Liftings of Homomorphisms Between Quotient Structures and Ulam Stability

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One of the trends in Descriptive Set theory during the last decade or so is the study of quotient structures over Polish spaces (see e.g., [27], [1], [15], [25], also [5, I.C, II.3, p. 466]). Typical such structures are the quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$, where $\mathcal{I}$ is an analytic ideal on the integers and where we consider $\mathcal{P}(\mathbb{N})$ with its Cantor-set topology, obtained by identifying subsets of $\mathbb{N}$ by their characteristic functions. In [8] (see also [12]) we have seen some success in studying these quotients as Boolean algebras. More precisely, we have proved that in this case the quotient structure $\mathcal{P}(\mathbb{N})/\mathcal{I}$ determines the ideal $\mathcal{I}$ up to the isomorphism, for a large class of nonpathological analytic P-ideals $\mathcal{I}$ (for definition see §4). In other words, the structure of the quotient algebra is uniquely determined by the corresponding ideal. These results can take both the Descriptive set-theoretic and the Combinatorial set-theoretic form, in [8, §2.1] we have proved that these two presentations are in fact equivalent.

In the present paper we investigate how this theory can be extended to different contexts, when $\mathcal{P}(\mathbb{N})$ is equipped with some weaker structure. For example, we consider these quotients as groups under the operation of symmetric difference or as semilattices under the intersection operation.

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1 An introduction

The power-set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ can be considered in any of the following disguises (and the list below is certainly not exhaustive):

BA) A Boolean algebra, with the operations $\cup, \cap, ^{\mathsf{c}}$ and constants $\emptyset, \mathbb{N}$.
G) A group, under the operation of symmetric difference Δ.

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SL) A semilattice, with the operation $\cap$.
P) A partially ordered set, with the relation $\subseteq$.
E) A set with no structure.

Also, any of the quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over an ideal $\mathcal{I}$ on $\mathbb{N}$ can be considered as a Boolean algebra, a group, etc. Note that the Boolean algebras have the richest structure, and therefore are easiest to analyze (see [8], [12]). In this note we will concentrate on studying the structure of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ in categories G and SL. We shall also restrict our attention to the ideals which are simple subsets of $\mathcal{P}(\mathbb{N})$. Recall that a subset of a Polish (i.e., separable and completely metrizable) space is *analytic* if it is a continuous image of a Borel subset of some Polish space. *Analytic ideals*, namely those ideals $\mathcal{I}$ which are analytic subsets of $\mathcal{P}(\mathbb{N})$, when it is given its Cantor-set topology, have emerged as a rather natural and rich class of simple ideals on $\mathbb{N}$, especially after the recent work of Todorcevic and Solecki ([41], [34], [43]). Essentially all natural examples of ideals on the integers occurring in the literature (the ideal Fin of all finite sets of integers, density ideals, summable ideals, countably generated ideals, ...) are analytic (see [8, §1]), and even Borel. We say that a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over an analytic ideal is an *analytic quotient*.

Understanding quotients requires understanding homomorphisms between them. For a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ by Φ_* we denote its *lifting*, i.e., the mapping $\Phi_*\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{\Phi_*} & \mathcal{P}(\mathbb{N}) \\ \big\downarrow{\scriptstyle \pi_{\mathcal{I}}} & & \big\downarrow{\scriptstyle \pi_{\mathcal{J}}} \\ \mathcal{P}(\mathbb{N})/\mathcal{I} & \xrightarrow{\Phi} & \mathcal{P}(\mathbb{N})/\mathcal{J} \end{array}$$

commutes ($\pi_{\mathcal{I}}, \pi_{\mathcal{J}}$ are the natural projections associated to the quotients over $\mathcal{I}$ and $\mathcal{J}$, respectively). We should remark that this terminology is different from one used in Analysis, where a lifting is required to be a homomorphism itself (see [17]). In this note (except in §7) we shall restrict our attention to those homomorphisms which have a Baire-measurable lifting. In our opinion, "Baire-measurable mapping of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$" is a natural formalization of "simply definable mapping of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$". This is because in Solovay's model for Analysis ([36]) all sets of reals have the property of Baire, and therefore all maps of $\mathcal{P}(\mathbb{N})$ into itself are Baire-measurable. Since Solovay's model contains all sets of reals which are definable by a formula having reals and ordinals as parameters, this can be interpreted as saying that every real function f definable in such a way is likely to be Baire-measurable. In [8] (see also [12]) we have studied analytic quotients as Boolean algebras, and proved that homomorphisms between "nonpathological" quotients (see §4 for the definition) are trivial. Namely, each of these homomorphisms is given by a *completely additive lifting*, i.e., one of the form

$$a \mapsto h^{-1}(a)$$

for some mapping $h\colon \mathbb{N} \to \mathbb{N}$. It is easy to see that in other categories, like G or SL, there are homomorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ which are not of this form (compare with Corollary 8 and 9).

The following definition of a trivial homomorphism is meaningful (and useful) in any of the categories BA, G, SL, and in the case of BA it coincides with the above definition.

Definition 1. *A homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is* trivial *if it has a lifting F which is itself a homomorphism of $\mathcal{P}(\mathbb{N})$ into itself. We say that such an F is* additive.

In this note we are mainly concerned with proving that certain homomorphisms are trivial. This is achieved as a result of a sequence of small modifications to a given lifting, eventually resulting in an additive lifting of the same homomorphism. The reader has to be warned that the technical side of this "trivialization" process is rather trivial itself (but the triviality is in the eye of the beholder—see [38] and [39]; also, these results tend to be quite useful—an unexpected application is given in §8).

Let us now give a short overview of the paper. In §§2–3 we work on simplifying Baire-measurable liftings. §4 gives the connection between the existence of liftings and stability in the sense of S. Ulam ([44, VI.1] [45, V.4]), which is then applied to make a further simplification of Baire-measurable liftings in §5. In §6 we describe how the simplest canonical liftings, obtained from Baire-measurable ones in §§2–5, look like. The question of what can be done in the case when liftings are not necessarily Baire-measurable is treated in §7. In §8 our results about semilattice homomorphisms are applied to give an application to Topology, following an idea of M. Bell ([2]). We conclude by suggesting a list of questions for further research in §9.

2 From Baire-measurable to continuous liftings

Throughout this section we shall assume $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $F: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a Baire-measurable function which is:

(a) Fin-*invariant with respect to* $\mathcal{I}$: If $A \Delta B$ is finite, then $F(A) \Delta F(B) \in \mathcal{I}$.

Proposition 1. *There is a sequence $1 = n_1 < n_2 < \dots$ and $s_i \subseteq [n_i, n_{i+1})$ such that F is continuous on $\{X \in \mathcal{P}(\mathbb{N}) : X \cap [n_i, n_{i+1}) = s_i$ for infinitely many $i\}$.*

Proof. Since F is Baire-measurable, it is continuous on a dense G_δ set, and every dense G_δ set includes one of the above form.

By m, n, i, j, k we shall denote integers and by $s, t, u, v, \dots$ finite sets of integers. For sequences $\{n_i\}$, $\{s_i\}$, as given by Proposition 1 let

$$A_{\text{even}} = \bigcup_i [n_{2i}, n_{2i+1}), \qquad A_{\text{odd}} = \bigcup_i [n_{2i+1}, n_{2i+2}),$$
$$S_{\text{even}} = \bigcup_i s_{2i}, \qquad S_{\text{odd}} = \bigcup_i s_{2i+1}.$$

The following result is well-known (see [47, p. 132], [43, Theorem 3], [8, Lemma 1.1]).

Corollary 1. *If a Boolean algebra homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ has a Baire-measurable lifting, then it has a continuous lifting.*

Proof. Apply Proposition 1 to a Baire-measurable lifting F of Φ to obtain sequences n_i, s_i. Then the map $\hat{F} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by

$$\hat{F}(X) = F((X \cap A_{\text{even}}) \cup S_{\text{odd}}) \cap F(A_{\text{even}}) \\ \cup (F((X \cap A_{\text{odd}}) \cup S_{\text{even}}) \cap F(A_{\text{odd}}))$$

is a lifting of Φ, and it is continuous since $(X \cap A_{\text{even}}) \cup S_{\text{odd}}$, $(X \cap A_{\text{odd}}) \cup S_{\text{even}}$ are members of the dense G_δ set on which F is continuous.

