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APPROXIMATE HOMOMORPHISMS II: GROUP HOMOMORPHISMS

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We investigate mappings between finite product groups which are approximately homomorphisms, with respect to some metric on the range. Our main positive result says that if the metric corresponds to a nonpathological submeasure, then the mapping can be approximated by a homomorphism within a constant error. We also use Ramsey's theorem to prove that this fails in the case when the nonpathologicity assumption is dropped. This note extends results of [2], where analogous results were proved for Boolean algebras.

In [2] we have studied approximate homomorphisms of finite Boolean algebras equipped with submeasures. It was proved that in the nonpathological case they can be approximated by strict homomorphisms modulo the fixed constant (for definitions see below, and note that in [2] we have studied only approximate homomorphisms of type II, as defined below). These results can be thought of as belonging to the general "stability program," proposed long ago by S. Ulam (see [12], [11], [5], an introduction to [2], also [8]), and as a part of the study of subsets of the discrete cube (see [9, Theorem 1.1], [10]). However, our original motivation came from a conjecture of Todorcevic about liftings of homomorphisms between Boolean algebras [3], to which the stability of approximate Boolean algebra homomorphisms happened to be equivalent. In the present paper we extend the results of [2] to a class of approximate group homomorphisms. These results have consequences to liftings of homomorphisms between quotients of compact metric groups analogous to those of [3] (see [4]), but we believe that they are in-

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teresting in their own right. They are also applied to give a quantitative improvement of [2, Theorem 4 and 5].

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1. Definitions

Let G, H be groups, let ν be a (left and right) unimodular probability measure on G, so that

$$\nu(aX) = \nu(X), \qquad \nu(Xa) = \nu(X), \quad \text{and} \quad \nu\{a^{-1} : a \in X\} = \nu(X), \tag{1}$$

for every measurable subset X of G and every $a \in G$. Note that in the case when G is finite the uniform probability measure satisfies these assumptions. Let d be a metric on H. If $\delta > 0$ then a map $f : G \to H$ is

1. a *δ-approximate homomorphism of type I with respect to ν* if

$$\nu^2\{(a,b) \in G^2 : f(a)f(b) \neq f(ab)\} \leq \delta \tag{2}$$

$$\nu\{a : f(a) \neq (f(a^{-1}))^{-1}\} \leq \delta. \tag{3}$$

2. a *δ-approximate homomorphism of type II with respect to d* if

$$d(f(a)f(b), f(ab)) \leq \delta \tag{4}$$

$$d(f(a), (f(a^{-1}))^{-1}) \leq \delta \tag{5}$$

for all $a, b \in G$.

If $H = \prod_{i \in I} H_i$ for some groups H_i $(i \in I)$ and φ is a strictly positive submeasure on the index-set I, then

$$d_\varphi(x, y) = \varphi\{i \in I : x(i) \neq y(i)\} \tag{6}$$

is a metric on H which induces the product topology on it (where each H_i is taken with the discrete topology), in particular the group operations of H are continuous with respect to d_φ. Recall that a submeasure φ is *pathological* if

$$\varphi \neq \sup_{\mu \leq \varphi} \mu,$$

where the supremum is taken over all measures μ dominated by φ.

2. Positive results

Theorem 2.1. *Assume* $G, H = \prod_{i\in I} H_i, \nu, \varphi$ *are as above,* G *is finite and* $f\colon G \to H$ *is an* ε*-approximate homomorphism of type II with respect to* d_φ. *If* φ *is nonpathological then there is a homomorphism* $g\colon G \to H$ *such that*

$$d_\varphi(g(a), f(a)) \le 24\varepsilon$$

for all $a \in G$.

Theorem 2.2. *Assume* G, H, ν *are as above,* G *is finite, and* $f\colon G \to H$ *is a* δ*-approximate homomorphism of type I with respect to* ν*, and* $\delta \le 1/11$. *Then there is a homomorphism* $g\colon G \to H$ *such that*

$$\nu\{x : g(x) \ne f(x)\} \le \frac{\delta}{1-3\delta},$$

and moreover, $\nu\{x : f(ax)f(x^{-1}) = g(a)\} \ge 1 - 3\delta$ *for all* $a \in G$.

