Automorphisms of the Calkin algebra

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My tutorial concentrated on presenting the proof of my result that TA implies all automorphisms of the Calkin algebra are inner. The structure of the proof (and even some of the terminology) is different from the one given in the paper. I hope that having a look at the proof from a different side will be beneficial for the readers.

$$
\begin{aligned}
H\colon &\quad \text{a complex, infinite-dimensional Hilbert space} \\
(e_n)\colon &\quad \text{an orthonormal basis of } H \\
(\xi|\eta)\colon &\quad \text{the inner product on } H \\
\|\xi\| = &\quad \sqrt{(\xi|\xi)} \\
a\colon H \to H\colon &\quad \text{a linear operator} \\
\|a\| = &\quad \sup\{\|a\xi\| \mid \xi \in H, \|\xi\| = 1\} \\
a \text{ is } \textit{bounded} &\quad \text{if } \|a\| < \infty. \\
(\mathcal{B}(H), +, \cdot, {}^*, \|\cdot\|)\colon &\quad \text{the algebra of all bounded operators on } H.
\end{aligned}
$$

The *adjoint*, a^*, is defined implicitly by

$$(a^*\xi|\eta) = (\xi|a\eta)$$

for all ξ, η in H.

Lemma

For all a, b we have

1. $(a^*)^* = a$,
2. $\|a\| = \|a^*\|$,
3. $\|ab\| \leq \|a\| \cdot \|b\|$,
4. $\|aa^*\| = \|a\|^2$.

Hence $\mathcal{B}(H)$ is a Banach algebra with involution. (4) is the "C equality."*

Example

If $H = L^2(X, \mu)$ *and* $f : X \to \mathbb{C}$ *is bounded and measurable, then*

$$H \ni g \mapsto m_f(g) = fg \in H$$

is a bounded linear operator. We have $\|m_f\| = \|f\|_\infty$ *and*

$$m_f^* = m_{\bar{f}}.$$

Hence $m_f^* m_f = m_f m_f^* = m_{|f|^2}$.

An operator a is *normal* if $aa^* = a^*a$.
If $\Phi\colon H_1 \to H_2$ is an isomorphism between Hilbert spaces, then

$$a \mapsto \operatorname{Ad}\Phi(a) = \Phi a \Phi^{-1}$$

is an isomorphism between $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$.

Theorem (Spectral Theorem)

If a is a normal operator then there is a finite measure space (X, μ), a measurable function f on X, and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\operatorname{Ad}\Phi(m_f) = a$.

An $a \in \mathcal{B}(H)$ is *self-adjoint* if $a = a^*$.
A $p \in \mathcal{B}(H)$ is a *projection* if $p^2 = p^* = p$.

Lemma

p is a projection iff it is an orthogonal projection to a closed subspace of H.

Pf. We have $p = m_f$ and $f = f^2 = \bar{f}$. Hence $f(x) \in \{0, 1\}$ for almost all x, and $m_f = \mathrm{proj}_{\{g \mid \overline{\mathrm{supp}}(g) \subseteq Y\}}$ with $Y = f^{-1}(\{1\})$. □

I is the identity operator on H.
An operator u is *unitary* if $uu^* = u^*u = I$.
An operator v is a *partial isometry* if

$$p = vv^* \text{ and } q = v^*v$$

are both projections.

Example

A partial isometry that is not a normal operator. Let (e_n) be the orthonormal basis of H. The unilateral shift S is defined by

$$S(e_n) = e_{n+1} \text{ for all } n.$$

Then $S^*(e_{n+1}) = e_n$ and $S^*(e_0) = 0$.

$$S^*S = I \neq \mathrm{proj}_{\overline{\mathrm{Span}}\{e_n | n \geq 1\}} = SS^*.$$

Concrete and abstract C* algebras

Definition (Concrete C* algebras)

If $X \subseteq \mathcal{B}(H)$ *let* $A = C^*(X)$ *be the smallest norm-closed subalgebra of* $(\mathcal{B}(H), +, \cdot, *)$.

Definition

A is an abstract C algebra if it is a Banach algebra with involution such that* $\|aa^*\| = \|a\|^2$ *for all* a.

Example

X: a locally compact Hausdorff space.

$$C_0(X) = \{f : X \to \mathbb{C} : f \text{ is continuous and vanishes at infinity}\}.$$

GNS

Theorem (Gelfand–Naimark)

Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

Theorem (Gelfand–Naimark–Segal)

Every C-algebra A is isomorphic to a concrete C*-algebra.*

Definition (Spectrum of an operator)

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda I \text{ is not invertible}\}.$$

Lemma

1. *$\sigma(a)$ is always a compact subset of $\mathbb{C}$.*
2. *$\sigma(a^*) = \{\overline{\lambda} \mid \lambda \in \sigma(a)\}$.*
3. *If q is normal then a is self-adjoint iff $\sigma(a) \subseteq \mathbb{R}$.*

Continuous function calculus

For $X \subseteq \mathcal{B}(H)$ let

$$C^*(X)$$

denote the C*-subalgebra of $\mathcal{B}(H)$ generated by X.

Lemma

If $a \in \mathcal{B}(H)$ is normal then

$$C^*(a, I) \cong C(\sigma(a)).$$

For every $f : \sigma(a) \to \mathbb{C}$ we can define $f(a) \in C^*(a, I)$. □

Here a corresponds to $\mathrm{id}_{\sigma(a)}$.
We can define $|a|$, $\sqrt{a}$ (if $\sigma(a) \subseteq [0, \infty)$), and what not.