The formula used to obtain $\hat{F}$ in the proof of Corollary 1 does not give us a lifting of Φ under the weaker assumption that Φ is a group homomorphism, since then Φ does not necessarily preserve unions and intersections. This is, however, not a substantial problem.

Corollary 2. *If a group homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ has a Baire-measurable lifting, then it has a continuous lifting.*

Proof. Apply Proposition 1 to a Baire-measurable lifting F of Φ to obtain n_i, s_i. Let $S = \bigcup_i s_i = S_{\text{even}} \cup S_{\text{odd}}$ and for $X \subseteq \mathbb{N}$ let $X_0 = (X \cap A_{\text{even}}) \Delta S$ and $X_1 = (X \cap A_{\text{odd}}) \Delta S$. Then

$$\hat{F}(X) = F(X_0) \Delta F(X_1)$$

is a lifting of Φ since $X_0 \Delta X_1 = X$. It is also continuous since X_0, X_1 are members of the dense G_δ set on which F is continuous.

In the case of semilattice homomorphisms we can only draw a conclusion weaker than one in Corollary 1 and 2.

Corollary 3. *If an SL-homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ has a Baire-measurable lifting, then we can partition $\mathbb{N}$ into two sets, A_0 and A_1, such that both restrictions of Φ to $\mathcal{P}(A_0)/\mathcal{I}$ and to $\mathcal{P}(A_1)/\mathcal{I}$ have continuous liftings.*

Proof. Apply Proposition 1 to a Baire-measurable lifting F of Φ to obtain n_i, s_i. Let $A_0 = A_{\text{even}}$ and $A_1 = A_{\text{odd}}$. Then define $\hat{F}_0\colon \mathcal{P}(A_0) \to \mathcal{P}(\mathbb{N})$ by

$$\hat{F}_0(X) = F(X \cup S_{\mathrm{odd}}) \cap F(A_0).$$

This is a clearly a continuous lifting of Φ on $\mathcal{P}(A_0)$, and

$$\hat{F}_1(X) = F(X \cup S_{\mathrm{even}}) \cap F(A_1)$$

is a continuous lifting of Φ on $\mathcal{P}(A_1)/\mathcal{I}$.

The conclusion of Corollary 3 can not be improved to obtain the result analogous to those in the Boolean algebra case or in the group case, as the following example shows.

Example 1. A semilattice homomorphism Φ of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself with a Borel-measurable lifting, but with no continuous lifting.

We shall define Φ by its lifting, F. Let $\mathcal{F}$ be a uniform Borel filter on $\mathbb{N}$ which contains all singletons. Let $\mathbb{N}_{\mathrm{even}} = \{2n : n \in \mathbb{N}\}$ and $\mathbb{N}_{\mathrm{odd}} = \{2n+1 : n \in \mathbb{N}\}$, and let $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be defined by

$$F(a) = \begin{cases} a \cup \mathbb{N}_{\mathrm{even}}, & \text{if } a \in \mathcal{F}, \\ a & \text{otherwise.} \end{cases}$$

Then F is a lifting of a semilattice homomorphism Φ, and it is Baire-measurable. On the other hand, Φ does not have a continuous lifting. Assume the contrary, that $G : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous lifting for Φ, and let

$$U_n = \{X \in \mathcal{P}(\mathbb{N}) : 2n \notin G(X)\}.$$

Observe that each U_n is clopen, and that for every $X \in \mathcal{P}(\mathbb{N})$ the set $\{n : X \in U_n\}$ is either finite or cofinite. Let T consist of all finite $s = \{i(1), \dots, i(k)\}$ (in an increasing order) such that the set

$$V_s = \bigcap_{j \text{ even}} U_{i(j)} \setminus \bigcup_{j \text{ odd}} U_{i(j)}$$

is nonempty. We consider T as a tree under the end-extension ordering. By the above remark, T does not have an infinite branch, and therefore it has a terminal node $\bar{s}$. Assume that the size of $\bar{s}$ is an even integer. Then $V_{\bar{s}}$ is a clopen subset of $\mathcal{P}(\mathbb{N})$ and every $X \in V_{\bar{s}}$ is in at most finitely many U_i's, therefore $2i \in G(X)$ for all but finitely many i. But we can choose $X \in V_{\bar{s}}$ to be finite, which contradicts to the assumption that F and G are liftings of the same map.

The case when the size of $\bar{s}$ is an odd integer leads to a contradiction by a similar argument, and this completes the proof.

It should be clear that this example can be easily made more complicated; Question 3 below asks how much more complicated.

3 From continuous to asymptotically additive liftings

In this section we continue an analysis of liftings done in previous sections under some stronger assumptions. Recall that an ideal $\mathcal{I}$ is a *P-ideal* if for every countable sequence of sets $\{A_n\}$ in $\mathcal{I}$ there is a single set A_∞ in $\mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all n. A submeasure φ is *lower semicontinuous* if

$$\lim_{n\to\infty} \varphi(A \cap [1,n]) = \varphi(A)$$

for all $A \subseteq \mathbb{N}$. Throughout this section we shall assume

(b) $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ and $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous function which is Fin-invariant with respect to $\mathcal{I}$.

Another way to think of this condition is

(b′) $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ and mapping $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a continuous lifting F.

Recall the following important result of Solecki ([34], [35]), which is extended in [42] to all P-ideals in $L(\mathbb{R})$.

Theorem 1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then $\mathcal{I}$ is an analytic P-ideal if and only if*

$$\mathcal{I} = Exh(\varphi) = \{A : \lim_n \varphi(A \setminus [1,n]) = 0\}$$

for some lower semicontinuous submeasure φ.

By Solecki's theorem we can assume that $\mathcal{I}$ given in (b′) is of the form $\mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ. A sequence $\{v_i\}$ of sets is *point-finite* if an intersection of any infinite subsequence is empty. In the following definition by $\varDelta_{i=1}^\infty s_i$ we denote the infinite product $s_1 \varDelta s_2 \varDelta s_3 \varDelta \dots$ (i.e., this is the infinite $\mathbb{Z}_2$-product). Note that this expression is well-defined if and only if the family $\{s_i\}$ is point-finite.

Recall that the additive liftings preserve algebraic operations, for example $F(X \varDelta Y) = F(X) \varDelta F(Y)$ in the case of group homomorphisms. The following definition gives a weakening of additivity which naturally occurs in our context.

Definition 2. *If $\{n_i\}$ is a strictly increasing sequence of integers, $\{v_i\}$ is a point-finite sequence of finite sets of integers, and $H_i : \mathcal{P}([n_i, n_{i+1})) \to \mathcal{P}(v_i)$, then define $\Psi_H : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by*

$$\Psi_H(X) = \prod_{i=1}^{\infty} H_i(X \cap [n_i, n_{i+1})).$$

We say that such Ψ_H is asymptotically additive.

Note that asymptotically additive liftings satisfy $F(s\Delta t) = F(s)\Delta F(t)$ if s and t are finite sets which are sufficiently "far apart", namely if $\max s < n_i \leq \min t$ for some i. Observe that in the case when v_i are pairwise disjoint we have

$$\Psi_H(X) = \bigcup_{i=1}^{\infty} H_i(X \cap [n_i, n_{i+1})),$$

and therefore the above definition of an asymptotically additive mapping Ψ_H generalizes "lifting of the form Ψ_H" introduced in [8, Definition 1.4.1] (see also [12]). We will now extend the following result given in [8, Theorem 1.4.3].

Proposition 2. *If, in addition to (b'), we assume that Φ is a Boolean algebra homomorphism, then Φ has an asymptotically additive lifting F.*

It should be noted that Proposition 2 has the following simple converse (but see also the beginning of §9).

Lemma 1. *If $\mathcal{I}$ is not a P-ideal, then there is a* BA- (G-, SL-) *homomorphism with a Baire-measurable lifting but with no asymptotically additive lifting.*

Proof. It will suffice to prove this in the case of BA-homomorphisms, since the notion of being "asymptotically additive" is independent on the algebraic structure of quotients.