Proof. (of Theorem 2.2) Let us first note a useful general fact.

Claim 1. *Assume that* $u(x,y)$ *and* v *are terms with only free variables* x, y, *possibly with some constants for elements in* G. *If for every pair* a, b *in* G *the system*

$$a = u(x,y) \qquad b = v(x,y) \tag{7}$$

has a solution in x, y, *then*

$$\nu^2\{(x,y) : f(u(x,y))f(v(x,y)) \ne f(u(x,y)v(x,y))\} \le \delta.$$

Proof. If the system (7) always has a solution, then it always has a unique solution (since G is finite). In other words, the transformation $(x,y) \mapsto (u(x,y), v(x,y))$ is measure-preserving. Now the conclusion follows by (2). ∎

For example, Claim 1 implies that for any $a \in G$ we have $\nu\{(x,y) \in G^2 : f(xa)f(a^{-1}y^{-1}) = f(xy^{-1})\} \le \delta$.

Claim 2. *We have* $\nu^2\{(x,y) \in G^2 : f(ax)f(x^{-1}) \ne f(ay)f(y^{-1})\} \le 3\delta$ *for all* $a \in G$.

Proof. If the pair (x,y) is in this set, call it A, then one of the following inequalities applies:

1. $(f(x^{-1}))^{-1} \ne f(x)$, or

2. $f(y^{-1})f(x)\neq f(y^{-1}x)$, or
3. $f(ay)f(y^{-1}x)\neq f(ax)$.

(This is because if (1)–(3) fail, then we have $f(ax)=f(ay)f(y^{-1})(f(x^{-1}))^{-1}$, which implies $(x,y)\notin A$.) But the set of (x,y) such that (i) holds has the ν^2-measure at most δ for each $i=1,2,3$. This follows from (3) for $i=1$ and from Claim 1 for $i=2,3$. Therefore $\nu(A)\leq 3\delta$, as required. ∎

Define a map $g\colon G\to H$ by letting $g(a)=p$ if

$$\nu\{x : f(ax)f(x^{-1}) = p\} > \frac{1}{2}.$$

This map is well-defined: If such a p exists, it is unique since $\nu(G)=1$. On the other hand, assume that such a p does not exist. Then for every fixed $b\in G$ we have

$$\nu\{y : f(ay)f(y^{-1}) = f(ab)f(b^{-1})\} \leq \frac{1}{2},$$

implying $\nu\{(x,y) : f(ax)f(x^{-1}) \neq f(ay)f(y^{-1})\} \leq 1/2$, and therefore contradicting Claim 2 and the assumption that $\delta \leq 1/11$. Therefore g is well-defined.

Note that Claim 2 implies that

$$g(a) = p \quad \text{if and only if} \quad \nu\{x : f(ax)f(x^{-1}) = p\} \geq 1-3\delta, \tag{8}$$

$$\text{if and only if} \quad \nu\{x : f(ax)f(x^{-1}) = p\} > 3\delta. \tag{9}$$

We claim that

$$\nu\{x : f(x) \neq g(x)\} \leq \frac{\delta}{1-3\delta}.$$

Assume the contrary, that $\nu\{x : f(x)\neq g(x)\} > \delta/(1–3\delta)$. By Fubini's theorem and (8), we have

$$\nu^2\{(x,y) : f(xy)f(y^{-1}) = g(x) \neq f(x)\} > \delta.$$

But this set has the ν-measure at most δ, by Claim 1.

Claim 3. *The map g is a homomorphism.*

Proof. We first fix $a,b\in G$ and prove that $f(a)f(b)=f(ab)$. By (8) we have

$$\nu^2\{(x,y) : f(ax)f(x^{-1})f(by)f(y^{-1}) = g(a)g(b)\} \geq (1-3\delta)^2.$$

By applying Claim 1 twice (first to $f(x^{-1})f(by)$ and then to $f(ax)f(x^{-1}by)$), we obtain $\nu^2\{(x,y) : f(aby)f(y^{-1})=g(a)g(b)\}\geq (1-3\delta)^2-2\delta$, and therefore