Continuous functional calculus: an application

Lemma

If a is normal and $\|a^2 - a\| < \varepsilon < 1/4$ then there is a projection $p \in C^*(a)$ such that $\|p - a\| < \sqrt{\varepsilon}$.

Proof.

By $\|a^2 - a\| < \varepsilon$ we have $|x(1-x)| < \varepsilon$ for all $x \in \sigma(a)$. Thus either $|x| < \sqrt{\varepsilon}$ or $|1-x| < \sqrt{\varepsilon}$.

Let f be the characteristic function of the $\sqrt{\varepsilon}$-ball around 1. Then $p = f(a)$ is in $C^*(a)$.

Finally $\|f(a) - a\| = \sup_{x \in \sigma(a)} |f(x) - a|$ and

$$|f(x) - x| = \min\{|x|, |1-x|\} < \sqrt{\varepsilon}.$$

□

The algebra of compact operators

$$\begin{aligned}\mathcal{K}(H) =& C^*(\{a \in \mathcal{B}(H) \mid a[H] \text{ is finite-dimensional}\}).\\ =&\{a \in \mathcal{B}(H) \mid a\overline{[\text{unit ball}]} \text{ is compact}\}\\ =&\{a \in \mathcal{B}(H) \mid a[\text{unit ball}] \text{ is compact}\}\end{aligned}$$

Fact

If $r_n = \mathrm{proj}_{\overline{\mathrm{Span}}\{e_j \mid j \leq n\}}$ *TFAE*

1. $a \in \mathcal{K}(H)$,
2. $\lim_n \|a(I - r_n)\| = 0$,
3. $\lim_n \|(I - r_n)a\| = 0$.

□

Note: if a is self-adjoint then

$$\|a(I - r_n)\| = \|(a(I - r_n))^*\| = \|(I - r_n)a\|.$$

The Calkin algebra

$\mathcal{K}(H)$ is a (self-adjoint, norm closed, two-sided) ideal of $\mathcal{B}(H)$.
The *Calkin algebra*:

$$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$$

$$\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$$

is the quotient map.
We sometimes write $\dot{a} = \pi(a)$.

Lifting elements in the Calkin algebra, I

a is *self-adjoint* is $a = a^*$.

Lemma

If $\mathbf{a}$ *is self-adjoint in* $\mathcal{C}(H)$*, then* $\mathbf{a} = \pi(a)$ *for a self-adjoint* a *in* $\mathcal{B}(H)$*.*

Pf. Fix any a_0 such that $\pi(a_0) = \mathbf{a}$. Let $a = (a_0 + a_0^*)/2$. □

Lifting elements in the Calkin algebra, II

p is a projection $p^2 = p^* = p$.

Lemma

If $\mathbf{p}$ is a projection in $\mathcal{C}(H)$, then $\mathbf{p} = \pi(p)$ for a projection p in $\mathcal{B}(H)$.

Pf. Fix a self-adjoint a such that $p = \pi(a)$. There are (X, μ) and $f \in L^\infty(X, \mu)$ and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\Phi(m_f) = a$. If

$$h(x) = \begin{cases} 1, & f(x) \geq 1/2 \\ 0, & f(x) < 1/2. \end{cases}$$

then m_h is a *projection* and $\pi(m_h) = \pi(m_f)$. □

Lifting elements in the Calkin algebra, III

a is *normal* if $aa^* = a^*a$.

Lemma

There is a normal (even a unitary) operator in $\mathcal{C}(H)$ that is distinct from $\pi(v)$ for any normal v in $\mathcal{B}(H)$.

Pf. The image $\dot{S}$ of the unilateral shift is a unitary in $\mathcal{C}(H)$, since $\dot{S}^*\dot{S} = I = \dot{S}\dot{S}^*$.
However, $S^*S = I \neq SS^*$.
Moreover, if $v - S$ is compact then v is not normal. (Fredholm index!) □

Embedding $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathcal{B}(H))$

$$\mathcal{P}(\mathcal{B}(H)) = \{p \in \mathcal{B}(H) : p \text{ is a projection}\}$$

Define a partial ordering on $\mathcal{P}(\mathcal{B}(H))$ by

$$p \leq q \text{ iff } pq = p.$$

Fix a basis $(e_i)_{i \in \mathbb{N}}$ of H.
Notation: For $X \in \mathcal{P}(\mathbb{N})$ let

$$P_X = \text{proj}_{\overline{\text{Span}}\{e_n | n \in X\}}$$

$$\mathcal{P}(\mathbb{N}) \ni X \mapsto P_X \in \mathcal{P}(\mathcal{B}(H)).$$

Hence $\mathcal{P}(\mathbb{N})$ is a Boolean subalgebra of $\mathcal{P}(\mathcal{B}(H))$.

The atomic masa

MASA: Maximal Abelian Self-Adjoint SubAlgebra.
Fix H and its orthonormal basis (e_n).

$$(\alpha_n) \in \ell^\infty$$

Diagonal operators

$$\sum_n \alpha_n P_{\{n\}} \in \mathcal{B}(H).$$

Lemma
$\mathcal{A}^{(\vec{e})} = \{\sum_n \alpha_n P_{\{n\}}\}$ *is a masa in* $\mathcal{B}(H)$. □

Automorphisms of C* algebras

$$\operatorname{Ad} u(a) = uau^*.$$

An automorphism Φ is *inner* if $\Phi = \operatorname{Ad} u$ for some unitary u.

