of $\dot{S}$, which ensures that the final image model will be correct about NS_{ω_1}.

We let η_0 be the least member of C and let H_0 be any $V[h]$-generic filter contained in $\mathbb{Q}_{<\eta_0}$.

For the successor stages of the construction we work in Z. Given $H_\alpha \subset \mathbb{Q}_{<\eta_\alpha}$, if $\eta_\beta \in A_\beta$ and the function f associated to A_β has domain in $\mathbb{Q}_{<\eta_\alpha}$, then we let a_α be the stationary subset of $[V_{\eta_\alpha+1}]^{\aleph_0}$ as guaranteed by Lemma 4.2. Then for any $\gamma > \eta_\alpha$ in C, letting

$$\mathbb{Q}_{<\gamma}(a_\alpha) = \{b \in \mathbb{Q}_{<\gamma} \mid b \leq a_\alpha\},$$

the quotient $(\mathbb{Q}_{<\gamma}(a_\alpha)/\mathbb{Q}_{<\eta_\alpha})^{V[h]}$ is a $\mathbb{Q}_{<\eta_\alpha}$-name in $V[h]$ for a partial order such that, in $V[h]$,

$$\mathbb{Q}_{<\eta_\alpha} * (\mathbb{Q}_{<\gamma}(a_{\eta_\alpha})/\mathbb{Q}_{<\eta_\alpha}) \cong \mathbb{Q}_{<\gamma}(a_{\eta_\alpha\gamma}),$$

and such that η_α is forced to be in the set represented by f.

Since each set of the form $\{h, x_\alpha, H_\alpha\}$ exists in a generic extension of V by a forcing in V_λ, if x_α is not in $V[h][H_\alpha]$ then x_α is generic over $V[h][H_\alpha]$ for some forcing in $(V[h][H_\alpha])_\lambda$, and by [21, Theorem A.0.9] that forcing can be subsumed by

$$(\mathbb{Q}_{<\gamma}(a_{\eta_\alpha})/\mathbb{Q}_{<\eta_\alpha})_{H_\alpha}^{V[h]}$$

(i.e., the realization of $(\mathbb{Q}_{<\gamma}(a_{\eta_\alpha})/\mathbb{Q}_{<\eta_\alpha})^{V[h]}$ by H_α) for some $\gamma > \eta_\alpha$ in C. Finally, pick the $<_w$-minimal pair $(\eta_{\alpha+1}, H_{\alpha+1})$ meeting the conditions (c)–(e). Since in this case the pair (η_α, H_α) is uniquely determined, (f) is automatic.

At limit stages α of our construction, if we let

$$\eta_\alpha = \sup\{\eta_\beta : \beta < \alpha\}$$

then η_α is a Woodin cardinal and therefore every predense set in $(\mathbb{Q}_{<\eta_\alpha})^{V[h]}$ is predense in $(\mathbb{Q}_{<\beta})^{V[h]}$ for a club set of $\beta < \eta_\alpha$. But each of these clubs intersects $\{\eta_\beta : \beta < \alpha\}$, which shows that $H_\alpha = \bigcup\{H_\beta : \beta < \alpha\}$ is $V[h]$-generic for $(\mathbb{Q}_{<\eta_\alpha})^{V[h]}$. Let $C^* = \{\eta_\alpha : \alpha < \omega_1\}$. Since C^* is a club subset of λ, $H = \bigcup\{H_\alpha : \alpha < \omega_1\}$ is $V[h]$-generic for $\mathbb{Q}_{<\lambda}$.

If $j^*\colon V[h] \to M^*$ is the generic embedding induced by H, then $\mathbb{R}^{M^*} \supseteq \mathbb{R}^Z$ by contruction. Since the generic ultrapower is closed under ω-sequences, we have

$\mathbb{R}^{M^*} = \bigcup_{\alpha<\omega_1} \mathbb{R}^{V[h][H_\alpha]} \subseteq Z$, hence $\mathbb{R}^{M^*} = \mathbb{R}^Z$. Since M^* is NS_{ω_1}-correct, the proof is complete. □

Proof of Theorem 4.1. We first prove the following case of Theorem 4.1:

> Suppose that some partial order $P \in V_\kappa$ forces the existence of an A-correct and NS_{ω_1}-correct model N of ϕ containing the reals. Then $\mathfrak{M} = \{M \cap H(\aleph_1) \mid M$ is an A-correct and NS_{ω_1}-correct model of $\phi\}$ contains a club in $\mathcal{P}_{\aleph_2}(H(\aleph_1))$ relative to IA.

We closely follow the proof in [21, p. 94]. Let A, κ, ϕ and P be as in the statement of the theorem. Let $\lambda < \kappa$ be a full Woodin cardinal with $P \in V_\lambda$. Since κ is a limit of Woodin limits of Woodin cardinals, we may fix a Woodin limit of Woodin cardinals δ greater than λ. Now let $G \subset \mathbb{P}_{<\delta}$ be V-generic with $a(\lambda) \in G$, and let $j\colon V \to M$ be the corresponding generic embedding. It suffices to show that the conclusion of the theorem holds in M for $j(A)$, since then it holds in V by elementarity.

By Lemma 4.6 find V-generic $h * H \subseteq P * \mathbb{Q}_{<\lambda}$ in M. If ζ is the first inaccessible above λ in V, then $V_\zeta \subseteq M$ and in M we can define $j^*\colon V_\zeta[h] \to M^*$ the generic embedding. We have $j^*(\omega_1) = \lambda$ by Lemma 4.5.