Since $\mathcal{I}$ is not a P-ideal, we can fix A_n ($n \in \mathbb{N}$), pairwise disjoint sets in $\mathcal{I}$ such that no set in $\mathcal{I}$ includes all A_n modulo finite. Then

$$B \mapsto \bigcup\{A_n : n \in B\}$$

is a simple lifting of a BA-homomorphism, and it is easy to check that it does not have an asymptotically additive lifting.

If $F, G \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, then $F =^{\mathcal{I}} G$ stands for $F(X)\Delta G(X) \in \mathcal{I}$ for all X. The following notion was extracted from work of Shelah ([33]) by Velickovic ([47]) and Just ([21]) and it is crucial for analysis of liftings (see e.g., [8, §1.4 and §3.10]).

Definition 3. *An $s \subseteq (n, n')$ is an (n, n')-*stabilizer *(of F with respect to φ) if there is a $k \in (n, n')$ such that for all $u, v \subseteq [1, n]$ and all $X \subseteq [n', \infty)$ we have*

(1) $(F(u \cup s \cup X)\Delta F(u \cup s \cup Y)) \cap [1, k] = \emptyset$,
(2) $\varphi(F(u \cup s \cup X)\Delta F(v \cup s \cup X) \setminus [1, k]) < 2^{-n}$.

The proof of the following lemma given below, different from the "usual" one (see e.g., [47], [8]), is due to V. Kanovei and it is reproduced here with his kind permission.

Lemma 2. *For every n and all large enough n' there is an (n, n')-stabilizer.*

Proof. Fix an n. By the uniform continuity of F, the condition (1) is satisfied for all large enough k, and it will therefore suffice to assure (2). For $u, v \subseteq [1, n)$ and $Z \subseteq [n, \infty)$ let

$$F_{uv}(Z) = F(u \cup Z) \Delta F(v \cup Z).$$

Then $G\colon \mathcal{P}([n,\infty)) \to \mathcal{I}$ defined by $G(Z) = \bigcup_{u,v\subseteq[1,n)} F_{uv}(Z)$ is a continuous map (where both spaces are taken with the Cantor-set topology inherited from $\mathcal{P}(\mathbb{N})$). Moreover for every $k > n$ the set

$$W_k = \{Z \subseteq [n,\infty) : \varphi(G(Z) \setminus [1,k)) < 2^{-n-1}\}$$

is closed, and $\mathcal{P}([n,\infty)) \subseteq \bigcup_k W_k$. By the Baire Category theorem there is a $k > n$ such that W_k includes a nonempty open set, and therefore there are $n' > n$ and a $s \subseteq [n, n')$ such that $\varphi(G(Z) \setminus [1,k)) \leq 2^{-n-1}$ for all $Z \subseteq [n,\infty)$ such that $Z \cap [n, n') = s$. Therefore k, n' and s are as required.

Proposition 3. *If, in addition to* (b'), *we assume that* Φ *is a group homomorphism, then* Φ *has an asymptotically additive lifting* F.

Proof. This proof follows closely the proof of [8, Theorem 1.4.2], modulo the necessary algebraic adjustments. By using Lemma 2 pick sequences n_i, k_i and $s_i \subseteq [n_i, n_{i+1})$ so that s_i is an (n_i, n_{i+1})-stabilizer of F, with k_i witnessing this fact. Like in §2, let $S_{\text{even}} = \bigcup_i s_{2i}$, $S_{\text{odd}} = \bigcup_i s_{2i+1}$, $A_{\text{even}} = \bigcup_i [n_{2i}, n_{2i+1})$, $A_{\text{odd}} = \bigcup_i [n_{2i+1}, n_{2i+2})$, and let $H_i\colon \mathcal{P}([n_i, n_{i+1}) \to \mathcal{P}([k_{i-1}, k_{i+1}))$ be defined by

$$H_i(t) = \begin{cases} (F(t\Delta S_{\text{even}})\Delta F(S_{\text{even}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is odd,} \\ (F(t\Delta S_{\text{odd}})\Delta F(S_{\text{odd}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is even.} \end{cases}$$

Then for $X \subseteq A_{\text{even}}$ and an even i by Definition 3.(2) we have

1. $\varphi((H_i(X \cap [n_i, n_{i+1}))\Delta F(S_{\text{odd}}))\Delta F(X\Delta S_{\text{odd}}) \cap [k_{i-1}, k_{i+1})) < 2^{-i}$.

Therefore if we let $G_{\text{even}}\colon \mathcal{P}(A_{\text{even}}) \to \mathcal{P}(\mathbb{N})$ be defined as

2. $G_{\text{even}}(X) = \Delta_{i \text{ even}} H_i(X \cap [n_i, n_{i+1}))$,

then by the above and $F(X) =^{\mathcal{I}} F(X\Delta S_{\text{even}})\Delta F(S_{\text{even}})$ for $X \subseteq A_{\text{even}}$ we have

3. $G_{\text{even}}(X)\Delta F(X) = G \in \text{Exh}(\varphi) = \mathcal{I}$, namely
4. $G_{\text{even}} =^{\mathcal{I}} F$ on $\mathcal{P}(A_{\text{even}})$.

An analogous argument gives that a map $G_{\text{odd}}\colon \mathcal{P}(A_{\text{odd}}) \to \mathcal{P}(\mathbb{N})$, defined by $G_{\text{odd}}(X) = \Delta_{i \text{ odd}} H_i(X \cap [n_i, n_{i+1}))$, satisfies

5. $G_{\text{odd}} =^{\mathcal{I}} F$ on $\mathcal{P}(A_{\text{odd}})$.

This implies that for all $X \subseteq \mathbb{N}$ we have

$$\begin{aligned} \Psi_H(X) = G_{\text{even}}(X \cap A_{\text{even}})\Delta G_{\text{odd}}(X \cap A_{\text{odd}}) \\ =^{\mathcal{I}} F(X \cap A_{\text{even}})\Delta F(X \cap A_{\text{odd}}) =^{\mathcal{I}} F(X), \end{aligned}$$

(since $(X \cap A_{\text{even}}) \Delta (X \cap A_{\text{odd}}) = X$, and therefore Ψ_H is the required asymptotically additive lifting of Φ.

Proposition 4. *If, in addition to (b'), we assume that Φ is a semilattice homomorphism, then there is a disjoint partition $\mathbb{N} = A_0 \cup A_1$ such that Φ has an asymptotically additive lifting on both A_0 and A_1.*

Proof. Let $n_i, k_i, S_{\text{even}}, S_{\text{odd}}, A_{\text{even}}, A_{\text{odd}}$, be defined exactly like in the proof of Proposition 3, and let $H_i : \mathcal{P}([n_i, n_{i+1})) \to \mathcal{P}([k_{i-1}, k_{i+1}))$ be defined by

$$H_i(t) = \begin{cases} (F(t \cup S_{\text{even}}) \cap F(A_{\text{odd}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is odd,} \\ (F(t \cup S_{\text{odd}}) \cap F(A_{\text{even}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is even.} \end{cases}$$

For $X \subseteq A_{\text{even}}$ we have

1. $\varphi(((H_i(X \cap [n_i, n_{i+1}))\Delta F(X \cup A_{\text{even}})) \cap F(A_{\text{even}})) \cap [k_{i-1}, k_{i+1})) < 2^{-i}$,

and therefore a map $G_{\text{even}} : \mathcal{P}(A_{\text{even}}) \to \mathcal{P}(\mathbb{N})$ defined by

2. $G_{\text{even}}(X) = \Delta_{i \text{ even}} H_i(X \cap [n_i, n_{i+1})) = \bigcup_{i \text{ even}} H_i(X \cap [n_i, n_{i+1})$
3. $G_{\text{even}}(X)\Delta F(X) \subseteq^{\mathcal{I}} (G_{\text{even}}(X)\Delta F(X \cup S_{\text{odd}})) \cap F(A_{\text{even}}) \in \text{Exh}(\varphi) = \mathcal{I}$. (We are using $(X \cup S_{\text{odd}}) \cap A_{\text{even}} = X$ for $X \subseteq A_{\text{even}}$.)