Lemma

If A is abelian then id *is its only inner automorphism.*

Pf. $uau^* = uu^*a = a$. □

Lemma

If $A = C(X)$ then each automorphism is of the form

$$f \mapsto f \circ \Psi$$

for an autohomeomorphism Ψ of X. □

Lemma

All automorphisms of $\mathcal{B}(H)$ are inner.

Hint. Prove 'all automorphisms of $\mathcal{P}(\mathbb{N})$ are trivial' (i.e., implemented by conjugation by some $h \in S_\infty$).

The essential spectrum of an operator in $\mathcal{B}(H)$

$$\sigma_e(a) = \begin{array}{l}\text{the set of all accumulation points of } \sigma(a)\\ \text{plus all points of } \sigma(a) \text{ of infinite multiplicity}\end{array}$$

a and b are *compalent* $\Leftrightarrow$ there exists c conjugate to a such that $c - b$ is compact.

Fact

If a and b are compalent then $\sigma_e(a) = \sigma_e(b)$.

Equivalence modulo compact operators

Theorem (Weyl–von Neumann–Berg)

The essential spectrum, $\sigma_e(a)$, provides a complete invariant for compalence of normal operators.

$$\mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}.$$

Example

For $g \in S_\infty$ the equations $v_g(e_n) = e_{g(n)}$ uniquely define a unitary operator v_g. If g has arbitrarily long cycles, or an infinite cycle, then $\sigma_e(v_g) = \mathbb{T}$ and v_g is compalent with the bilateral shift of the basis.

The Calkin algebra

$\mathcal{B}(H)$: The algebra of all bounded operators on H.
$\mathcal{K}(H)$: Its ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the *Calkin algebra*.
$\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$: The quotient map.

Fact

Operators a and b are compalent if and only if $\pi(a)$ and $\pi(b)$ are conjugate in the Calkin algebra.

Corollary

For normal operators a and b TFAE.

1. $\sigma_e(a) = \sigma_e(b)$,
2. *a and b are compalent,*
3. *$\pi(a)$ and $\pi(b)$ are conjugate in $\mathcal{C}(H)$,*
4. $(\exists \Phi \in \mathrm{Aut}(\mathcal{C}(H)))\, \Phi(\pi(a)) = \pi(b)$.

Let $S(e_n) = e_{n+1}$ and $\dot{S} = \pi(S)$. Then $\sigma_e(S) = \mathbb{T} = \sigma_e(S^*)$, but S and S^* are not conjugate.

Positive answers to the following three questions are decreasing in strength.

Question (Brown–Douglas–Fillmore, 1977)

Is there an automorphism Φ of $\mathcal{C}(H)$ such that

1. $\Phi(\dot{S}) = \dot{S}^*$*?*
2. *there are a and b in $\mathcal{C}(H)$ such that $\Phi(a) = b$ but $uau^* \neq b$ for all u?*
3. *Is there an outer automorphism of $\mathcal{C}(H)$?*

The answer to (2) is negative if a and b are images of normal operators in $\mathcal{B}(H)$ (BDF).

Proposition

An automorphism of $\mathcal{C}(H)$ is inner iff its restriction to $\mathcal{C}(H_0)$ for some (any) infinite-dimensional subspace H_0 of H is implemented by a unitary.

Proof.

($\Leftarrow$) Fix u such that $\Phi(b) = ubu^*$ for $b \in \mathcal{C}(H_0)$.
Fix $v \in \mathcal{C}(H)$ so that $vv^* = \dot{P}_{H_0}$ and $v^*v = \dot{I}$. Then

$$\begin{aligned}\Phi(a) =& \Phi(v^*)\Phi(vav^*)\Phi(v)\\ =& \Phi(v^*)uvav^*u^*\Phi(v)\end{aligned}$$

With $w = \Phi(v^*)uv$ we have $\Phi(a) = waw^*$. □

Theorem (W. Rudin)

CH implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has $2^{\aleph_1}$ automorphisms (most of them nontrivial).

Proposition

An automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ extends to an automorphism of $\mathcal{C}(H)$ if and only if it is trivial.

Pf. Assume Φ is an automorphism of $\mathcal{C}(H)$ which sends $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ to itself.

Then Φ defines an automorphism of the group $S_\infty/\operatorname{Fin}$; by a result of Alperin–Covington–McPherson, (almost) every automorphism of this group is inner. □

Theorem (Phillips–Weaver, 2006)

CH implies there is an outer automorphism of the Calkin algebra.

The proof uses very nontrivial methods developed for analysis of a deep extension of BDF theory, Kasparov's KK-theory, for separable C* algebras.

I will present my own elementary proof of the Phillips–Weaver theorem.

FDD algebras

For $0 = n_0 < n_1 < \dots$ define $\vec{J} = (J_i)i \in \mathbb{N}$ by

$$J_i = [n_i, n_{i+1}).$$

Let $\mathbb{P}$ be the set of all such $\vec{J}$.

Fact

If $\vec{J} \in \mathbb{P}$ then $(\vec{E})$:

$$E_n = \overline{\mathrm{Span}}\{e_i \mid i \in J_n\}$$

is a decomposition of H into finite-dimensional orthogonal subspaces:

$$H = \bigoplus_n E_n.$$

FDD algebras

$$\mathcal{D}[\vec{J}] = \{a \in \mathcal{B}(H) \mid a[E_i] \subseteq E_i\}.$$

If $\vec{K}$ refines $\vec{J}$, then $\mathcal{D}[\vec{J}] \supseteq \mathcal{D}[\vec{K}]$.