There is a model N of ϕ in M^* with the additional symbol in ϕ interpreted as $j^*(A_h)$ (where A_h is the reinterpretation of A in $V[h]$) such that N contains the reals of M^* and is correct in M^* about $j^*(A_h)$, ω_1 and NS_{ω_1}. The universally Baire set A is absolute between M and M^* by Corollary 3.5 (see also Remark 3.6), in particular $j^*(A_h) = A \cap M^*$. Also, M^* is correct in M about ω_1 and NS_{ω_1}, hence model N is correct about ω_1, NS_{ω_1} and A in M as well. The family of intersections with $H(\aleph_1)$ of such models includes a club in $\mathcal{P}_{\aleph_2}(H(\aleph_1))$ relative to IA, and the elementarity implies the claim in V.

If CH holds then $\mathfrak{M}$ contains a club relative to IA if and only if there is an A-correct, NS_{ω_1}-correct model of ϕ containing all reals. Hence we can complete the proof using a standard fact. By [21, Theorem 2.5.10] (due to Woodin), if δ is a Woodin cardinal and some sentence ψ holds in some initial segment of the universe below δ, then ψ can be forced to hold in some initial segment of the universe by a partial order of cardinality δ after forcing with any partial order in V_δ. By Theorem 3.4 and the proof of Theorem 2.5.10 in [21], this fact holds even when one adds a predicate for a given universally Baire set to the language. Thus if $\mathfrak{M}$ includes a club relative to IA in some forcing extension then in every forcing extension satisfying CH there exists an A-correct and NS_{ω_1}-correct model of ϕ containing the reals. □

5. Universally Baire sets and $L(Q)$

If Γ is a pointclass let $\mathcal{L}^\Gamma$ be an extension of the first-order logic with formulas in which we have

(1) the quantifier Qx, interpreted as 'there exists uncountably many,'
(2) a predicate NS for the nonstationary ideal on ω_1,
(3) predicates for all sets in Γ,
(4) constants for all hereditarily countable sets.

If Γ is the class of all universally Baire sets then we write $\mathcal{L}^{\mathrm{uB}}$ for $\mathcal{L}^\Gamma$.

Definition 5.1. If ϕ is a sentence of $\mathcal{L}^{\mathrm{uB}}$ we say that a model $\mathfrak{X}$ of ϕ is *correct* if the conditions below are satisfied.

(5) The universe X of $\mathfrak{X}$ is an element of $H(\aleph_2)$,
(6) all universally Baire predicates and NS_{ω_1} as well as all constants occurring in ϕ are interpreted correctly,
(7) this model is *standard*: it interprets Qx as 'there exist uncountably many.'

For a $\mathcal{L}^\Gamma$ sentence ϕ such that Γ=Borel and Γ does not involve the predicate NS the statement 'ϕ has a correct model' is absolute between transitive models of ZFC. This folklore consequence of Keisler's completeness theorem for $L_{\omega_1\omega}(Q)$ ([19]) was stated explicitly in [9]. An apparently stronger result in which ϕ is allowed to have countable conjunctions is still true. We drop infinitary sentences since in our context where we are dealing only with ω-models they would not add to the expressive power of the language.

Note that countable sets of reals are universally Baire, so our definition of correct model allows for real parameters in ϕ. An easy Löwenheim–Skolem argument shows that every correct model for ϕ has a correct elementary submodel of cardinality $\aleph_1$. Also, if A is a universally Baire set coding all the universally Baire sets appearing in ϕ and (N, I) is an A-iterable pair such that $N \models \phi$, then as in Subsection 2.3 there is an iteration of (N, I) of length $\omega_1 + 1$ whose last model is correct for ϕ.

Theorem 5.2. *Suppose $\kappa < \delta$ are Woodin cardinals and ϕ is a sentence in $\mathcal{L}^{\mathrm{uB}}$. Then ϕ has a correct model in a forcing extension by a partial order in V_κ if and only if ϕ has a correct model in V.*

Proof. Let A code all universally Baire sets occurring in ϕ. Since 'ϕ has a correct model' is a Σ_1 sentence in $\langle H(\aleph_2); \mathrm{NS}_{\omega_1}, A, \in\rangle$, this is a consequence of Theorem 2.1. □

In the absence of large cardinals there can exist a formula ϕ with a correct model in V such that some P forces ϕ has no correct models even with $\Gamma = \Pi^1_1$ (Proposition 7.8). By augmenting the large cardinal strength we obtain a stronger transfer theorem.

Theorem 5.3. *Suppose $\kappa < \delta$ are Woodin cardinals, κ is full, and ϕ is a sentence in $\mathcal{L}^{\mathrm{uB}}$. Then the following statement is absolute between forcing extensions by posets in V_κ:*

> *the set $\mathfrak{M} = \{M \cap H(\aleph_1) \mid M$ is a correct model of $\phi\}$ contains a club in $\mathcal{P}_{\aleph_2}(H(\aleph_1))$ relative to IA.*

Proof. Let A be the universally Baire set occurring in ϕ. If N is a model of ZFC that correctly computes A, ω_1 and NS_{ω_1}, then a correct model for ϕ in N is also correct in V. Therefore this is a consequence of Theorem 4.1. By the proof of Theorem 4.1, one needs only that κ is a full Woodin cardinal below a Woodin cardinal. □

Corollary 5.4. *If there exists a proper class of Woodin cardinals and ϕ is a formula of $\mathcal{L}^{\mathrm{uB}}$, then the set of reals x such that $\phi(x)$ has a correct model is universally Baire.*

Proof. Let X be the set of all reals r such that $\phi(r)$ has a correct model. Assume that $G \subseteq \mathbb{Q}_{<\delta}$ is a generic filter and $j \colon V \to M$ is the induced generic embedding. If $r \in \mathbb{R}^M$, then $\phi(r)$ has a correct model in M if and only if it has a correct model in $V[G]$. This follows from the fact that M and $V[G]$ have the same reals, and Theorems 2.6 and 3.1, with P in Theorem 2.6 as the trivial forcing. (The existence of a correct model in either of M and $V[G]$ gives rise to a countable A-iterable model - for the relevant universally Baire set A - which can be iterated ω_1 times as in Subsection 2.3 to produce a correct model in the other.)