Therefore G_{even} is an asymptotically additive lifting of Φ on $\mathcal{P}(A_{\text{even}})$, and we can find an asymptotically additive lifting G_{odd} on $\mathcal{P}(A_{\text{odd}})$ by an analogous argument. This completes the proof.

Corollary 3 and Proposition 4 together show that for every Baire-measurable lifting of an SL-homomorphism between quotients over analytic P-ideals there is a partition of $\mathbb{N}$ into four pieces such that Φ is trivial on each one. However, it is not difficult to check that these two constructions can be performed simultaneously, and therefore $\mathbb{N}$ can always be partitioned into two pieces such that Φ is trivial on each one. Note that by Example 1 this number can not be lowered to one (see also Question 3). The following slightly stronger version of Proposition 4 can be obtained by a more careful analysis. Since we do not have a good use of this strengthening (other than one mentioned at the beginning of §9), we shall omit the proof.

Proposition 5. *If, in addition to (b'), we assume that Φ is a semilattice homomorphism, then Φ has an asymptotically additive lifting.*

4 Ulam stability

We will now prove that a natural question about the structure of liftings is equivalent to the stability of solutions of some functional equations. The definitions

(1)–(3) below can be generalized to arbitrary algebras with metrics in an obvious way, and in [24] an interesting non-metric generalization was studied.

(1) If $\mathcal{A}, \mathcal{B}$ are two Boolean algebras, $\varepsilon > 0$, and d is a metric on $\mathcal{B}$, then a mapping $f : \mathcal{A} \to \mathcal{B}$ is an *ε-approximate* BA-*homomorphism* if

$$d(f(a \cup b), f(a) \cup f(b)) \leq \varepsilon \qquad \text{and} \qquad d(f(a^{\complement}), f(a)^{\complement}) \leq \varepsilon$$

for all $a, b \in \mathcal{A}$.

(2) If $\mathcal{A}, \mathcal{B}$ are two groups, $\varepsilon > 0$, and d is a metric on $\mathcal{B}$, then a mapping $f : \mathcal{A} \to \mathcal{B}$ is an *ε-approximate* G-*homomorphism* if

$$d(f(ab), f(a)f(b)) \leq \varepsilon \qquad \text{and} \qquad d(f(a^{-1}), f(a)^{-1}) \leq \varepsilon$$

for all $a, b \in \mathcal{A}$.

(3) If $\mathcal{A}, \mathcal{B}$ are two semilattices, $\varepsilon > 0$, and d is a metric on $\mathcal{B}$, then a mapping $f : \mathcal{A} \to \mathcal{B}$ is an *ε-approximate* SL-*homomorphism* if

$$d(f(a \wedge b), f(a) \wedge f(b)) \leq \varepsilon$$

for all $a, b \in \mathcal{A}$.

Assume that $\mathcal{A}, \mathcal{B}, f$, and d are as in 1, 2, or 3 above. If $\delta > 0$, then f is *δ-approximated* by a $g : \mathcal{A} \to \mathcal{B}$ if

$$d(f(a), g(a)) \leq \delta$$

for all $a \in \mathcal{A}$. If $\mathcal{D}$ is a class of metrics, we say that *approximate homomorphisms with respect to metrics in $\mathcal{D}$ are stable* if there is a universal constant K such that every ε-approximate homomorphism with respect to some metric in $\mathcal{D}$ can be $K\varepsilon$-approximated by a strict homomorphism. (This definition applies to BA-, G-, and SL-homomorphisms.) The term "stability" was used in this context by S. Ulam and a reader interested in the general setup may wish to consult [44, VI.1], [45, V.4] and the introduction to [10].

If φ is a submeasure on $[1, m]$ then we define a metric d_φ on $\mathcal{P}([1, m])$ by

$$d_\varphi(a, b) = \varphi(a \Delta b).$$

If f is an ε-approximate homomorphism with respect to some d_φ, then we say that f is an *ε-approximate homomorphism* with respect to φ. Recall that a submeasure φ is *nonpathological* if $\varphi = \sup_{\nu \leq \varphi} \nu$, where ν ranges over all measures that are pointwise dominated by φ. (This is not exactly the standard definition. What we call nonpathological is sometimes called *weakly nonpathological*, while a submeasure is said to be pathological if it does not dominate a non-zero measure at all. However, in the case when the support of a submeasure is finite—i.e., in our case of interest—only a trivial submeasure can be pathological. For more on pathological submeasures see [23], [13].) An ideal $\mathcal{I}$ is *nonpathological* if $\mathcal{I} = \mathrm{Exh}(\varphi)$ (see Theorem 1) for some lower semicontinuous nonpathological submeasure φ. Essentially all natural examples of analytic P-ideals occurring in the literature are nonpathological.

We say that a homomorphism is *Baire* if it has a Baire-measurable lifting. In [8] (see also [12]) the following was proved (using a slightly different terminology):

Proposition 6. *The following are equivalent*:

(1) *The approximate* BA*-homomorphisms with respect to* (*nonpathological*) *submeasures are stable,*
(2) *All* BA*-Baire homomorphisms between quotients over* (*nonpathological*) *analytic P-ideals are trivial.*

The (2) of the above proposition was conjectured by Todorcevic in [43]. The nonpathological case of (1) was proved in [8, Theorem 1.8.1], thus confirming Todorcevic's conjecture for a large class of quotients. However, it was also showed in [8, Theorem 1.8.2] that (1), and therefore Todorcevic's conjecture, fails in general. Our aim is to extend Proposition 6 to the categories G and SL, and we will do so in Proposition 7 and 8 below. It should be noted that this connection is not new. For example, N. Kalton related a three-body problem in local convexity of quasi-Banach spaces with the stability of approximately additive set-mappings (see [23, §5], [22]).

The proof of Proposition 7 given below is due to Vladimir Kanovei and it is included with his kind permission. Our original proof used some metamathematical methods.

Proposition 7. *The following are equivalent*

(1) *The approximate* G*-homomorphisms between finite powers of* $\mathbb{Z}_2$ *with respect to* (*nonpathological*) *submeasures are stable.*
(2) *Every Baire* G*-homomorphism between quotients of* $\mathbb{Z}_2^{\mathbb{N}}$ *over* (*nonpathological*) *analytic P-ideals is trivial.*

Proof. Assume (2). Then we can find sequences $1 = n_1 < n_2 < \dots$, $1 = k_1 < k_2 < \dots$, and $H_i : \mathbb{Z}_2^{[n_i, n_{i+1})} \to \mathbb{Z}_2^{[k_i, k_{i+1})}$ and a submeasure φ_i on $[k_i, k_{i+1})$ such that H_i is an 2^{-i}-approximate homomorphism with respect to d_{φ_i} which can not be 1-approximated by a strict homomorphism. (If the approximate homomorphisms with respect to nonpathological submeasures are not stable, then we can pick φ_i as above to be nonpathological.) Let

$$\varphi = \sup_i \varphi_i \text{ and } \mathcal{I} = \mathrm{Exh}(\varphi),$$

and let F be the asymptotically additive mapping Ψ_H given by H_i's.

Claim. If each φ_i is nonpathological, so is φ.

Proof. For $s \subseteq \mathbb{N}$ and $\varepsilon > 0$ find an i such that $\varphi(i) \leq \varphi_i(s \cap [n_i, n_{i+1})) + \varepsilon/2$. Since φ_i is nonpathological, we can find a measure ν pointwise dominated by φ_i (and therefore by φ) such that $\varphi_i(s) \leq \nu(s) + \varepsilon/2$. Therefore $\varphi(s) \leq \nu(s) + \varepsilon$, and since s and ε were arbitrary, φ is equal to the supremum of all measures pointwise dominated by φ, and it is nonpathological.

It is a routine to check that F is a continuous lifting of a homomorphism $\Phi\colon \mathbb{Z}_2^{\mathbb{N}}/\mathrm{Fin} \to \mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$ and that Φ has no simple lifting (see the proof of [8, Theorem 1.8.2]), and this completes the proof that (2) implies (1).