Fact

The unilateral shift S does not belong to $\mathcal{D}[\vec{J}]$ for any $\vec{J} \in \mathbb{P}$. Moreover, $(\mathrm{Ad}\, u)S$ does not belong to $\mathcal{D}[\vec{J}]$ for any $\vec{J}$ and any u.

The key triviality (commutative version)

Lemma

If $g : \mathbb{N} \to \mathbb{N}$ is a finite-to-one function, then there is $\vec{J} \in \mathbb{P}$ such that with

$$J_i^{even} = J_{2i} \cup J_{2i+1} \text{ and } J_i^{odd} = J_{2i+1} \cup J_{2i+2}$$

we can write $g = g^{even} \cup g^{odd}$ so that

$$g^{even}[J_i^{even}] \subseteq J_i^{even} \text{ and } g^{odd}[J_i^{odd}] \subseteq J_i^{odd}$$

for all i.

Pf. Pick n_i so that the g-image of $[0, n_i)$ and the g-preimage of $[0, n_i)$ are both included in $[0, n_{i+1})$. Let $J_i = [n_i, n_{i+1})$ and

$$g^{\text{even}} = g \cap \bigcup_i (J_{2i} \cup J_{2i+1})^2 - (J_{2i+1})^2$$

$$g^{\text{odd}} = g \cap \bigcup_i (J_{2i+1} \cup J_{2i+2})^2 - (J_{2i+2})^2$$

□

The second talk starts here.

The key triviality (quantized version)

Lemma

For every $a \in \mathcal{B}(H)$ *there is* $\vec{J} \in \mathbb{P}$ *such that we can write*

$$a = a^e + a^o + c$$

so that $a^e \in \mathcal{D}[\vec{J}^{even}]$, $a^o \in \mathcal{D}[\vec{J}^{odd}]$ *and* $c \in \mathcal{K}(H)$.

Proof of the key triviality

Pf. Given n_i find n_{i+1} large enough so that

$$\|(I - P_{n_{i+1}})aP_{n_i}\| < 2^{-i}$$

and

$$\|(I - P_{n_{i+1}})a^* P_{n_i}\| < 2^{-i}.$$

Let $J_i = [n_i, n_{i+1})$, $Q_i = P_{J_i}$, and

$$a^e = \sum_i (Q_{2i} + Q_{2i+1})a(Q_{2i} + Q_{2i+1}) - Q_{2i+1}aQ_{2i+1}$$

$$a^o = \sum_i (Q_{2i+1} + Q_{2i+2})a(Q_{2i+1} + Q_{2i+2}) - Q_{2i+2}aQ_{2i+2}.$$

Then

$$a - a^e - a^0 = \sum_{|m-n|\geq 2} Q_m a Q_n$$

is compact. □

The center of $\mathcal{D}[\vec{J}]$

$$\mathcal{Z}(A) = \{a \in A : ab = ba \text{ for all } b \in A\}.$$

Fact

1. $\mathcal{Z}(\mathcal{B}(H)) = \{\alpha I : \alpha \in \mathbb{C}\}$.
2. $\mathcal{Z}(\mathcal{D}[\vec{J}]) = \{\sum_n \alpha_n P_{J_n}, \vec{\alpha} \in \ell^\infty\}$.

Lemma

If $u \in \mathcal{D}[\vec{E}]$ and $v \in \mathcal{D}[\vec{E}]$ are unitaries, TFAE:

1. $\operatorname{Ad} u$ *and* $\operatorname{Ad} v$ *agree on* $\mathcal{D}[\vec{E}]$.
2. $v = bu$ *for* $b \in \mathcal{Z}(\mathcal{D}[\vec{E}])$.

Proof.

$$
\begin{aligned}
(1) \Leftrightarrow & \operatorname{Ad}(u^* v) = \mathrm{id} \\
\Leftrightarrow & (u^* v) a (u^* v)^* = a, \quad (\forall a \in \mathcal{D}[\vec{J}]) \\
\Leftrightarrow & u^* v \in \mathcal{Z}(\mathcal{D}[\vec{E}]].
\end{aligned}
$$

□

$$\mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}.$$

Lemma (Turning the knobs)

Assume u is a unitary in $\mathcal{D}[\vec{J}]$ and $\alpha_n \in \mathbb{T}$ for all n. If

$$v = \sum_n \alpha_n P_{J_n} u$$

then $\operatorname{Ad} u$ *and* $\operatorname{Ad} v$ *agree on* $\mathcal{D}[\vec{J}]$.

Unitaries in the atomic masa

For $\alpha \in (\mathbb{T})^{\mathbb{N}}$

$$u_\alpha = \sum_n \alpha(n) P_{\{n\}}$$

is a unitary.
For $X \subseteq \mathbb{N}$ and α, β in $(\mathbb{T})^{\mathbb{N}}$ let

$$\Delta_X(\alpha, \beta) = \inf_{z \in \mathbb{T}} \|(u_\alpha^* u_\beta - z \cdot I) P_X\|.$$

Corollaries to the key triviality

[In the talk I have claimed the following

> *If* $\Phi \in \mathrm{Aut}(\mathcal{B}(H))$ *then* Φ *is uniquely determined by its restriction to* $\bigcup_{\vec{J} \in \mathbb{P}} \mathcal{D}[\vec{J}]$.

and then I realized that it is neither obvious nor necessary. It is probably true, though.]
With $\mathcal{K}[\vec{J}] = \mathcal{D}[\vec{J}] \cap \mathcal{K}(H)$ and $\mathcal{C}[\vec{J}] = \mathcal{D}[\vec{J}]/\mathcal{K}[\vec{J}]$ we have:

Corollary

If $\Phi \in \mathrm{Aut}(\mathcal{C}(H))$ *then* Φ *is uniquely determined by its restriction to* $\bigcup_{\vec{J} \in \mathbb{P}} \mathcal{C}[\vec{J}]$.