By Theorem 5.2, $\phi(r)$ has a correct model in $V[G]$ if and only if it has a correct model in $V[r]$, an extension by a forcing in V_δ. By Woodin's *Tree Production Lemma* ([21, Theorem 3.3.15] or [34, Theorem 4.2]) applied with $\psi(x)$ being '$\phi(x)$ has a correct model,' we conclude that $X = \{r \mid \phi(r) \text{ has a correct model}\}$ is universally Baire. □

Corollary 5.5 (CH). *If there exists a proper class of full Woodin cardinals and ϕ is a formula of $\mathcal{L}^{\mathrm{uB}}$, then the sets of reals x such that $\phi(x)$ has a correct model containing all the reals is universally Baire.*

Proof. Analogous to the proof of Corollary 5.4, but using Theorem 5.3. The one key difference is in showing that $V[r]$ and M agree about whether $\phi(r)$ has a correct model, in the situation where λ is a full Woodin cardinal, $j\colon V \to M'$ is the generic embedding induced by a V-generic filter $G' \subset \mathbb{Q}_{<\lambda}$ and $r \in \mathbb{R}^{M'}$. Forcing over $V[G']$, we may fix for some Woodin cardinal $\delta > \lambda$ a V-generic $G \subset \mathbb{P}_{<\delta}$ with $G \cap V_\lambda = G'$ and $a(\lambda)$ (as in Section 4) in G. Then there is an elementary embedding from M' to M with critical point greater than λ which fixes the images of the universally Baire sets from V. Furthermore, M' contains all subsets of λ in $V[G']$. If there exists a correct model of ϕ in $V[r]$, then the proof of Lemma 4.6 shows that there is one in M and so there exists one in M'. If there exists a correct model N in M', then it suffices to show that N is correct in $V[G']$. Since M, M' and $V[G]$ agree about the universally Baire sets from V, the only issue here is whether $\mathrm{NS}_{\omega_1}^{V[G']} \cap N = \mathrm{NS}_{\omega_1}^N$. This follows from the fact that $\mathrm{NS}_{\omega_1}^M \cap N = \mathrm{NS}_{\omega_1}^N$ and every club subset of λ in $V[G']$ is in M. □

The following well-known fact (e.g., [21, Theorem 3.3.10]) follows easily from Corollary 5.5 by using standard arguments.

Corollary 5.6. *Assume there are class many full Woodin cardinals and CH. Then every Σ^2_1 set X is universally Baire. Moreover, we can allow arbitrary universally Baire sets as parameters in the definition of X.* □

In [10] we add $\diamond$ to our assumptions and prove variant of Theorem 5.2 for the *Magidor–Malitz logic*: the logic with Ramseyan quantifiers asserting the existence of uncountable homogeneous sets for partitions of any $[\omega_1]^n$ (and even $[\omega_1]^{<\omega}$).

A Σ^2_1 well-ordering of the reals can be forced starting from any model of ZFC ([2]) and therefore an additional assumption such as CH is necessary in Corollary 5.6. Also, a Δ^2_2 well-ordering of the reals in type ω_1 can be forced starting from any model of ZFC ([1]) and therefore Σ^2_1 in Corollary 5.6 cannot be improved much unless an axiom stronger than CH is assumed.

6. Isolating cardinal invariants

We now give an application of results in §5.

Definition 6.1. Assume ϕ is an $\mathcal{L}^{\mathrm{uB}}$-formula in a language with an additional predicate Z. Let $E \subset \mathbb{R}^2$ be universally Baire. A set $A \subset \mathbb{R}$ is *(ϕ, E)-covering* if there is a correct model $\mathfrak{X}$ for ϕ such that $Z^{\mathfrak{X}} = A$ and moreover $(\forall x \in \mathbb{R})(\exists y \in A)(x, y) \in E$.

The minimal cardinality of a (ϕ, E)-covering set will be denoted by $\mu_{\phi,E}$. A cardinal invariant of this form is an *abstract covering number.*

The set A as occurring in Definition 6.1 is not assumed to be unviersally Baire.

Every *tame* cardinal invariant as in [42, §B.2] is an abstract covering number. This includes a large number of standard cardinal invariants of the continuum such as $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \dots$, as well as all cardinal invariants occurring in Cichon's diagram (see [6], [5] for the theory of cardinal invariants). The definition of a tame cardinal invariant in [42] requires ϕ to be arithmetic and E to be projective. With these additional assumptions, Theorem 6.3 as well as the auxiliary lemmas below are due to Zapletal, whose proof used Σ^2_1-absoluteness and relied on [7].

Not every abstract covering number is provably equal to a covering number of some absolutely defined σ-ideal. For example, the minimal length of a nonextendible tower in $\mathcal{P}(\omega)/\mathrm{Fin}$, $\mathfrak{t}$, is an abstract covering number but not a covering number of any σ-ideal. This is because the value of $\mathfrak{t}$ can be collapsed to $\aleph_1$ without adding reals (see [42, Example B.1.8]). The standard cardinal invariants known not to be tame, such as $\mathfrak{g}$ ([42, B.26]) or $\mathfrak{h}$ ([42, B.25]) are not abstract covering numbers, as Theorem 6.3 fails for them.

Let us introduce some of the machinery from [42, §3]. The following definition applies to most σ-ideals used in the theory of forcing.

Definition 6.2. A σ-ideal I on the reals such that the collection of all codes for analytic sets in I is given by a universally Baire definition is a *universally Baire ideal*. An ideal I is *iterable and almost full* if the following hold.

(1) It is a universally Baire σ-ideal.
(2) Every positive universally Baire set has a positive Borel subset.
(3) The forcing $\mathtt{Borel}/I$ is proper in all forcing extensions.