$$\nu\{y : f(aby)f(y^{-1}) = g(a)g(b)\} \geq (1-3\delta)^2 - 2\delta,$$

On the other hand, by (8) we have

$$\nu\{y : f(aby)f(y^{-1}) = g(ab)\} \geq 1 - 3\delta.$$

Since $\delta \leq 1/11$ implies $(1-3\delta)^2-2\delta+1-3\delta > 1$, we can find y in the intersection of these two sets, and therefore

$$g(a)g(b) = f(aby)f(y^{-1}) = g(ab).$$

Now fix a. To see that $g(a^{-1})g(a)=1_G$, note that by (8) we have

$$\nu^2\{(x,y) : f(ax)f(x^{-1})f(a^{-1}y)f(y^{-1}) = g(a)g(a^{-1})\} \geq (1-3\delta)^2,$$

and that

$$\nu^2\{(x,y) : f(ax)f(x^{-1})f(a^{-1}y)f(y^{-1}) \neq 1_G\} \leq 3\delta$$

(by applying Claim 1 first to $f(x^{-1})f(a^{-1}y)$, then to $f(x^{-1}a^{-1}y)f(y^{-1})$, and finally to $f(ax)f(x^{-1}a^{-1})$). Since $\delta \leq 1/11$ implies $(1-3\delta)^2 - 3\delta > 0$, we can find a pair x,y such that $1_G = f(ax)f(x^{-1})f(a^{-1}y)f(y^{-1}) = g(a)g(a^{-1})$, and this completes the proof. ∎

This completes the proof that g is a homomorphism which approximates f as required. The moreover part is given in (8). ∎

Proof. (of Theorem 2.1) Let $f\colon G \to \prod_{i\in I} H_i$ be an ε-approximate homomorphism. For $i \in I$ let $f_i\colon G \to H_i$ be $f_i(a) = f(a)(i)$. Fix $\delta \leq 1/11$. Let us say that an i such that f_i is a δ-approximate homomorphism of type I with respect to ν is *good*.

Claim 4. *$\varphi\{i \in I : i$ is not good$\} \leq 2\varepsilon/\delta$.*

Proof. Assume otherwise. Since φ is nonpathological, Fubini's theorem holds for the product of ν and φ (see [1]). Applying it to sets $\{(i,a) : f_i(a) \neq (f_i(a^{-1}))^{-1}\}$ and $\{(i,(a,b)) : f_i(a)f_i(b) \neq f_i(ab)\}$, we can find either an $a \in G$ such that

$$\varphi\{i : f_i(a) \neq (f_i(a^{-1}))^{-1}\} > \varepsilon$$

or $a,b \in G$ such that

$$\varphi\{(a,b) : f_i(a)f_i(b) \neq f_i(ab)\} > \varepsilon.$$

But both cases are incompatible with our assumption on f. ∎

By Theorem 2.2, for each good i there is a homomorphism $g_i\colon G\to H_i$ such that

$$\nu\{x\in G: g_i(x)\neq f_i(x)\}\leq \frac{\delta}{1-3\delta}$$

and moreover for every $a\in G$ we have

$$\nu\{y: f_i(ay)f_i(y^{-1})\neq g_i(a)\}\leq 1-3\delta.$$

For a bad i let $g_i\colon G\to H_i$ be the homomorphism which sends all elements of G to 1_{H_i}, and let $g\colon G\to\prod_i H_i$ be the diagonal of g_i's,

$$g(a)=\langle g_i(a)\rangle_{i\in I}.$$

Assume there is an $\bar a\in G$ such that

$$d_\varphi(f(\bar a),g(\bar a))>\frac{\varepsilon}{1-3\delta}+\frac{2\varepsilon}{\delta}=M,$$

or in other words, $\varphi(A)>M$ for $A=\{i: f_i(\bar a)\neq g_i(\bar a)\}$. By Claim 4, we can assume that $\varphi(A)>\varepsilon/(1-3\delta)$ and that all $i\in A$ are good. By the "moreover" part of Theorem 2.1, we have