Lemma

If $\vec{J} \in \mathbb{P}$ *and* $\alpha, \beta \in \mathbb{T}^{\mathbb{N}}$ *then*

1. $\mathrm{Ad}\, u_\alpha$ *and* $\mathrm{Ad}\, u_\beta$ *agree on* $\mathcal{D}[\vec{E}]$ *if and only if* $\sup_n \Delta_{J_n}(\alpha, \beta) = 0$.
2. $\mathrm{Ad}\, \dot{u}_\alpha$ *and* $\mathrm{Ad}\, \dot{u}_\beta$ *agree on* $\mathcal{D}[\vec{E}]$ *modulo* $\mathcal{K}(H)$ *if and only if* $\limsup_n \Delta_{J_n}(\alpha, \beta) = 0$.

For (2) use the fact that $a \in \mathcal{D}[\vec{J}]$ is compact if and only if $\limsup_n \|aP_{J_n}\| = 0$.

An order on $\mathbb{P}$

For $\vec{J}$, $\vec{K}$ in $\mathbb{P}$ let

$$\vec{J} \ll^* \vec{K} \quad \text{iff} \quad (\forall^\infty m)(\exists n) J_m \cup J_{m+1} \subseteq K_n \cup K_{n+1}.$$

Fact

The ordering $\ll^*$ is σ-directed. (It is Tukey-equivalent to $\mathbb{N}^\mathbb{N}, \leq^*$.)

For $\vec{J} \in \mathbb{P}$ we define

$$J_i^{\text{even}} = J_{2i} \cup J_{2i+1} \text{ and } J_i^{\text{odd}} = J_{2i+1} \cup J_{2i+2}$$

$\approx_{\vec{J}}$ and $\sim_{\vec{J}}$

$$\mathcal{D}D[\vec{J}] = \mathcal{D}[\vec{J}^{\text{even}}] \cup \mathcal{D}[\vec{J}^{\text{odd}}].$$

Then $\vec{J} \ll^* \vec{K}$ iff $\mathcal{D}D[\vec{J}] \subseteq \mathcal{D}D[\vec{K}]$.
Let $u \approx_{\vec{J}} v$ if

$$(\text{Ad}\, u)a - (\text{Ad}\, v)a$$

is compact for all $a \in \mathcal{D}[\vec{J}]$.

Lemma

If $u_\alpha \approx_{\vec{J}} u_\beta$ then there is a γ such that $u_\alpha \approx_{\vec{J}^{even}} u_\gamma$ and $u_\beta \approx_{\vec{J}^{odd}} u_\gamma$.

Proof.

Turn the knobs of α. □

Let $u \sim_{\vec{J}} v$ if

$$(\text{Ad}\, u)a - (\text{Ad}\, v)a$$

is compact for all $a \in \mathcal{D}D[\vec{J}]$.

Coherent families of unitaries

A family $\mathcal{F}$ of pairs $(\vec{J}, u)$ is a *coherent family of unitaries (cfu)* if

1. For all $(\vec{J}, u) \in \mathcal{F}$, $\vec{J} \in \mathbb{P}$ and u is a unitary,
2. $\mathcal{F}_0 = \{\vec{J} \mid (\exists u)(\vec{J}, u) \in \mathcal{F}\}$, is $\ll^*$-cofinal,
3. For $\vec{J} \ll^* \vec{K}$ in $\mathcal{F}_0$ we have $u_{\vec{J}} \sim_{\mathcal{D}D[\vec{J}]} u_{\vec{K}}$.

Lemma

If $\mathcal{F}$ is a coherent family of unitaries then there an automorphism $\Phi_{\mathcal{F}}$ of $\mathcal{C}(H)$ such that $\Phi_{\mathcal{F}}(\pi(a)) = \pi(uau^*)$ for all $(\vec{J}, u) \in \mathcal{F}$ and all $a \in \mathcal{D}[\vec{J}]$.
Such $\Phi_{\mathcal{F}}$ is unique.

Pf. Use the Key Triviality. □

A coherent family of unitaries is *trivial* if $\Phi_{\mathcal{F}}$ is inner.

Theorem (Farah, 2007)

Assume CH. Then there is a nontrivial coherent family of unitaries. Hence there exists an outer automorphism of $\mathcal{C}(H)$.

Pf. Enumerate all unitaries in $\mathcal{B}(H)$ as $v(\xi)$, for $\xi < \omega_1$.
Recursively find an increasing $\vec{J}^\xi$, for $\xi < \omega_1$, cofinal in $\mathbb{P}, \ll^*$ and $\alpha(\xi) \in \mathbb{T}^{\mathbb{N}}$ so that

$$\mathcal{F} = \{(\vec{J}^\xi, \alpha(\xi)) : \xi < \omega_1\}$$

is a cfu and $u_{\alpha(\xi)} \not\sim_{\mathcal{D}D[\vec{J}^\xi]} v(\xi)$ for all ξ. □

Improvements

Stefan Geschke: $\mathfrak{d} = \aleph_1 + 2^{\aleph_1} > 2^{\aleph_0}$ implies the existence of a nontrivial cfu
Juris Steprāns: $\mathfrak{d} = \aleph_1$ implies the existence of a nontrivial cfu.