By (2), forcings $\mathtt{Borel}/I$ and $\mathcal{P}(\mathbb{R})/I$ are forcing equivalent in $L(\mathbb{R})$ and other standard inner models of AD, in the presence of a proper class of Woodin cardinals. Futhermore (in the presence of class many Woodin cardinals), Definition 6.2 describes ideals that are both *iterable* (see [42, 3.1.1]) and *almost full* ([42, 2.2.6]) in Zapletal's terminology. This follows from [42, Claim 3.1.3].

Recall that $\mathrm{cov}(I)$ is the smallest size of a subset of I whose union covers the real line. Let $\mathrm{cov}^+(I)$ be the smallest size of a subset of I whose union covers an I-positive Borel set. If the ideal I is *homogeneous*, in the sense that for every Borel I-positive set B there is a Borel isomorphism $f\colon B \to \mathbb{R}$ such that $f[A] \in I$ iff $A \in I$ for all $A \subseteq B$, then clearly $\mathrm{cov}(I) = \mathrm{cov}^+(I)$.

Let $\mu_{E,I}$ be an abstract covering number. A set A is *(ϕ, E, I)-covering* if for every I-positive Borel set B and a Borel function $f\colon B \to \mathbb{R}$ there is $x \in A$ such that

$$\{y \in B : (f(y), x)) \in E\}$$

contains an I-positive Borel set. The following is the main result of this section.

Theorem 6.3. *Assume there are class many full Woodin cardinals. For an abstract covering number $\mu_{\phi,E}$ and an iterable and almost full I the following are equivalent.*

(1) Any countable support iteration of $\mathtt{Borel}/I$ *of length at least ω forces $\mu_{\phi,E} \leq \aleph_1$.*
(2) In some forcing extension we have $\mu_{\phi,E} < \mathrm{cov}^+(I)$.
(3) In some forcing extension there exists a (ϕ, E, I)-covering set.

Along similar lines one might expect that the consistency of $\aleph_1 < \mathrm{cov}^+(I)$ implies the properness of $\mathtt{Borel}/I$, but J. Zapletal pointed out that there is a universally Baire σ-ideal I for which this is false; see [43].

If P is a forcing notion then we say A is *(ϕ, E, P)-covering* if for every P-name for a real τ and every $p \in P$ there is a regular subordering Q of $\{q \in P : q \leq p\}$ that is forcing-equivalent to $\mathtt{Borel}/I$ for a universally Baire ideal I such that τ is a Q-name and A is (ϕ, E, I)-covering. Typically, P will be a countable support iteration of some quotients $\mathtt{Borel}/I$.

Lemma 6.4. *Assume there are class many Woodin cardinals, P is a proper forcing notion, and A is (ϕ, E, P)-covering. Then P forces A is (ϕ, E)-covering.*

Proof. Assume τ is a P-name for a real and $p \in P$. Find a regular suborder Q of P such that τ is a Q-name and forcing equivalent to $\mathtt{Borel}/I$ for some universally Baire ideal I below p. Since Q is proper, by [42, Lemma 2.2.1] there is a Borel function $f\colon \mathbb{R} \to \mathbb{R}$ such that Q forces $\tau = f(\dot{r})$, where $\dot{r}$ is the canonical generic real for $\mathtt{Borel}/I$. Find $y \in A$ such that $\{x \in \mathbb{R} \mid (f(x), y) \in E\}$ contains an I-positive Borel set, B. Then B forces $(\tau, y) \in E$ because the set $C = \{z \mid (f(z), y) \in E\}$ is universally Baire (this follows from our large cardinal assumption since C is projective in E, see Corollary 5.4 or [21, Exercise 3.3.11]), $B \subseteq C$ and $\dot{r}$ is forced to belong to B. □

Let I be a σ-ideal on $\mathbb{R}$ and $\alpha < \omega_1$. For $B \subseteq \mathbb{R}^\alpha$ let

$$T_B = \{s \in \textstyle\bigcup_{\beta<\alpha} \mathbb{R}^\beta : s = x \restriction \beta \text{ for some } x \in B\}.$$

A set $B \subseteq \mathbb{R}^\alpha$ is *I-perfect* ([42, 3.1.4, 3.2.2] if

(1) For every $s \in T_B$ the set $\{r \in \mathbb{R} : s^\frown r \in T_B\}$ is I-positive, and
(2) T_B is *σ-closed*: if s_n $(n \in \omega)$ is an increasing sequence in T_B then $\bigcup_{n\in\omega} s_n \in T_B$.

There is another natural way to define when $B \subseteq \mathbb{R}^\alpha$ is 'large' with respect to I introduced in [42, 3.2.1]. Consider a game $G(B)$ of length α in two players, Adam and Eve. Adam plays first at all limit stages and his move is always a set $A_\xi \in I$. Eve plays reals x_ξ $(\xi < \alpha)$ so that $\langle x_\xi : \xi < \eta\rangle \in T_B$ for all η and $x_\xi \notin A_\xi$ for all ξ. If at some stage Eve is unable to make a move, then Adam wins. Otherwise, after α moves Eve wins. The family of all sets for which Adam has a winning strategy is denoted by I^α. Clearly $I^1 = I$. It is not difficult to see that I^α is a σ-ideal and that all I-perfect subsets of $\mathbb{R}^\alpha$ are I^α-positive.