$$\nu\{x: f_i(\bar a x)f_i(x^{-1})=g_i(\bar a)\neq f_i(\bar a)\}\geq 1-3\delta$$

for every $i\in A$. By applying Fubini's theorem to the set

$$\{(i,x): f_i(\bar a x)f_i(x^{-1})=g_i(\bar a)\neq f_i(\bar a)\}$$

(again using the fact that φ is nonpathological and [1]), we can find a $b\in G$ such that

$$\varphi\{i: f_i(\bar a b)f_i(b^{-1})\neq f_i(\bar a)\}>\varepsilon,$$

i.e. $d_\varphi(f(\bar a b)f(\bar b^{-1}),f(\bar a))>\varepsilon$, contradicting our assumption on f. We have proved that g is an $\varepsilon/(1-3\delta)+2\varepsilon/\delta$-approximation of f for $\delta\leq 1/11$, and this clearly implies that g is a 24ε-approximation to f. ∎

3. Negative results

In this section we prove a result which shows that some assumption on the submeasure φ is necessary in Theorem 2.1. We shall use some basic Ramsey theory; see e.g., [6] for the definitions. In particular, recall that $(m)^2_n$ is the minimal integer k such that for every $f\colon [k]^2 \to [n]$ there is a subset X of $[k]$ such that f is constant on $[X]^2$. Such k exists by Ramsey's theorem. Using this notation, let us define a fast-growing function

$$h(n) = (n+2)^2_n.$$

By $|X|$ we shall denote the cardinality of a set X.

Theorem 3.3. *Let K be a finite nontrivial abelian group. Then for every $M \in \mathbb{N}$ and $G = K^{\bar{k}}$, where $\bar{k} = \lceil \log_{|K|} h(|K|^{3M}) \rceil$, there is a submeasure φ on some finite set I and a 3-approximate homomorphism $g\colon G \to K^I$ of type II with respect to d_φ which can not be $3M$-approximated by a homomorphism.*

Proof. Let $I = K^G$, the set of all mappings from G into K. Define φ by letting

(10)
$$\varphi(\mathcal{A}) = \min\{|X| : X \subseteq G \text{ and for no } f \in \mathcal{A} \setminus \operatorname{Hom}(G,K),\ f \restriction X \text{ can be extended to a homomorphism}\},$$

It is easy to see that φ is a submeasure (see [2]). For $X \subseteq G$, let

$$\mathcal{C}_X = \{f \in K^G : f \restriction X \text{ can not be extended to a homomorphism}\};$$

then clearly $\varphi(\mathcal{A}) \leq k$ if and only if $\mathcal{A} \subseteq \mathcal{C}_X \cup \operatorname{Hom}(G,K)$ for some $X \subseteq G$ of size $\leq k$. Define the mapping $g\colon G \to K^I$ by

$$g(a) = \langle f(a) \rangle_{f \in I}.$$

The mapping g is a 3-approximate homomorphism of type II with respect to φ, because for all $a, b \in G$ we have

$$d_\varphi(g(a)g(b), g(ab)) = \varphi\{f \in G^K : f(a)f(b) \neq f(ab)\} \leq \varphi(\mathcal{C}_{\{a,b,ab\}}) = 3$$
$$d_\varphi(g(a^{-1}), g(a)^{-1}) = \varphi\{f \in G^K : f(a^{-1}) \neq (f(a))^{-1}\} \leq \varphi(\mathcal{C}_{\{a,a^{-1}\}}) = 2.$$

It remains to prove that g can not be $3M$-approximated by a homomorphism. Let $\Phi\colon G \to K^I$ be a map such that

$$\varphi(\{f \in K^G : \Phi(a)(f) \neq g(a)(f)\}) \leq 3M$$

for all $a\in G$. We will prove that Φ is not a homomorphism. Let $X_a\subseteq G$ be a set of size $\leq 3M$ such that the above set is included in X_a. Since all elements in G are of order $\leq |K|$ and G is abelian, the subgroup Y_a of G generated by X_a, which we denote by $\langle X_a\rangle$, is of size $\leq |K|^{3M}$. (This was the only place where we used the assumption that $G=K^I$; but we shall use the fact that it is abelian and that for every $a\in G\setminus\{1_G\}$ there is a homomorphism $g\colon G\to K$ such that $g(a)\neq 1_K$.) Let us write

$$n=|K|^{3M}.$$

Claim 5. *There are distinct $a,b\in G$ such that $b\notin Y_a$, $ab\notin\langle (Y_a\cup Y_b)\cap Y_{ab}\rangle$ and $a\neq 1$.*