It is not known whether $\mathfrak{d} = \aleph_1$ implies the existence of a trivial automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

Theorem (Steprāns)

In the Silver model $\mathfrak{d} = \aleph_1$ and every automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is somewhere trivial.

An axiom formerly known as OCA

Todorcevic's Axiom, TA: If $G = (V, E)$ is a graph such that

$$E = \bigcup_{n=0}^{\infty} X_n \times Y_n,$$

then G is either

1. *countably chromatic*: $V = \bigcup_{n \in \mathbb{N}} V_n$ so that $[V_n]^2 \cap E = \emptyset$ for each n, or
2. it has an uncountable *clique*: $H \subseteq V$ such that $[H]^2 \subseteq E$.

TA is among the axioms that are sometimes called OCA.

TA and automorphisms of quotient Boolean algebras

Building on seminal work of Shelah (consistency) and Shelah–Steprans (implication from PFA):

Theorem (Velickovic, 1990)

TA+MA imply that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *are trivial.* □

Just, Farah: TA+MA imply rigidity for a number of quotient Boolean algebras of the form $\mathcal{P}(\mathbb{N})/J$, where J is an analytic ideal.

Conjecture (Rigidity Conjecture)

Assume TA+MA. If I *and* J *are analytic ideals, then* $\mathcal{P}(\mathbb{N})/I \cong \mathcal{P}(\mathbb{N})/J$ *if and only if* $I \cong J$*. Moreover, every isomorphism is given by* $h\colon \mathbb{N} \to \mathbb{N}$*.*

TA and automorphisms of the Calkin algebra

We shall prove that TA implies all automorphisms of the Calkin algebra are inner. The following lemma is the first step.

Lemma

TA implies that every coherent family of unitaries $\mathcal{F}$ is trivial. In particular, if Φ is implemented by a unitary on each

$$\mathcal{C}[\vec{J}] = \mathcal{D}[\vec{J}]/(\mathcal{K}(H) \cap \mathcal{D}[\vec{J}])$$

then it is inner.

Proving ‘TA implies every coherent family of unitaries is trivial

Lemma

Assume $\mathcal{F}$ is a cfu and $(\vec{J}_0, u_0) \in \mathcal{F}$ with $\vec{J}_0$ being the partition of $\mathbb{N}$ into singletons. Then

$$\mathcal{F}' = \{(\vec{J}, u_0^* u) : (\vec{J}, u) \in \mathcal{F} \text{ and } \vec{J}_0 \ll^* \vec{J}\}$$

is a cfu, each unitary appearing in $\mathcal{F}$ is diagonalized, and $\mathcal{F}$ is trivial iff $\mathcal{F}'$ is trivial.

Hence wlog each cfu can be identified with a family of pairs $(\vec{J}, u_\alpha)$ for $\alpha \in \mathbb{T}^{\mathbb{N}}$.

Proving 'TA implies every coherent family of unitaries is trivial, II

Recall

$$\Delta_J(\alpha,\beta) = \inf_{z\in\mathbb{T}} \|(u_\alpha^* u_\beta - z\cdot I)P_J\|,$$

$$\Delta_\emptyset(\alpha,\beta) = 0.$$

For $k \in \mathbb{N}$ let G_k be the graph on $\mathcal{F}_0$ in which $\{\vec{J},\vec{K}\} \in E_k$ iff

$$\sup_{m,n} \Delta_{J_m\cap K_n}(\alpha(\vec{J}),\alpha(\vec{K})) > 2^{-k}$$

Lemma

There are no uncountable cliques for any k.

Pf. Assume $\mathcal{X} \subseteq \mathcal{F}_0$ is of size $\aleph_1$ and such that

$$\sup_{m,n} \Delta_{J_m \cap K_n}(\alpha(\vec{J}), \alpha(\vec{K})) > 2^{-k}$$

for all distinct $\vec{J}$ and $\vec{K}$ in $\mathcal{X}$.
Since TA implies $\mathfrak{b} > \aleph_1$ there is is $\vec{L}$ such that $\vec{J} \ll^* \vec{L}$ for all $\vec{J} \in \mathcal{X}$.
Fix $\beta \in (\mathbb{T})^{\mathbb{N}}$ such that $(\vec{L}^{\text{even}}, u_\beta) \in \mathcal{F}_0$ and $(\vec{L}^{\text{odd}}, u_\beta) \in \mathcal{F}_0$.

Still proving there are no uncountable cliques.

For all $\vec{J} \in \mathcal{X}$ and all but finitely many n we have $J_n \subseteq L_{2k} \cup L_{2k+1}$ or $J_n \subseteq L_{2k+1} \cup L_{2k+2}$ for some k. Therefore

$$\limsup_n \Delta_{J_n}(\alpha(\vec{J}), \beta) = 0.$$

But this implies that for all $\vec{J}, \vec{K}$ in $\mathcal{X}$

$$\limsup_m \limsup_n \Delta_{J_n \cap K_m}(\alpha(\vec{J}), \alpha(\vec{K})) = 0.$$

Using the separability of $\mathbb{T}$ and the uncountability of $\mathcal{X}$ we can find distinct $\vec{J}, \vec{K}$ in $\mathcal{X}$ such that

$$\sup_{m,n} \Delta_{J_m \cap K_n}(\alpha(\vec{J}), \alpha(\vec{K}))$$

is as small as we wish. $\square$

An old warhorse

Lemma

If $(\mathbb{Q}, \preceq)$ *is a* σ*-directed poset and* $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} X_n$*, then* $X_{\bar{n}}$ *is cofinal in* $\mathbb{Q}$ *for some* $\bar{n}$*.*

Proof.