Let us now prove that (1) implies (2). Let $\mathcal{I}, \mathcal{J}$ be analytic P-ideals, and let $\Phi\colon \mathbb{Z}_2^{\mathbb{N}}/\mathcal{I} \to \mathbb{Z}_2^{\mathbb{N}}/\mathcal{J}$ be a homomorphism with a Baire measurable lifting. By Corollary 2, we can assume that it has a continuous lifting.

By Lemma 2, there is a sequence $1 = n_1 < k_1 < n_2 < k_2 < \dots$ of positive integers and $s_i \subseteq [n_i, n_{i+1})$ such that each s_i is an (n_i, n_{i+1})-stabilizer, with a witness k_i. Define maps $H_i\colon \mathcal{P}([n_i, n_{i+1})) \to \mathcal{P}([k_{i-1}, k_{i+1}))$ like in the proof of Proposition 3, by

$$H_i(t) = \begin{cases} (F(t \Delta S_{\mathrm{even}}) \Delta F(S_{\mathrm{even}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is odd,} \\ (F(t \Delta S_{\mathrm{odd}}) \Delta F(S_{\mathrm{odd}})) \cap [k_{i-1}, k_{i+1}), & \text{if } i \text{ is even.} \end{cases}$$

Let Ψ_H be an asymptotically additive mapping determined by the sequence $\{H_i\}$. In addition, for $i < j$ define a map $H_{i,j}\colon \mathcal{P}([n_i, n_j)) \to \mathcal{P}([k_{i-1}, k_j))$ by

$$H_{i,j}(t) = \Delta_{i \leq l < j} H_l(t \cap [n_l, n_{l+1})),$$

in particular, $H_{i,i+1} \equiv H_i$. Let φ be a lower semicontinuous submeasure such that $\mathcal{J} = \mathrm{Exh}(\varphi)$ (by Theorem 1).

Claim. For every $\varepsilon > 0$ there is an i_ε such that $H_{i,j}$ is an ε-approximate homomorphism (with respect to φ) whenever $i_\varepsilon \leq i < j$.

Proof. Assume this fails for some $\varepsilon > 0$. Then we can recursively find $i(1) < j(1) \leq i(2) < j(2) < \dots$ such that for every l the mapping

$$H'_l = H_{i(l), j(l)}$$

is not an ε-approximate homomorphism with respect to φ. Namely, there are $u_l, v_l \subseteq [n_{i(l)}, n_{j(l)})$ such that

$$\varphi(H_l(u_l \Delta v_l) \Delta H'_l(u_l) \Delta H'_l(v_l)) > \varepsilon$$

for all l. Let $U = \bigcup_l u_l$, $V = \bigcup_l v_l$ and $W = U \Delta V = \bigcup_l (u_l \Delta v_l)$. Note that $\Psi_H(U)$ ($\Psi_H(V)$, $\Psi_H(W)$) is the disjoint union of $H_l(u_l)$ ($H_l(v_l)$, $H_l(u_l \Delta v_l)$, respectively) Therefore

$$\liminf_m \varphi((\Psi_H(W) \Delta \Psi_H(U) \Delta \Psi_H(V)) \setminus [1, m]) \geq \varepsilon > 0,$$

and this set does not belong to $\mathcal{I}$, contrary to the fact that Ψ_H is a lifting of Φ. This contradiction concludes the proof of Claim.

By Claim, we can find a sequence $1 = i(1) < i(2) < i(3) < \dots$ such that for every $l \geq 1$ and $i(l) \leq i < j$ the map $H_{i,j}$ is a 2^{-l}-approximate homomorphism with respect to φ. In particular,

$$H'_l = H_{i(l),i(l+1)}$$

is a 2^{-l}-approximate homomorphism. By the assumption (1) there is a constant K such that for all l there is a strict homomorphism

$$G_l : \mathcal{P}([n_{i(l)}, n_{i(l+1)})) \to \mathcal{P}([k_{i(l)-1}, k_{i(l+1)}))$$

which is a $K\varepsilon$-approximation of H'_l with respect to φ. (If we are assuming that the approximate homomorphisms with respect to nonpathological submeasures are stable, note that if a submeasure is nonpathological then its restriction to every finite set is nonpathological. In particular, if we have a nonpathological ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$, then every H'_l is a 2^{-l}-approximate homomorphism with respect to a nonpathological submeasure $\varphi \restriction [k_{i(l-1)}, k_{i(l+1)})$, and therefore we can find G_l as above.) Let $F' : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be defined by

$$F'(X) = \mathit{\Delta}_l\, G_l(X \cap [n_{i(l)}, n_{i(l+1)}))$$

for $X \subseteq \mathbb{N}$. Then F' is a homomorphism, and we also have $F'(X) \mathit{\Delta} \Psi_H(X) \in \mathcal{J}$, since $\Psi_H(X) = \mathit{\Delta}_l\, H'_l(X \cap [n_{i(l)}, n_{i(l+1)}))$. Therefore F' is the required lifting of Φ, and this completes the proof.

Proposition 8. *The following are equivalent*

(1) *The approximate* SL-*homomorphisms between* $\mathcal{P}([1, m])$*'s with respect to* (*nonpathological*) *submeasures are stable.*
(2) *Every continuous* SL-*homomorphism between quotients of* $\mathbb{Z}_2^{\mathbb{N}}$ *over* (*nonpathological*) *analytic P-ideals is trivial.*

Proof. The proof is identical, modulo the necessary algebraic adjustments, to that of Proposition 7 (compare the proofs of Proposition 3 and Proposition 4), and we can therefore omit it.

Let us say that an SL-homomorphism Φ is *semi-trivial* if $\mathbb{N}$ can be partitioned into finitely many sets $A_1 \cup \dots A_k$ such that $\Phi \restriction \mathcal{P}(A_i)$ is trivial for all $i \leq k$.

Corollary 4. *The following are equivalent*

(1) *The approximate* SL-*homomorphisms between* $\mathcal{P}([1, m])$*'s with respect to* (*nonpathological*) *submeasures are stable.*
(2) *Every Baire* SL-*homomorphism between quotients of* $\mathbb{Z}_2^{\mathbb{N}}$ *over* (*nonpathological*) *analytic P-ideals is semi-trivial.*

Proof. This follows from Corollary 3, Proposition 4 and Proposition 8 (see also paragraph after the proof of Proposition 4).

5 From asymptotically additive to additive liftings

Propositions 6, 7 and 8 are giving us a curious connection between Set-theoretic phenomena (the triviality of homomorphisms between quotient structures) and that of finite combinatorics (the Ulam stability). In order to apply these results to analyze the structure of analytic quotients we need to have some Ulam-stability results. In [10] it was proved that the approximate BA-homomorphisms with respect to nonpathological submeasures are stable, and by Proposition 6 this implies the following (see [8, Theorem 1.8.1]).

Theorem 2. *All Baire* BA*-homomorphisms between quotients over nonpathological analytic P-ideals are trivial.*

In [11, Theorem 2.1] the following was proved (note that approximate homomorphisms in the sense of §4 are the approximate homomorphisms of type II in the terminology of [11]).

Theorem 3. *The approximate* G*-homomorphisms between finite powers of* $\mathbb{Z}_2$ *with respect to nonpathological submeasures are stable.*

Corollary 5. *Every* G*-Baire-homomorphism between quotients of* $\mathbb{Z}_2^{\mathbb{N}}$ *over nonpathological analytic P-ideals is trivial.*

Proof. This follows immediately from Proposition 7 and Theorem 3.

This result is, in a sense, stronger than Theorem 2, since the group $\mathbb{Z}_2^{\mathbb{N}}$ and its quotients has less structure than the Boolean algebra $\mathcal{P}(\mathbb{N})$ (and its quotients). (Also, Theorem 3 implies the analogous result for Boolean algebras and even gives a better upper bound for the numerical value of the "stability constant"; see [11, §4].) However, Corollary 5 is less natural than Theorem 2, since $\mathbb{Z}_2^{\mathbb{N}}$ and its quotients form a less natural class of groups than $\mathcal{P}(\mathbb{N})$ and its quotients form a class of Boolean algebras. It would be interesting to find more general results about homomorphisms between quotients of Polish groups, or at least compact metric groups, over their "simple" subgroups. It was A. Kechris who has suggested to us the investigation of this problem shortly after he was informed about the results from [8].