Proof. Fix a linear ordering $<$ on G, and let $\{a,b\}_<$ denote the unordered pair $\{a,b\}$ such that $a<b$. For every $a\in G$ fix an enumeration

$$Y_a=\{d_1^a,\dots,d_n^a\},$$

and define a partition $p\colon [G]^2\to[3M+3]$ as follows:

$$p(\{a,b\}_<)=\begin{cases}1, & \text{if } b\notin Y_a \quad\text{and}\quad ab\notin\langle (Y_a\cup Y_b)\cap Y_{ab}\rangle,\\ 2, & \text{if } b\in Y_a,\\ 3, & \text{if } b\notin Y_a \quad\text{and}\quad ab\in Y_b,\\ i+3, & \text{if } b\notin Y_a, ab\notin Y_b, \text{ and } i \text{ is the minimal such that}\\ & ab=d_i^a b' \text{ for } d_i^a\in Y_{ab} \text{ and some } b'\in Y_{ab}.\end{cases}$$

For all $a\neq b$ in G the value $p(\{a,b\})$ is well-defined: If neither of the first two cases applies, then $ab\in\langle (Y_a\cup Y_b)\cap Y_{ab}\rangle$, but since G is abelian this means that we can write $ab=a'b'$ for $a'\in Y_a\cap Y_{ab}$ and $b'\in Y_b\cap Y_{ab}$.

Subclaim 1. *There exists no set $A\subseteq G$ of size $n+2$ which is l-homogeneous for some $l\neq 1$.*

Proof. There exists no set $A\subseteq G$ which is 2-homogeneous of size $n+2$; otherwise, if a is the $<$-minimal element of A, then $A\setminus\{a\}\subseteq Y_a$, and this contradicts to fact that $|Y_a|\leq n$. Similarly, there exists no 3-homogeneous $A\subseteq G$ of size $n+2$.

It remains to take care on the case when $l\geq 3$. Assume that A is l-homogeneous for $l\geq 3$ and of size $n+2$. Let $A=\{a_1,\dots,a_{n+2}\}$ be its $<$-increasing enumeration. Then for $a_i'=d_l^{a_i}$ and all $j>i$ we have $c_{ij}'\in Y_{a_j}\cap Y_{a_ia_j}$ such that $a_i'c_{ij}'=a_ia_j$ (observe that a_i' does not depend on j). We claim that $a_i'\neq a_j'$ whenever $i<j<3M+2$; otherwise we have $a_j'c_{ij}=a_i'c_{ij}'=a_ia_j$, and therefore $a_ia_j\in Y_{a_j}$, contradicting the fact that $p(\{a_i,a_j\}_<)\neq 3$. But this means that $Y_{a_{n+2}}$ has $n+1$ many elements, a contradiction. ∎

Since the size of G is $\geq h(n) = (n+2)^2_n$, by Subclaim it follows that there are $a < b$ in G such that $p(\{a,b\}) = 1$; such a,b are as required. ∎