Lemma 6.5 (Zapletal, [42, 3.4.1])**.** *Assume there are class many Woodin cardinals. If I is iterable and almost full then every universally Baire set in $\mathbb{R}^\alpha$ is either in I^α or it contains a Borel I-perfect subset.*

Proof. We shall only point to the main ingredients of the proof, as the reader can find the details in [42]. First, the large cardinal assumption implies all perfect information games of fixed countable length in which the opponents are playing reals with a universally Baire payoff are determined, and moreover the winning strategy can be coded as a universally Baire set of reals ([28, Exercise 2E.11]). Adam's moves in the game $G(A)$ need not be coded as reals, but Zapletal defines an auxiliary game $H(A)$ with this property that is equivalent to $G(A)$. Therefore $A \subseteq \mathbb{R}^\alpha$ is not in I^α if and only if Eve has a winning strategy in $G(A)$ if and only if Eve has a universally Baire winning strategy in $H(A)$. Second, Eve has a universally Baire winning strategy in $H(A)$ if and only if A contains an I-perfect Borel subset. □

Lemma 6.6. *Assume I is a σ-ideal on $\mathbb{R}$ such that for every $\alpha < \omega_1$ every I^α-positive universally Baire set contains a Borel I-perfect set. Then*

(1) $\mathrm{cov}^+(I) = \mathrm{cov}^+(I^\alpha)$ *for all* $\alpha < \omega_1$.

(2) *If* $\mu_{\phi,E}$ *is an abstract covering number then every* (ϕ, E, I)*-covering set is* (ϕ, E, I^α)*-covering for all* $\alpha < \omega_1$.

Proof. We prove only (1) since the proof of (2) is almost identical. It is obvious that the sequence $\mathrm{cov}^+(I^\alpha)$ is nonincreasing in α. We prove $\mathrm{cov}^+(I^\alpha) \geq \mathrm{cov}^+(I)$ by induction on α. Assume $\lambda < \mathrm{cov}^+(I)$, $B \subseteq \mathbb{R}^\alpha$ is an I-perfect Borel set, and B_ξ ($\xi < \lambda$) are Borel sets in I^α. For $\xi < \lambda$ let τ_ξ be a winning strategy for Adam in the I-perfect game for B_ξ. In order to prove $B \not\subseteq \bigcup_{\xi<\lambda} B_\xi$, we construct a sequence $\langle x_\xi : \xi < \alpha\rangle$ of Eve's simultaneous responses to all τ_ξ in $G(B)$.

Let $T_B = \{s \in \bigcup_{\beta<\alpha} \mathbb{R}^\beta : s = x \restriction \beta \text{ for some } x \in B\}$. If $s = \langle x_\eta : \eta < \xi\rangle$ has been chosen, let $Z_\eta = \bigcup_{\xi<\lambda} Z_{\eta,\xi}$, where $Z_{\eta,\xi} \in I$ is the set provided to Adam by τ_ξ as a response to Eve's moves, coded by $s \in T_B$. Note that this is not necessarily a legal move in $G(B)$ for Adam since Z_η need not be in I. The set $X = \{r : s^\frown r \in T_B\}$ is universally Baire and I-positive, and by assumption it contains a Borel I-positive set. Since each $Z_{\eta,\xi}$ is in I and $\lambda < \mathrm{cov}^+(I)$, we have that Z_η does not cover X. Eve extends s by picking $x_\xi \in X \setminus Z_\xi$. This shows that Eve can play at any successor stage of the game. Since T_B is σ-closed and B is I-perfect, Eve wins the game. The obtained sequence belongs to $B \setminus \bigcup_{\xi<\lambda} B_\xi$. Since B and B_ξ ($\xi < \lambda$) were arbitrary, this shows that an I-perfect set cannot be covered by fewer than $\mathrm{cov}^+(I)$ Borel sets in I^α. □

Lemma 6.7. *Assume there are class many Woodin cardinals. Assume that $\mu_{\phi,E}$ is an abstract covering number and I is iterable and almost full. Then every (ϕ, E, I)-covering set is (ϕ, E, P)-covering, where P is any countable support iteration of* `Borel`$/I$.

Proof. Lemma 6.5 and Lemma 6.6 together imply that every (ϕ, E, I)-covering set is (ϕ, E, I^α)-covering for all $\alpha < \omega_1$. It is well-known (see [42, §3.1]) that for every condition B in a countable support iteration P_κ of `Borel`$/I$ of any length there is $\alpha < \omega_1$ and a regular subalgebra Q of P_κ forcing-equivalent to the algebra of Borel subsets of $\mathbb{R}^\alpha$ modulo I^α. It is also known that every positive set contains an I-perfect positive subset. Since the assumption on I being iterable implies the iteration is proper, Q can be chosen so that any given name for a real is a Q-name. □

Proof of Theorem 6.3. Since the countable support iteration of `Borel`$/I$ of ω_2 forces $\mathrm{cov}^+(I) > \aleph_1$, (1) implies (2). Also, (2) readily implies (3). Assume (3). Let ψ be the $\mathcal{L}^{\mathrm{uB}}$-sentence in the language with an additional predicate A stating that A is a (ϕ, E, I)-covering. Then it is consistent that ψ has a correct model containing all the reals. Consider the extension by countable support iteration of `Borel`$/I$ of length $\geq \omega$. A countable support iteration of forcings adding reals of length ω forces CH. Using Theorem 5.3 in this intermediate model we can find a correct model $\mathfrak{X}$ for ψ containing all reals of this model. Since the factor forcing is a countable support iteration of `Borel`$/I$, Lemma 6.7 implies that the (ϕ, E, I)-covering set provided by $\mathfrak{X}$ is (ϕ, E, P)-covering. By Lemma 6.4, (1) follows. □

Given $\alpha < \omega_1$ and a seequence I_ξ ($\xi < \alpha$) of σ-ideals one can define ideal $\prod_{\xi<\alpha} I_\xi$ analogously to the iterated power I^α. If all of these ideals are iterable and almost full and there are class many Woodin cardinals then the analogue of Lemma 6.5 holds and the proof of Lemma 6.6 shows that $\mathrm{cov}^+(\prod_{\xi<\alpha} I_\xi) = \sup_{\xi<\alpha} \mathrm{cov}^+(I_\xi)$. So we have the following extension of Theorem 6.3.