Consider the class of quotients over Polish groups with the property that all Baire homomorphisms between them are trivial. It wold be of interest to determine which quotients belong to this class. By Corollary 5, all $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$, for $\mathcal{I}$ a nonpathological ideal, do have this desirable property. By Theorem 4 below this class can not even be extended to include $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$ for all analytic P-ideals $\mathcal{I}$.

Theorem 4. *There is a nontrivial Baire group homomorphism between quotients of* $\mathbb{Z}_2^{\mathbb{N}}$ *over analytic P-ideals.*

Proof. In [11, Theorem 3.1] it was proved that approximate group homomorphisms between finite powers of $\mathbb{Z}_2$ are not stable in general, and therefore an application of Proposition 7 gives the (un)desired conclusion.

Another, very simple, example of a nontrivial Baire homomorphism was given by V. Kanovei ([24]). In the positive direction, we note that in [10] the following stronger form of Theorem 3 was proved.

Theorem 5. *Approximate* G*-homomorphisms between finite products of finite groups with respect to nonpathological submeasures are stable.*

Consider groups of the form $G = \prod_{i=1}^{\infty} G_i$, where all G_i are finite, with the product topology (every G_i is considered with its discrete topology). These groups are compact metric, and a zero-dimensional metric group is compact if and only if it is a closed subgroup of a group of this form (see [1]). Now if K_i is a fixed normal subgroup of G_i and $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$, then

$$K_{\mathcal{I}} = \{X \in G : \{n : X(n) \notin K_i\} \in \mathcal{I}\}$$

is a normal analytic subgroup of G.

Corollary 6. *Every* G*-Baire-homomorphism between quotients* $\prod_i G_i/K_{\mathcal{I}}$*, for nonpathological P-ideals* $\mathcal{I}$*, is trivial.*

Proof. It is straightforward to check that the reductions of Corollary 2 and Proposition 7 can be performed when $\mathbb{Z}_2^{\mathbb{N}}$ is replaced with a countable product of finite groups. Therefore the conclusion follows by Theorem 5.

Of course, the subgroups of $\mathbb{Z}_2^{\mathbb{N}}$ of the form $K_{\mathcal{I}}$ are rather special, even if we consider only the subgroups which are Polishable (a group is G *Polishable* if there is a complete metric on G which makes it into a separable topological group). The class of Polishable subgroups of $\mathcal{P}(\mathbb{N})$ is quite rich. For example, by a result of Hjorth, there are Polishable subgroups of $\mathcal{P}(\mathbb{N})$ of arbitrarily high Borel complexity (see [16, pages 88–91]). (Note that, by [34], the Polishable analytic ideals on $\mathbb{N}$ are analytic P-ideals, which are $F_{\sigma\delta}$ subsets of $\mathcal{P}(\mathbb{N})$. Therefore it is even more curious to note that Hjorth's subgroups are closely associated with Borel ideals on $\mathbb{N}$; namely they are of the form $\mathcal{I} \cup \mathcal{I}^*$ ($\mathcal{I}^*$ stands for the dual filter of $\mathcal{I}$) for some Borel ideals $\mathcal{I}$.)

The following related result was recently proved by V. Kanovei ([24]).[1]

Theorem 6. *If* G_0, G_1 *are subgroups of* $\langle \mathbb{R}, + \rangle$ *and* G_1 *is countable, then all Baire homomorphisms of* $\mathbb{R}/G_0$ *into* $\mathbb{R}/G_1$ *are trivial.*

6 The structure of additive liftings

In order to make the lifting results of previous sections useful, we need to understand how do the Baire-measurable homomorphisms F of $\mathcal{P}(\mathbb{N})$ into itself look

[1] **Added in proof.** Very recently, V. Kanovei and M. Reeken have proved that for essentially all natural Borel ideals $\mathcal{I}$ (which are not necessarily P-ideals) all Baire G-homomorphisms of $\mathsf{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are trivial.

like. Note first that if $F: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a BA- (G, SL, respectively) homomorphism, then for every $n \in \mathbb{N}$ the set

$$\mathcal{X}_n = \{X : n \notin F(X)\}$$

is a maximal ideal (or subgroup of index two, or complement of a filter, respectively). Note that the following (well-known) lemma does not say the same as Corollary 1 and Corollary 2.

Lemma 3. *Every Baire-measurable* BA- *or* G-*homomorphism* $F: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *is automatically continuous.*

Proof. Let us prove the BA-case first. Note that $\mathcal{I}_n = \{X : F(X)(n) = 0\}$ is a maximal ideal on $\mathbb{N}$, and it has the property of Baire. Therefore every $\mathcal{I}_n$ has to be a principal ideal (this is an old result of Sierpiński, but it also easily follows from a result of Jalali-Naini and Talagrand ([18], [37]) used in the proof of Theorem 7), and this easily implies that F is continuous.

Similarly, in the G-case, $\mathcal{I}_n$ as defined above has to be a subgroup of $\mathbb{Z}_2^{\mathbb{N}}$ of index two with the Property of Baire. Such subgroups have to be clopen (see [26]), therefore F is continuous.

The following two facts are simple and well-known, but we include them for the sake of completeness.

Corollary 7. *Every Baire-measurable* BA-*homomorphism* $F: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *is of the form*

$$X \mapsto h^{-1}(X)$$

for some $h: \mathbb{N} \to \mathbb{N}$.

Corollary 8. *Every Baire-measurable* G-*homomorphism* $F: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *is of the form*

$$F(X) = \{n : |X \cap s_n| \text{ is odd}\}$$

for some sequence $\{s_n\}$ *of finite subsets of* $\mathbb{N}$.

Proof. This follows from Lemma 3 and [11, Proposition 4.4].

Note that maps from Corollary 7 are special cases of maps from Corollary 8: If $h: \mathbb{N} \to \mathbb{N}$, then let $s_n = \{h(n)\}$. Since there are nonprincipal filters with the property of Baire, the conclusion of Lemma 3 fails for SL-homomorphisms (see Example 1).

Definition 4. (1) *An* SL-*homomorphism is* strongly trivial *if it has a lifting which is a continuous* SL-*homomorphism of* $\mathcal{P}(\mathbb{N})$ *into itself.*

(2) *An* SL-*homomorphism* Φ *is* semi-strongly trivial *if* $\mathbb{N}$ *can be partitioned into finitely many sets* $A_1 \cup \cdots \cup A_k$ *such that* $\Phi \restriction \mathcal{P}(A_i)$ *is strongly trivial for all* $i \leq k$.

A topological application of the following Corollary is given in §8 below.

Corollary 9. *Every Baire* SL*-homomorphism* Φ *of* $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ *into itself is semi-strongly trivial.*

Proof. By Proposition 4 (from where we adopt the notation) we can assume that Φ has an asymptotically additive lifting. Let us fix a sequence $\{H_i\}$ which determines this lifting.

Claim. All but finitely many of the H_i are SL-homomorphisms from $\mathcal{P}([n_i, n_{i+1}))$ into $\mathcal{P}(v_i)$.

Proof. Assume the contrary, that for infinitely many i there are s_i, t_i such that

$$H_i(s_i \cap t_i) \neq H_i(s_i) \cap H_i(t_i).$$

Since $\{v_i\}$ is a point-finite sequence, by going to a subsequence we can assume that the sets $H_i(s_i \cup t_i) \cup H_i(s_i) \cup H_i(t_i) \subseteq v_i$ are pairwise disjoint. This is done to avoid interference between different i's. Then if $X = \bigcup_i s_i$ and $Y = \bigcup_i t_i$, we have $\Psi_H(X) \cap \Psi_H(Y) \neq^* \Psi_H(X \cap Y)$, contradicting the fact that Ψ_H is a lifting of Ψ on A.

Therefore we can assume that every H_i is an SL-homomorphism.

Claim. If u, v are finite sets and $H\colon \mathcal{P}(u) \to \mathcal{P}(v)$ is a semilattice homomorphism, then there are $s_j \subseteq u$ for each $j \in v$ such that $H(t) = \{j : t \supseteq s_j\}$ for all $t \subseteq u$.