If a,b are as in the above Claim, let $i \leq \bar{k}$ be such that $a(i) \neq 1_K$ (recall that $a \in G = K^{\bar{k}}$), and let $p_i\colon G \to K$ be the projection to this coordinate. Define $f_1\colon Y_a \to K$ to be the restriction of p_i to Y_a. Now find a homomorphism $f_2\colon Y_b \to K$ so that f_1 and f_2 agree on $Y_a \cap Y_b$ and $f_2(b) = 1_K$ (this is possible, since $Y_a \cap Y_b$ is a subgroup of Y_b). We can extend $f_1 \cup f_2$ to a homomorphism $f'\colon \langle Y_a \cup Y_b \rangle \to K$. This is done by letting $f'(c) = f_1(c_a) f_2(c_b)$ if $c_a \in Y_a$, $c_b \in Y_b$ are such that $c = c_a c_b$. Although c_a, c_b are not unique for the given c, it is easy to see that the value $f_1(c_a) f_2(c_b)$ depends only on c, and that this formula does define a homomorphism which extends f_1 and f_2. Let $f_3\colon Y_{ab} \to K$ be a homomorphism which extends $f' \restriction Y_{ab}$ and such that $f_3(ab) = 1_K$. The maps f_1, f_2 and f_3 are constructed so that they pairwise agree on the intersections of their domains, therefore there is a function $f\colon G \to K$ which extends all three of them. Note that f is not a homomorphism, since $f(a)f(b) \neq f(ab)$, but $f \in \mathcal{C}_{Y_a} \cup \mathcal{C}_{Y_b} \cup \mathcal{C}_{Y_{ab}}$ (as witnessed by f_1, f_2 and f_3, respectively), and therefore $g(a)(f) = \Phi(a)(f)$, $g(b)(f) = \Phi(a)(f)$ and $g(ab)(f) = \Phi(ab)(f)$. But this implies that $\Phi(a)\Phi(b)$ and $\Phi(ab)$ differ on f-th coordinate, and therefore Φ is not a homomorphism. ∎

4. Boolean algebras revisited

In this section we connect the stability of approximate group homomorphisms with the stability of approximate Boolean algebra homomorphisms, studied in [2]. As an application, we use the results of §2 to give a quantitative improvement of [2, Theorems 4 and 5].

Definition 4.1. Assume $[m]$ supports a submeasure φ and fix $\varepsilon, \delta > 0$. A mapping $f\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is an *ε-approximate Boolean algebra homomorphism (of type II)* (with respect to φ) if for all s,t we have

$$\varphi((f(s) \cup f(t)) \Delta f(s \cup t)) < \varepsilon$$
$$\varphi(f(s^{\complement}) \Delta f(s)^{\complement}) < \varepsilon$$

A homomorphism $\Phi\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is a *δ-approximation for H* if

$$\varphi(\Phi(s) \Delta f(s)) < \delta$$

for all s.

We can naturally identify the group $\langle\{0,1\}^{[m]},\Delta\rangle$ with the group $\langle\mathbb{Z}_2^{[m]},+_2\rangle$, and a submeasure φ on $[m]$ with a pseudometric d_φ on $\mathbb{Z}_2^{[m]}$ (see (6) from the introduction). Therefore an (approximate) Boolean algebra homomorphism becomes an (approximate) group homomorphism (but not vice versa). Let us be more precise.

Lemma 4.1. *If $f\colon \{0,1\}^{[m]}\to\{0,1\}^{[n]}$ is a δ-approximate Boolean algebra homomorphism of type I (type II, respectively), then it is a 7δ-approximate group homomorphism of type I (type II, respectively).*

Proof. Since in $\mathbb{Z}_2^{[m]}$ each element is idempotent, we trivially have $f(s^{-1})=f(s)^{-1}$ for all u. Since $s\Delta t=(s^{\complement}\cup t)^{\complement}\cup(s\cup t^{\complement})^{\complement}$, we need no more than seven applications of $^{\complement}$ and $\cup$ to express Δ. Therefore $\varphi(f(s\Delta t)\Delta(f(s)\Delta f(t)))\leq 7\varepsilon$. ∎

Definition 4.2. For an n-tuple $\mathbf{X}=\langle X_j : j\in[n]\rangle$ of nonempty subsets of $[m]$ define $g_{\mathbf{X}}\colon \mathbb{Z}_2^{[m]}\to\mathbb{Z}_2^{[n]}$ by

$$g_{\mathbf{X}}(a)=\{j : |a\cap X_j| \text{ is odd}\}.$$

Proposition 4.1. *Every group homomorphism $g\colon \mathbb{Z}_2^{[m]}\to\mathbb{Z}_2^{[n]}$ is of the form $g_{\mathbf{X}}$ for some n-tuple $\mathbf{X}$.*

Proof. Since g is a homomorphism if and only if each g_j is a homomorphism, it will suffice to prove this in the case when $j=1$. But $g\colon \mathbb{Z}_2^{[m]}\to\mathbb{Z}_2$ is a homomorphism if and only if $A=\{s\in\mathbb{Z}_2^{[m]}: g(s)=0\}$ is a subgroup of $\mathbb{Z}_2^{[m]}$ of order 2.