Theorem 6.8. *Assume there are class many full Woodin cardinals. Further assume $\mathfrak{Z}$ is a set of abstract covering numbers and $\mathcal{F}$ is a family of iterable and almost full ideals. Then the following are equivalent.*

(1) Every countable support iteration of of length at least ω consisting of partial orders of the form $\mathtt{Borel}/I$ for some $I \in \mathcal{F}$ forces that $\mu \leq \aleph_1$ for all $\mu \in \mathfrak{Z}$.
(2) There is a forcing extension in which $\mu < \mathrm{cov}^+(I)$ for all $\mu \in \mathfrak{Z}$ and $I \in \mathcal{F}$.
(3) For each $\mu \in \mathfrak{Z}$ and $I \in \mathcal{F}$ there is a forcing extension in which $\mu < \mathrm{cov}^+(I)$.
(4) For each $\mu_{\phi,E} \in \mathcal{F}$ and each $I \in \mathcal{F}$ there is a forcing extension in which a (ϕ, E, I)-covering set exists. □

7. Concluding remarks

7.1. **Applications.** In this section let Γ be a pointclass such that the conclusion of Theorem 2.1 holds for all ϕ whose predicates for universally Baire sets are restricted to sets in Γ. If $\Gamma \subseteq \mathtt{Borel}$ then the results below are provable in ZFC.

For $B \subseteq \omega_1$ let $B^* = \{x \in \mathbb{R} : x \text{ codes an ordinal in } B\}$ be its code.

Proposition 7.1. *Assume $B \subseteq \omega_1$. If $B^* \in \Gamma$ then there is a club $C \subseteq \omega_1$ that is either contained in or disjoint from B.*

Proof. Assume B is stationary. Then in an ω_1-preserving forcing extension in which no new reals are added there is a club $D \subseteq B$. Then there is a function $f : \omega_1 \to \omega_1$ such that for every $\alpha < \omega_1$ closed under f we have $\alpha \in B^*$. By Theorem 2.1 such a function exists in V and therefore B contains a club. □

Proposition 7.2. *(1) If $\Gamma \supseteq \mathcal{P}(\mathbb{R})^M$ for some inner model M such that $\omega_1^M = \omega_1$ then ω_1 is measurable in M.*
(2) If Γ contains all codes for subsets of ω_1 constructible from x then $x^\#$ exists.

Proof. (1) By Proposition 7.1 the club filter on ω_1 in M is an ultrafilter. (2) By Proposition 7.1, the club filter on ω_1 is an $L[x]$-ultrafilter. □

If $n \geq 1$ then $[X]^n = \{s \subseteq X \mid |s| = n\}$. For $K \subseteq [X]^n$ a set $Y \subseteq X$ is *K-homogeneous* if $[Y]^n \subseteq K$.

Proposition 7.3. *For every partition $[\mathbb{R}]^n = K_0 \cup K_1$ such that K_0 is a continuous image of a set in Γ the statement 'there is an uncountable K_0-homogeneous set' is forcing-absolute.*

Proof. Let A be such that $f[A] = K_0$ for some continuous f. Note that the graph of f is in Γ since it is Borel. Let ϕ be a sentence 'there exist an uncountable $X \subseteq \mathbb{P}$ and $Y \subseteq A$ such that $[X]^n \subseteq f[Y]$' and apply Theorem 2.1. □

The ZFC case of Proposition 7.3, when $\Gamma = \mathtt{Borel}$ was proved in [20]. By Proposition 7.8 this theorem is optimal since even the Π^1_1 case cannot be proved in ZFC. It is known (see [29]) that for a Borel $K \subseteq [\mathbb{R}]^2$ 'there exists an uncountable K-homogeneous set' is strictly weaker than 'there exists a perfect K-homogeneous set.' Therefore Proposition 7.3, as well as the following two propositions, do not follow from forcing-absoluteness for universally Baire sets of reals.

A forcing notion $\mathbb{P}$ is in Γ if $\mathbb{P} \subseteq \mathbb{R}$ and $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are all in Γ. Proposition 7.3 immediately implies the following.

Proposition 7.4. *If P is a ccc forcing in $p[\Gamma]$ (the pointclass of continuous images of sets in Γ) then it is productively ccc.* □

In particular, assuming there exists a proper class of Woodin cardinal all universally Baire ccc posets are productively ccc. In [4] productiveness of various projective ccc forcings with ccc forcings in Solovay's model was proved, starting from more modest large cardinal assumptions. The case when Γ is $\mathtt{Borel}$ is a well-known result of Shelah ([16]). Apparently the only known proof of Shelah's theorem uses Keisler's completeness theorem for $L_{\omega_1\omega}(Q)$, and our proof is modeled on it.

A variant of Proposition 7.3 may be worth stating. If $A \subseteq X \times Y$ and $x \in X$ then $A_x = \{y \mid (x, y) \in A\}$. First note that if there are class many Woodin cardinals and $A \subseteq \mathbb{R}^2$ is universally Baire, then the set $B = \{x \mid A_x \text{ is uncountable}\}$ is universally Baire. This is because under our assumption every set projective in a universally Baire set is universally Baire [21, Exercise 3.3.11]) and $x \in B$ if and only if $(\exists y \in \mathbb{R}^{\mathbb{N}})(\forall z \in A_x)(\exists n \in \mathbb{N}) z = y(n)$. A higher-dimensional variant of this fact is also true.