Proof. Note that $\mathcal{F}_j = \{t \subseteq u : H(t) \ni j\}$ is a filter, and let $s_j = \bigcap \mathcal{F}_j$. Then $j \in H(t)$ if and only if $t \in \mathcal{F}_j$ if and only if $s_j \subseteq t$.

Applying the above claim to each H_i, we complete the proof of Corollary 9.

7 Arbitrary homomorphisms

In [33] (see also [48]) it was proved that all automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ can be trivial. In [8] this was extended to all quotient algebras over nonpathological analytic P-ideals. This result is quite useful, since it reduces the logical complexity of a question whether two quotients are isomorphic from Σ^2_1 ("There exists $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which is a lifting … ") to much lower Σ^1_2 ("There exists $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \mapsto h^{-1}(A)$ is a lifting … "). (Some applications of this phenomenon are given in [8] and [12].) It is therefore natural to ask whether analogous results can be proved in the categories of groups, or semilattices.

In the case of groups, the situation is different from that in the case of Boolean algebras, since Hausdorff gaps do not stand on our way and nothing prevents the straightforward recursive construction of nontrivial automorphisms. We can say even more.

Proposition 9. *If $\mathcal{I}, \mathcal{J}$ are analytic ideals on $\mathbb{N}$, then the groups $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$ and $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{J}$ are isomorphic, and each one of them has $2^{\mathfrak{c}}$ many automorphisms.*

Proof. These groups can be considered as vector spaces over $\mathbb{Z}_2$, and let us first compute the dimension of $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$. By [29], we can find disjoint finite sets u_i such that any infinite union of u_i's is $\mathcal{I}$-positive. Then if A_ξ ($\xi < \mathfrak{c}$) is an family of subsets of $\mathbb{N}$ whose pairwise intersections are finite, then $B_\xi = \bigcup_{i \in A_\xi} u_i$ form a linearly independent family of vectors in vector space $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$. Therefore the dimension is at least continuum. Since the size of $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$ is equal to the continuum, the dimension can not be bigger, and therefore it is exactly continuum.

Since two vector spaces over the same field and of the same dimension are isomorphic, the first statement follows. Since every permutation of the basis corresponds to an automorphism of the space and $2^{\mathfrak{c}}$ is the maximal number, the second statement follows.

In the case of semilattices, the situation is quite the opposite. Namely, if $\mathcal{B}$ is a Boolean algebra, and f is an automorphism of $\mathcal{B}$ considered as a semilattice, then f is automatically a Boolean algebra automorphism. Therefore if all Boolean algebra automorphisms of $\mathcal{B}$ are trivial, so are its semilattice automorphisms. By results of [8] (see also [12]), we have the following (MA stands for the Martin's Axiom, and for the definition of OCA see §8).

Proposition 10. *Assume OCA and MA, and let $\mathcal{I}, \mathcal{J}$ be nonpathological analytic P-ideals. Then every SL-isomorphism $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is trivial.*

We should note that W. Just has found a forcing extension of the universe in which every SL-homomorphism between any two analytic quotients is somewhere trivial (see [20], also [21, Theorem 11]). In the following section we will give a topological application of this result and Corollary 9.

8 An application: Exp($\mathbb{N}^*$) is not a remainder of $\mathbb{N}$

In this section we give a topological application of our analysis of semilattice homomorphisms, in particular of Corollary 9. Recall that $\mathbb{N}^*$ is the Čech–Stone remainder of the integers, and that $\mathrm{Exp}(\mathbb{N}^*)$ is its exponential space, namely the space of all closed subsets of $\mathbb{N}^*$ equipped with the Vietoris topology (see [7]). By a result of van Douwen ([46]), $\mathbb{N}^*$ is not a continuous image of $\mathrm{Exp}(\mathbb{N}^*)$ (since every infinite continuous image of some exponential space has a nontrivial converging sequence). On the other hand, the Continuum Hypothesis implies that $\mathbb{N}^*$ maps onto $\mathrm{Exp}(\mathbb{N}^*)$ (by [30]). We shall prove that the Continuum Hypothesis can not be removed from the assumption of this result.

Let us recall the statement of Open Coloring Axiom, OCA, introduced by Todorcevic in [40] (although we shall need only its consequence, Theorem 8 below). Let X be a separable metric space, and by $[X]^2$ denote the family of all unordered pairs of its elements, $\{\{x, y\} | x \neq y \text{ and } x, y \in X\}$. Subsets of $[X]^2$ can be naturally identified with the symmetric subsets of $X \times X$ minus the diagonal. A partition $[X]^2 = K_0 \cup K_1$ is *open* if K_0, when identified with a symmetric subset of X^2, is open in the product topology. We say that a subset Y of X is K_i-*homogeneous* if $[Y]^2$ is included in K_i $(i = 0, 1)$.

(OCA) If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is an open partition, then X either has an uncountable K_0-homogeneous subset or it can be covered by a countable family of K_1-homogeneous sets.

The idea on which the following result is based is essentially due to M. Bell ([2]). (See also [3, §6], where some similar ideas are exposed.)

Theorem 7. *Assume OCA. Then* $\mathbb{N}^*$ *does not map onto its exponential space,* $\mathrm{Exp}(\mathbb{N}^*)$.

Proof. By $\mathcal{F}$ we shall always denote a filter on $\mathbb{N}$ which includes the filter of all cofinite sets. For such an $\mathcal{F}$, $\mathcal{F}^+$ denotes the coideal of all $\mathcal{F}$-positive sets, i.e., all sets A such that $\mathbb{N} \setminus A$ is not in $\mathcal{F}$. Let $\mathcal{B}$ the Boolean subalgebra of $\mathcal{P}(\{\mathcal{F} : \mathcal{F}$ filter on $\mathbb{N}\})$ generated by sets of the form (for A and $B_1, \dots, B_k$ infinite subsets of $\mathbb{N}$):

$$\langle A; B_1, \dots, B_k \rangle = \{\mathcal{F} : A \in \mathcal{F} \qquad \text{and} \qquad B_i \in \mathcal{F}^+ \text{ for } i = 1, \dots, k\}.$$

Claim. $\mathrm{Exp}(\mathbb{N}^*)$ is homeomorphic to the Stone space of $\mathcal{B}$.

Proof. If we identify filters on $\mathbb{N}$ with closed subsets of $\mathbb{N}^*$ in a natural way, then every set of the form $\langle A; \boldsymbol{B} \rangle$ is clopen in the Vietoris topology. Moreover, for every $K \in \mathrm{Exp}(\mathbb{N}^*)$ and its Vietoris open neighborhood U there is a set of the form $\langle A; \boldsymbol{B} \rangle$ which has K as its element and is included in U. Therefore the Stone spaces of $\mathcal{B}$ and clopen algebra of $\mathrm{Exp}(\mathbb{N}^*)$ coincide, and this concludes the proof.

Assume that $\Phi \colon \mathcal{B} \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is an embedding of Boolean algebras; this is exactly the negation of the assertion of Theorem 7. For an infinite $A \subseteq \mathbb{N}$ let

$$\hat{A} = \langle A; \langle\rangle \rangle = \{\mathcal{F} : A \in \mathcal{F}\}$$

(here $\langle\rangle$ denotes the empty sequence of B_i's), and let $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a mapping such that

1. $F(A) = B$ if and only if $\Phi(\hat{A}) = [B]_{\mathrm{Fin}}$.

Note that

2. $\hat{A} \cap \hat{B} = \widehat{A \cap B}$, but
3. $\hat{A} \cup \hat{B} = \widehat{A \cup B}$ if and only if $\hat{A} \subseteq \hat{B}$ or $\hat{B} \subseteq \hat{A}$ if and only if $A \subseteq^* B$ or $B \subseteq^* A$.

Therefore, since the kernel of Φ is trivial, F satisfies the following:

4. $F(A) \cap F(B) =^* F(A \cap B)$,
5. F is Fin-*invariant*, namely $A =^* B$ implies that $F(A) =^* F(B)$, and
6. $F(A \cup B) \setminus (F(A) \cup F(B))$ is infinite whenever both A and B are infinite and $A \cap B = \emptyset$.