Claim 6. *If A is a subgroup of $Z_2^{[m]}$ of order two, then $A=\{s\in\mathbb{Z}_2^{[m]}: s\cap X_A$ is even$\}$ for some $X_A\subseteq[m]$.*

Proof. Let $X_A=\{i\in[m]:\{i\}\notin A\}$. Since A is of order two, for all $s\in\mathbb{Z}_2^{[m]}$ and $i\in[m]$ we have

$$s\Delta\{i\}\in A \text{ if and only if } s\in A$$

exactly when $i\in X_A$. Since $\emptyset\in A$, the conclusion follows by an induction on the size of s. ∎∎

Lemma 4.2. *If $g\colon \{0,1\}^{[m]} \to \{0,1\}$ is a group homomorphism which is not a Boolean algebra homomorphism, then g is not a $3/8$-approximate Boolean algebra homomorphism of type I (with respect to the uniform probability measure ν on $\{0,1\}^{[m]}$).*

Proof. We shall prove that

$$\alpha = \nu^2\{\langle s,t\rangle : g(s) \cup g(t) \neq g(s \cup t)\}$$

is $\geq 3/8$. For a set B let $|B|_2$ be its cardinality modulo 2, i.e., $|B|_2 = 1$ if $|B|$ is odd and 0 otherwise. By Proposition 4.1 let X be such that $g(s) = |s \cap X|_2$. Then we have

$$\alpha = \nu\{\langle s,t\rangle : \max(|s \cap X|_2, |t \cap X|_2) \neq |(s \cup t) \cap X|_2\},$$

and it is a routine to check that $\alpha \geq 3/8$ if $|X| \geq 3$. ∎

Lemma 4.3. *If $f\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is a δ_1-approximate Boolean algebra homomorphism which is δ_2-approximated by some g, then g is a $3\delta_2 + \delta_1$-approximate homomorphism.*

Proof. $g(s \cup t) \Delta (g(s) \cup g(t)) \subseteq (g(s \cup t) \Delta f(s \cup t)) \cup f(s \cup t) \Delta (f(s) \cup f(t)) \cup (f(s) \Delta g(s)) \cup (f(t) \Delta g(t))$, and similarly for ${}^{\mathsf{c}}$. ∎

Theorem 4.4. *Every ε-approximate Boolean algebra homomorphism with respect to some nonpathological submeasure can be 160ε-approximated by a Boolean algebra homomorphism.*

Proof. This proof closely follows that of Theorem 2.2. Let $f\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ be an ε-approximate Boolean algebra homomorphism with respect to some nonpathological submeasure φ on $[n]$. For $j \in [n]$, let $f_j\colon \{0,1\}^{[m]} \to \{\emptyset, \{j\}\}$ be defined by

$$f_j(u) = f(u) \cap \{j\}.$$

Let ν be the uniform probablitity measure on $\{0,1\}^{[m]}$. Fix $0 < \delta_0 \leq 1/79$, and let us say that $j < n$ is *good* if f_j is a δ_0-approximate Boolean algebra homomorphism of type I.

Claim 7. $\varphi\{j : f_j$ *is not good*$\} \leq 2\varepsilon/\delta_0$.

Proof. By an application of Fubini's theorem, analogous to one in the proof of Theorem 2.1. ∎

By Lemma 4.1, f_j is a $7\delta_0$-approximate group homomorphism of type I for every good j. Since $7\delta_0 < 1/11$, the assumptions of Theorem 2.2 (with $\delta = 7\delta_0$) are satisfied. Thus there is a group homomorphism $g_j\colon \{0,1\}^{[m]} \to \{0,\{j\}\}$ (where the range is isomorphic to $\mathbb{Z}_2$) which is a $7\delta_0/(1-21\delta_0)$ approximation to f_j, and which moreover satisfies

$$\nu\{x : f_j(s\Delta x)f_j(x) = g_j(s)\} \geq 1 - 21\delta_0, \qquad \text{for all } s \in \{0,1\}^{[m]}. \tag{11}$$

By Lemma 4.3, g is a $21\delta_0/(1-21\delta_0)+\delta_0$-approximate Boolean algebra homomorphism. Since $\delta_0 \leq 1/79$, this expression is less than $3/8$, therefore by Lemma 4.2 g is a Boolean algebra homomorphism.