Proposition 7.5. *Assume there are class many measurable Woodin cardinals. If $n \geq 1$ and $A \subseteq \mathbb{R} \times [\mathbb{R}]^n$ is universally Baire, then the set $\{x \mid$ there is an uncountable $Y \subseteq \mathbb{R}$ such that $[Y]^n \subseteq A_x\}$ is universally Baire.*

Proof. With $\phi(x, A)$ being an $\mathcal{L}^{\mathrm{uB}}$-formula 'there is an uncountable A_x-homogeneous set of reals' this is an immediate consequence of Corollary 5.4. □

We leave other applications, such as those involving homogeneous sets for square bracket partition relations or homogeneous clubs, to an interested reader.

7.2. **Limitations.** The case of Theorem 5.3(5.2) when $\phi \in \mathcal{L}^{\mathtt{Borel}}$ and it does not include a predicate for NS_{ω_1} is a well-known ZFC consequence of Keisler's completeness theorem for $L_{\omega_1\omega}(Q)$ ([19]). It was apparently stated explicitly for the first time in [9]. This was proved by associating an $L_{\omega_1\omega}(Q)$ sentence ϕ^M in an expanded language to each $\mathcal{L}^{\mathtt{Borel}}$ sentence ϕ such that a (reduction of a) standard model for ϕ^M is a correct model for ϕ. By Proposition 7.7 below this method cannot be applied if Γ includes all Π^1_1 sets. The proof relies on the following observation, communicated to us by Boban Velickovic and included here with his kind permission ($\mathbf{A}, \mathbf{E}$ are unary and binary relation symbols and A, E are their interpretations).

Lemma 7.6. *Assume that ϕ is an $L_{\omega_1\omega}(Q)$ sentence in a language including $\{\mathbf{A}, \mathbf{E}\}$ such that $(A, E)^{\mathfrak{X}}$ is an uncountable well-ordering in some correct model $\mathfrak{X}$ of ϕ. Then $(A, E)^{\mathfrak{Y}}$ is not a well-ordering in some corect model $\mathfrak{Y}$ of ϕ.*

Proof. We assume familiarity with the theory of $L_{\omega_1\omega}(Q)$, as presented in [19]. Assume the contrary. Let $\alpha = \sup\{\mathrm{o.t.}(A, <)^{(\mathfrak{Y}, q)} | (\mathfrak{Y}, q)$ a weak model of $\phi\}$. Since the set of all weak models of ϕ is Borel, by the Boundedness Theorem ([18, Theorem 31.2]) α is countable. Let M be a countable transitive model of a large enough fragment of ZFC such that $\omega_1^M > \alpha$. By Keisler's theorem, there exists a correct model $\mathfrak{Y}$ of ϕ in M. If $q = \{B \subseteq Y | B$ is definable and uncountable in $\mathfrak{Y}\}$ then $(\mathfrak{Y}, q)$ is a weak model of ϕ and $\mathrm{o.t.}(A, E)^{\mathfrak{Y}} > \alpha$, a contradiction. □

Proposition 7.7. *There is a Π^1_1 set A such that there is no $L_{\omega_1\omega}(Q)$ sentence ϕ in language containing a predicate $\mathbf{A}$ for A with the property that all standard models of ϕ are correct about A.*

Proof. Assume the contrary, that for every Π^1_1 set A there is a sentence ϕ such that every standard model of ϕ is correct about A. Let A be the Π^1_1 set of all subsets $\mathbb{N}^2$ that code a well ordering. Let ϕ be a sentence of $L_{\omega_1\omega}(Q)$ such that each standard model of ϕ is correct for A. Let ψ be the conjunction of ϕ and the sentence '$\mathbf{E}$ is an ω_1-like ordering of $\mathbf{B}$ and every proper initial segment is order-isomorphic to some $x \in A$.' Then in every model of ψ we have that E is a well-ordering of B in type ω_1, contradicting Lemma 7.6. □

Proposition 7.7 in particular shows that the approach of [9] cannot be applied to prove the version of Theorem 5.3(5.2) for $\mathcal{L}^{\Pi^1_1}$. The following Proposition shows that the version of Theorem 5.3(5.2) for $\mathcal{L}^{\Pi^1_1}$ cannot be proved in ZFC.

Proposition 7.8. *If the conclusion of Theorem 2.1 holds for all ϕ in $\mathcal{L}^{\mathbf{\Pi}^1_1}$ then ω_1 is inaccessible in L.*

Proof. Assume otherwise. Let x be a real such that $\omega_1^{L[x]} = \omega_1$. Then there exists an uncountable $\Pi^1_1(x)$ set A with no perfect subset. Then 'A has an uncountable subset' is a sentence of $\mathcal{L}^{\mathbf{\Pi}^1_1}$ true in V but false in its extension obtained by collapsing ω_1. □

Assume ϕ is a sentence of $L_{\omega_1\omega}(Q)$ whose language includes $\{\mathbf{X}, \mathbf{E}\}$ such that the interpretation of $(\mathbf{X}, \mathbf{E})$ is an ω_1-like linear order in every model of ϕ. Many arguments using $L_{\omega_1\omega}(Q)$ in set theory would be simplified if the statement

$W(\phi)$: Sentence ϕ has a model in which the interpretation of $(\mathbf{X}, \mathbf{E})$ is a well-ordering

were known to be forcing-absolute.

Proposition 7.9. *Assume there are class many full Woodin cardinals. Using the notation from the last paragraph, $W(\phi)$ is forcing-absolute.*

Proof. By [21, Theorem 2.5.10] it will suffice to show that if $W(\phi)$ holds in a forcing extension then it holds in V. Assume $W(\phi)$ holds in some forcing extension. By the construction from Theorem 4.1(1) there is a transitive model M^* of ZFC with correct ω_1 such that $W(\phi)$ holds in M^*. Then a standard well-founded model of ϕ is a standard well-founded model of ϕ in V. □

The conclusion of Proposition 7.9 has some large cardinal strength.