If not, for each n pick a_n not dominated by any element of X_n. Since $\mathbb{Q}$ is σ-directed, there is b such that $a_n \preceq b$ for all n. But then $b \in X_m$ for some m – which one? □

Lemma (Infinitely branching node)

If $\mathcal{X} \subseteq \mathbb{P}$ *is* $\ll^*$*-cofinal, then there is* m *such that for every* $k \geq m$ *there is* $\vec{J} \in \mathcal{X}$ *and* i *such that*

$$[m, k) \subseteq J_i. \qquad \square$$

Use TA $\Rightarrow (\mathcal{F}_0, E_k)$ is countably chromatic for every k

Pick $\mathcal{F} \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ so that for all k

1. $[\mathcal{X}_k]^2 \cap E_k = \emptyset$ and
2. $\mathcal{X}_k$ is $\ll^*$-cofinal.

Find $0 = m_0 < m_1 < m_2 < \dots$ so that

1. $(\forall n \geq m_k)(\exists \vec{J} = J(k,n) \in \mathcal{X}_k)(\exists p)[m_k, n) \subseteq J_p$

Fix k. Consider $\alpha_{k,n}$ such that

$$(\vec{J}(k,n), \alpha_{k,n}) \in \mathcal{X}_k.$$

Go to a subsequence so that in $(\mathbb{T})^{\mathbb{N}}$ we have

$$\lim_n \alpha_{k,n} = \alpha_k$$

Define

$$\gamma_0 = \alpha_0$$
$$\gamma_{k+1} = \gamma_k(m_{k+1})\overline{\alpha_{k+1}(m_{k+1})}\alpha_{k+1}.$$

(Note that $\gamma_{k+1}(m_{k+1}) = \gamma_k(m_{k+1})$.)
Let $L_k = [m_k, m_{k+1})$ and define γ by

$$\gamma \upharpoonright L_k = \gamma_k \upharpoonright L_k$$

Lemma

For all k we have

1. $\gamma_k(m_{k+1}) = \gamma_{k+1}(m_{k+1})$
2. $(\forall(\vec{J}, \beta) \in \mathcal{X}_k)(\forall n \geq k)(\forall i)\Delta_{J_i \setminus m_n}(\beta, \gamma_n) \leq 2^{-k}$.
3. $(\forall n \geq k)\Delta_{[m_n,\infty)}(\gamma_k, \gamma_n) \leq 2^{-n}$.
4. $\sup_{i \geq m_k} |\gamma_k(i) - \gamma(i)| \leq 2^{-k}$.
5. $(\forall(\vec{J}, \beta) \in \mathcal{X}_k)(\forall i)(\Delta_{J_i \setminus M_k}(\beta, \gamma) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}$.

Fix $\vec{K} \in \mathcal{F}_0$. We need to show that

$$\limsup_m \Delta_{K_m}(\alpha(\vec{K}), \gamma) = 0.$$

Fix k. Find $\vec{L} \in \mathcal{X}_k$ such that $\vec{K} \ll^* \vec{L}$. Then

$$\limsup_i \Delta_{K_i}(\alpha(\vec{K}), \alpha(\vec{L})) = 0.$$

The other consequence of TA

Proposition

TA implies that Φ *is inner on each* $\mathcal{C}[\vec{J}]$.

Theorem

TA implies all automorphisms of $\mathcal{C}(H)$ *are inner.*

For $M \subseteq \mathbb{N}$ let

$$\mathcal{D}_M[\vec{J}] = \{a \in \mathcal{D}[\vec{J}] \mid aP_{J_n} = 0 \text{ for all } n \notin M\}.$$

Lemma

Assume $|J_n| \leq |J_{n+1}|$ for all n. If Φ is implemented by a unitary on $\mathcal{C}_M[\vec{J}]$ for some infinite M then it is implemented by a unitary on $\mathcal{C}[\vec{J}]$,

Proof.

Find v such that $v^*v = I$ and $v\mathcal{D}[\vec{J}]v^* \subseteq \mathcal{D}_M[\vec{J}]$.
Then for $a \in \mathcal{D}[\vec{J}]$:

$$\begin{aligned} \Phi(a) &= \Phi(v^*)\Phi(vav^*)\Phi(v) \\ &= \Phi(v^*)uvav^*u^*\Phi(v) \end{aligned}$$

hence with $w = \Phi(v^*)uv$ we have that $\operatorname{Ad} w$ is a representation of Φ on $\mathcal{D}[\vec{J}]$. □

The third talk starts here.

Representations (liftings)

Some $\Psi \colon \mathcal{B}(H) \to \mathcal{B}(H)$ is a representation of $\Phi \colon \mathcal{C}(H) \to \mathcal{C}(H)$ if

$$\begin{CD}
\mathcal{B}(H) @>{\Psi}>> \mathcal{B}(H) \\
@V{\pi}VV @VV{\pi}V \\
\mathcal{C}(H) @>>{\Phi}> \mathcal{C}(H).
\end{CD}$$

Let Ψ be a representation of Φ

Wlog, for all a:

1. a is a projection/unitary/self-adjoint iff $\Psi(a)$ is projection/unitary/self-adjoint.
2. $\|a\| = \|\Psi(a)\|$.
3. $\Psi(a^*) = \Psi(a)^*$.