It will suffice to prove that under OCA there is no F satisfying (4)–(6). We shall prove a slightly more precise statement, after a definition. Note that (4) and (5) say that $\Psi \colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ defined by

7. $\Psi([A]_{\mathrm{Fin}}) = [F(A)]_{\mathrm{Fin}}$

is a semilattice homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself.

Theorem 8 (Just). *Assume OCA. Then every semilattice homomorphism Ψ of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself has a Borel-measurable lifting on some infinite set.*

Proof. This was proved in [21, Theorem 11], but we do need a word of explanation of this claim to the reader, who is assumed to have a copy of [21] handy. First plug in $\mathcal{I} = \mathrm{Fin}$, $\mathcal{M} = \emptyset$ (this is a "closed approximation" to Fin, in the terminology of [21]), take $\mathcal{A}$ to be any uncountable family of pairwise almost disjoint sets, and let F be an arbitrary lifting of Ψ. Now note that $\emptyset \oplus \emptyset = \emptyset$ and that "$F \restriction \mathcal{P}(a)$ is semi-$\emptyset$ precise" means that we can split a into finitely many sets such that on each one of them Ψ has a Borel-measurable lifting.

By Theorem 8 and Corollary 9 we conclude that Φ is strongly trivial on some infinite set. To complete the proof of Theorem 7 we shall use the proof of Corollary 9. (Of course, it is possible to use Corollary 9 directly, but this will save us some space.) Let Ψ be a semilattice homomorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself; as noted before, it will suffice to prove that (6) fails for a lifting F of Ψ. Let s_i be a sequence as obtained by applying Corollary 9. Note that its proof also gives $k'_j = k_{2j+1}$ such that (let $I_j = [k'_j, k'_{j+1})$)

8. if $i \in I_j$ then $s_i \subseteq I_j$.

Now for each j pick an arbitrary $m(j) \in I_j$, and let

$$t_j = \{i \in I_j : s_i \subseteq s_{m(j)}\}.$$

Then (9) implies that $t_1 < t_2 < \dots$ and that $s_i \subseteq \bigcup_{j=1}^{\infty} s_{m(j)}$ if and only if $i \in \bigcup_{j=1}^{\infty} t_j$. In particular, the sets

$$B = \bigcup_{j=1}^{\infty} s_{m(2j)} \qquad \text{and} \qquad C = \bigcup_{j=1}^{\infty} s_{m(2j+1)}$$

satisfy $F_2(B \cup C) = \bigcup_{j=1}^{\infty} t_j = F_2(B) \cup F_2(C)$ although B and C are infinite and disjoint. This contradicts (6) and completes the proof of Theorem 7.

We should remark that Just used an analysis of semilattice embeddings of Theorem 8 to prove that $\mathbb{N}^*$ does not map onto $(\mathbb{N}^*)^2$ ([19]). In [8, Chapter 3] (see also [12]) we have shown that this—and more—can be proved already by considering Boolean algebra homomorphisms.

9 Concluding remarks

In this note we have concentrated on quotients over analytic P-ideals. Lemma 1 indeed says that this restriction is necessary if we are looking for asymptotically additive liftings. However, in [8, §4] it was proved that for every countably generated ideal $\mathcal{I}$ on $\mathbb{N}$ every Baire BA-homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is trivial. (In the terminology of [8], every countably generated ideal has the *Radon–Nikodym property*.) A similar proof gives the following.

Theorem 9. *Let $\mathcal{I}$ be a countably generated ideal on $\mathbb{N}$. Then*

(1) *Every Baire G-homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is trivial.*
(2) *Every Baire SL-homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is semi-strongly-trivial.*

Proof. (1) The proof follows the proofs of [8, §4] very closely.

(2) First use Corollary 3, and then mimick the proof of [8, Theorem 4.2] while using Proposition 5.

Question 1. Are the Radon–Nikodym property, (1), and (2) of Theorem 9 equivalent for every analytic ideal $\mathcal{I}$?

Let us say a word about the two remaining categories mentioned in the introduction: P, analytic quotients as partially ordered sets and E, analytic quotients as sets without any structure. The latter was extensively studied in the context of Borel equivalence relations and Borel (or Baire) measurable reductions between them (see [15]). (In our terminology, a *Borel (Baire) reduction* between equivalence relations E and F on $\mathcal{P}(\mathbb{N})$ is an injection $\Phi\colon \mathcal{P}(\mathbb{N})/E \to \mathcal{P}(\mathbb{N})/F$ with a Borel- (Baire-, respectively) measurable lifting.) In this context there are many results and interesting ways of reducing liftings (a nice example is given in [28]), but we are still far from having a reasonably general lifting theory. Baire-homomorphisms in the category P of quotients as partially ordered sets did not seem to attract very much attention. On the other hand, the question of the existence of embeddings (with arbitrary liftings) between analytic quotients, considered as partially ordered sets, and in particular of embeddings of partially ordered sets into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, is very relevant to apparently distant fields (see e.g., [6]). Some positive results were obtained in [32] and in [9, §3] forcing was used to give a situation in which one analytic quotient does not embed into another.

To understand lattice embeddings between analytic quotients, by Proposition 8 we need an answer to the following question.

Question 2. Are the approximate SL-homomorphisms between $\mathcal{P}([1,m])$'s with respect to (nonpathological) submeasures stable?

Another question about the lattice homomorphisms is a ispired by comparing Example 1 with the known positive results in the BA- and G-context. Recall that a set of reals is *universally Baire* if for every continuous $f : \mathbb{R} \to \mathbb{R}$ its f-preimage has the Property of Baire. A function is *universally Baire-measurable* if all preimages of open sets are universally Baire.

Question 3. Is every SL-homomorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself which has a universally Baire-measurable lifting trivial?

Note that Example 1 does not give a negative answer to this question, since a homomorphism defined there is trivial. If $\mathcal{F}_n$ $(n \in \mathbb{N})$ are Borel (or universally measurable) filters, then $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by

$$F(X) = \{n : X \in \mathcal{F}_n\}$$

is a trivial universally Baire-measurable SL-homomorphism of $\mathcal{P}(\mathbb{N})$ into itself. Question 3 asks whether this is the most complicated form of a universally Baire-measurable lifting (compare with [8, Example 3.2.2, Question 3.13.1]).

In [8] we have proved that every BA-Baire isomorphism between quotients over nonpathological analytic P-ideals has a lifting of the form $A \mapsto h''A$ for some bijection $h : \mathbb{N} \to \mathbb{N}$. This in particular implies that two quotients over such ideals are Baire-isomorphic if and only if the ideals are *isomorphic*, i.e., if there is a permutation of the integers which sends sets in one ideal into sets in another ideal. This is an easy consequence of our analysis of liftings, Corollary 7, and a well-known lemma about mappings. The criterion given by Corollary 8 is much less convenient, and we do not know whether an analogous statement remains true if we consider quotients as groups.

Problem 1. Give a simple criterion on analytic P-ideals $\mathcal{I}, \mathcal{J}$ for deciding when the groups $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{I}$ and $\mathbb{Z}_2^{\mathbb{N}}/\mathcal{J}$ are Baire-isomorphic.

Note that Corollary 2 implies that if $\mathcal{I}$ and $\mathcal{J}$ are of different descriptive complexity, then their quotients are not isomorphic.

Let us now turn to a question which seems to require an extension of our methods. Recall that *Čech function* is a mapping $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that (i) $F(A) \supseteq A$ for all A, (ii) $F(A \cup B) = F(A) \cup F(B)$, for all A, B, (iii) F is onto, and (iv) F is not equal to the identity. E. Čech ([4]) asked whether there are Čech functions, and they are known to exist under Continuuum Hypothesis, Martin's Axiom, and essentially in every known model of Set-theory. It is also known that a Čech function has to be Fin-invariant (for more information and proofs see [31], [14]). If F is a Čech function, then the dual function $\hat{F}(X) = \mathbb{N} \setminus F(\mathbb{N} \setminus X)$ is a lifting of a semilattice homomorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into itself.

Question 4. Can there be a universally Baire Čech function?

A positive answer to this question may give an absolute definition of Čech function and therefore a positive answer to Čech's question. On the other hand, a negative answer would imply that in Solovay's model ([36]) there are no Čech functions, and therefore that the use of Axiom of Choice is necessary in constructing them.

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