Proposition 7.10. *Assume $W(\phi)$ is forcing-absolute for every ϕ in $L_{\omega_1\omega}(Q)$. Then ω_1 is inaccessible in L.*

Proof. Assume $\omega_1 = \omega_1^{L(x)}$ for a real x. There exists an uncountable $\Pi^1_1(x)$ set B with no perfect subset (see e.g., [27, 5.A.8] or [24, 3.19]). Let A be the Π^1_1 set of codes for countable ordinals. Since A is Π^1_1-complete, we have $B = f^{-1}(A)$ for some continuous function f. Let ϕ be an $L_{\omega_1\omega}(Q)$ formula in the language $\{\mathbf{A}, \mathbf{B}, \mathbf{f}, \mathbf{X}, \mathbf{E}\}$ saying:

(1) $\mathbf{A}, \mathbf{B}$ are sets of reals,
(2) each element of $\mathbf{B}$ codes a countable linear ordering,
(3) $\mathbf{E}$ is an ω_1-well-ordering of a set $\mathbf{X}$,

(4) each initial interval $(-\infty, x]$ of $(\mathbf{X}, \mathbf{E})$ is coded by some element of $\mathbf{B}$,
(5) The interpretation of $\mathbf{f}$ is the restriction of $\mathbf{f}$ to the reals of the model,
(6) $\mathbf{A}$ is the $\mathbf{f}$-preimage of $\mathbf{B}$.

If in a model of ϕ the interpretation of $(\mathbf{B}, \mathbf{E})$ is well-founded, then it is correct about B and A. But such a model contains uncountably many elements of A, therefore if $\omega_1^{L(x)}$ is collapsed then ϕ cannot have such models. □

Problem 7.11. Using large cardinals develop a forcing-absolute syntax for $\mathcal{L}^{\mathrm{uB}}$ and prove the corresponding completeness theorem.

A positive solution to this problem would be a substantial improvement of Theorem 5.2.

7.3. **Special trees in $L(\mathbb{R})$.** A consequence of Theorem 2.1 is that if $T \subseteq \mathbb{R}^{<\omega_1}$ is a tree coded by a universally Baire set, then large cardinals imply that the statement 'T has an ω_1-branch' is forcing-absolute. Such an ω_1-branch constructs an ω_1-sequence of reals and therefore cannot exist in $L(\mathbb{R}, T)$ (again, assuming sufficient large cardinals to ensure that universally Baire sets satisfy the regularity properties), but it is natural to ask whether it is possible to determine whether T has an ω_1-branch in V by working in the model $L(\mathbb{R}, T)$. It turns out that this question has already been considered.

After the first author presented some of our results at the Annual Meeting of ASL at Stanford in March 2005, John Steel communicated Theorem 7.13 to us. The following definition is also due to him.

Assume $T \subseteq \mathbb{R}^{<\omega_1}$ is coded by a universally Baire set. Write

$$T^+ = \{p \in \mathbb{R}^{<\omega_1} \mid (\forall \alpha < \mathrm{dom}(p)) p \restriction \alpha \in T\}.$$

If there is a function g such that $g(p)$ is a surjection from ω onto $\mathrm{dom}(p)$ for every $p \in T^+$ we say that T is *specialized by* g. Every tree that is covered by countably many antichains is specialized by a function definable from the sequence of antichains. In the presence of the Axiom of Choice every tree is specialized by a function, but the following justifies the name in the context of the inner models of determinacy. A 'universally Baire function' is a function whose graph is a universally Baire set of pairs of reals.

Lemma 7.12. *Assume there are two Woodin cardinals $\delta < \lambda$ and T is a universally Baire tree specialized by a universally Baire function g. Then T has no ω_1 branches.*

Proof. Assume the contrary. By Theorem 2.3 both T and g are weakly homogeneously Suslin, and by Lemma 2.6 there is a T, g-iterable model $(N, \mathrm{NS}_{\omega_1}^N)$ satisfying 'T has an ω_1-branch,' p. (We only need $\mathrm{NS}_{\omega_1}^N$ to be precipitous and the generic ultrapower to be correct about T and g.) Let $j \colon N \to N^*$ be the generic ultrapower embedding. Then $p = j(p) \restriction \omega_1^N$ belongs to N^*, hence by elementarity and T-correctness $p \in T$. Thus $g(p)$, a surjection from ω onto ω_1^N, is a real in N^*. If τ is a $\mathrm{NS}_{\omega_1}^+$-name in N for $g(p)$, then it is forced that $\tau = g(p)$, and therefore $g(p) \in N$, a contradiction to the assumption that p is uncountable. □

Lemma 7.12 has a converse.

Theorem 7.13 (Steel)**.** *Suppose that there exist infinitely many Woodin cardinals below a measurable cardinal. Let $T \subset \mathbb{R}^{<\omega_1}$ be a tree in $L(\mathbb{R})$. Then exactly one of the following holds:*

- *T has an uncountable branch in V.*
- *T is specialized by a function in $L(\mathbb{R})$.* □

The relevance of the theorem is that for each Σ_1 statement ϕ for $\langle H(\aleph_2); A, \in\rangle$, where A is a set of reals in $L(\mathbb{R})$ as in Theorem 2.1 there is a tree in $L(\mathbb{R})$ which has an uncountable branch in V if and only if ϕ holds (such trees are typically called the *tree of attempts* to produce a witness for ϕ). The second conclusion of the theorem is a statement in $L(\mathbb{R})$, and thus forcing-absolute. The theorem then gives an absolute impediment witnessing the failure of a given Σ_1 statement. At a key place in the proof of Theorem 7.13 a rather deep result from the inner model theory is employed to construct a T-iterable model (see §2.3) of 'T has an ω_1-branch.' As pointed out to us by Stuart Zoble, extending to arbitrary universally Baire sets necessitates a different proof since the required inner model theory result is not known to be true in this context. A version of Steel's theorem for any tree T coded by a universally Baire set is proved in [10, §5].

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