Let $\mathcal{G} = \{j \in [n] : j \text{ is good}\}$ and define $g\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ by

$$g(s) = \bigcup_{j \in \mathcal{G}} g_j(s). \tag{12}$$

We claim that

$$\varphi(g(s)\Delta f(s)) \leq \frac{\varepsilon}{1-21\delta_0} + \frac{2\varepsilon}{\delta_0} \tag{13}$$

for all $s \in \{0,1\}^{[m]}$. It will suffice to prove that $\varphi((g(s)\Delta f(s)) \cap \mathcal{G}) \leq \varepsilon/(1-21\delta_0)$ for all s. This can be done by applying Fubini's theorem, exactly like in Theorem 2.1.

Since $\delta_0 \leq 1/79$, we have $1/(1-21\delta_0)+2/\delta_0 \leq 160$, and this concludes the proof. ∎

5. Concluding remarks

It is rather clear that the numerical estimations appearing in Theorem 2.1, and especially Theorem 4.4, can be improved. For example, the following (among other results) was shown in [7], by a proof closely following our proof of Theorem 2.1:

Theorem 5.5. *An ε-approximate homomorphism $f\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ can be 7.5ε-approximated by a strict homomorphism.*

Although it would be interesting to know the true values of "stability constants", the problem of identifying classes of homomorphisms in which Ulam stability occurs is much more interesting and important. In particular, the main objective of §4 is not to improve the numerical constants, but instead to point out the connection between two instances of stability. It should be noted that this connection is not symmetric; see [4, §7].

By Theorem 2.1, the submeasure φ constructed in Theorem 3.3 is pathological for a large enough M. This can be be proved directly, in a way similar to [2, Theorem 8].

It would be interesting to know whether a counterexample like the one in §3 can be obtained when φ is an arbitrary "sufficiently pathological" submeasure. This question can be formalized in exactly the same way as it was in [2, Questions 11 and 12], to where we direct the reader for further information. Let us conclude with a question, a positive answers to which would most likely imply the positive answer to the corresponding [2, Question 14]. An approximate homomorphism $f\colon G \to H$ is an *ε-approximate epimorphism* if every element of H is within ε-distance from some element in the range of f.

Question 1. Is there a universal constant K such that every ε-approximate epimorphism can be $K\cdot\varepsilon$-approximated by a homomorphism?

References

[1] J. P. R. Christensen: Some results with relation to the control measure problem, in: *Vector Measures and Applications*, volume 644 of *Lecture Notes in Mathematics*, pages 125–158. Springer-Verlag, 1978.

[2] I. Farah: Approximate homomorphisms, *Combinatorica*, **18** (1998), 335–348.

[3] I. Farah: Completely additive liftings, *The Bulletin of Symbolic Logic*, **4** (1998), 37–54.

[4] I. Farah: Liftings of homomorphisms between quotient structures and Ulam stability, to appear in Proceedings of the Logic Colloquium '98, 1999.

[5] G. L. Forti: Hyers–Ulam stability of functional equations in several variables, *Aequationes Math.*, **50** (1995), 143–190.

[6] R. L. Graham, B. L. Rotschild, and J. Spencer: *Ramsey Theory*, John Wiley and Sons, 1980.

[7] V. Kanovei and M. Reeken: On Ulam's problem of approximation of non-exact homomorphisms, preprint, 1999.

[8] D. Kazhdan: On ε-representations, *Israel Journal of Mathematics*, **43** (1983), 315–323.

[9] M. Talagrand: Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem, *Geometric and Functional Analysis*, **3** (1993), 295–314.

[10] M. Talagrand: On boundaries and influences, *Combinatorica*, **17** (1997), 275–285.

[11] S. M. Ulam: *Problems in Modern Mathematics*, John Wiley & Sons, 1964.

[12] S. M. Ulam and D. Mauldin: Mathematical problems and games, *Advances in Applied Mathematics*, **8** (1987), 281–344.

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