Uniformizations (selections)

A subset of a Polish space is *analytic* if it is a continuous image of a Borel set. Sets in the σ-algebra generated by analytic sets are *C-measurable*.

Theorem (Jankov, von Neumann)

If X and Y are Polish spaces and $B \subseteq X \times Y$ is analytic, then B can be uniformized by a C-measurable function.

The strong operator topology on $\mathcal{B}(H)_{\leq 1} = \{a \mid \|a\| = 1\}$ is Polish.

Theorem

If Φ has a C-measurable representation on $\mathcal{B}(H)_{\leq 1}$, then it is inner.

Proof.

Although this proof is rather nontrivial, long and interesting, there is not much set theory in it so I skip it. □

Corollary

TFAE:

1. Φ *is inner,*
2. Φ *has a C-measurable representation on* $\mathcal{B}(H)_{\leq 1}$
3. $\Gamma_\Phi = \{(a, b) \mid \|a\| \leq 1, \|b\| \leq 1, \Phi(\dot{a}) = \dot{b}\}$ *is analytic.*

ε-approximations

$\Psi\colon \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$ is an *ε-approximation* to Φ if $\|\pi(\Psi(a)) - \Phi(\pi(a))\| \leq \varepsilon$ for all $a \in \mathcal{B}(H)_{\leq 1}$.

$$\begin{CD}
\mathcal{B}(H)_{\leq 1} @>{\Psi}>> \mathcal{B}(H)_{\leq 1} \\
@V{\pi}VV @VV{\pi}V \\
\mathcal{C}(H)_{\leq 1} @>>{\Phi}> \mathcal{C}(H)_{\leq 1}.
\end{CD}$$

Proposition

TFAE:

1. Φ *is inner.*
2. *For every* $\varepsilon > 0$ Φ *has a C-measurable* ε*-approximation.*
3. *For every* $\varepsilon > 0$ *the set* $\Gamma^{\varepsilon}_{\Phi} = \{(a, b) \mid \|a\| \leq 1, \|b\| \leq 1, \|\Phi(\dot{a}) - \dot{b}\| \leq \varepsilon\}$ *is analytic.*

Pf. (1) implies (2): $\operatorname{Ad} u$ is continuous.
(2) implies (3): If Ψ_{ε} is an ε-approximation then

$$\Gamma^{\varepsilon}_{\Phi} = \bigcap_{n} \Gamma^{1/n}_{\Psi_{\varepsilon}}.$$

(3) implies (1): $\Gamma = \bigcap_n \Gamma^{1/n}_{\Phi}$ is analytic and for every a there is b such that $(a, b) \in \Gamma$.
Now apply Jankov, von Neumann to get a C-measurable representation.

Lemma

Assume $\vec{J}$ is a partition of $\mathbb{N}$ into finite intervals such that $|J_n| \leq |J_{n+1}|$ for all n. If $M \subseteq \mathbb{N}$ is infinite and Φ has a C-measurable ε-approximation on $\mathcal{D}_M[\vec{J}]$, then Φ has a C-measurable ε-approximation on $\mathcal{D}[\vec{J}]$.

Proof.

Fix v such that $v^*v = I$ and $v\mathcal{D}[\vec{J}]v^* \subseteq \mathcal{D}_M[\vec{J}]$... □

A set $\Delta \subseteq \mathcal{B}(H)_{\leq 1} \times \mathcal{B}(H)_{\leq 1}$ is ε-*narrow* if

$$(\forall a, b, c)((a, b) \in \Delta \text{ and } (a, c) \in \Delta \text{ implies } \|\dot{b} - \dot{c}\| \leq \varepsilon).$$

Fact

1. *Γ_Φ^ε is 2ε-narrow*
2. *If $\mathcal{X} \supseteq \Psi$ and $\mathcal{X}$ is ε-narrow, then $\mathcal{X} \subseteq \Gamma_\Phi^\varepsilon$.* □

Fix $\vec{J}$.

$\mathcal{J}^\varepsilon = \{M \subseteq \mathbb{N} \mid \Gamma_\Phi$ can be covered by countably many ε-narrow analytic sets on $\mathcal{D}_M[\vec{J}]\}$

$= \{M \subseteq \mathbb{N} \mid$ there are ε-narrow analytic sets Δ_i, $i \in \mathbb{N}$, such that

$$\{(a, b) \in \Gamma_\Phi \mid a \in \mathcal{D}_M[\vec{J}]\} \subseteq \bigcup_i \Delta_i\}$$

Proposition

If $M = \dot{\bigcup}_n M_n$ is in $\mathcal{J}^\varepsilon$, then Γ_Φ has a C-measurable 4ε-approximation on $\mathcal{C}_{M_n}[\vec{J}]$ for some n.

Proof.

Diagonalization. □

Corollary

Assume that $\mathcal{J}^\varepsilon$ contains an infinite set for every $\varepsilon > 0$. Then Φ is inner on $\mathcal{C}[\vec{J}]$.

Pf. If $M \in \mathcal{J}^\varepsilon$ is infinite, then Φ has a C-measurable 4ε-approximation on $\mathcal{D}_N[\vec{J}]$ for some infinite $N \subseteq M$. Therefore Φ has a C-measurable 4ε-approximation on $\mathcal{D}[\vec{J}]$. □

Lemma

For all $\varepsilon > 0$ *and all* $\vec{J}$*, TA implies that* $\mathcal{J}_{\Phi}^{\varepsilon}$ *contains an infinite set.*

Proof. Index $\vec{J}$ as J_s, $s \in 2^{<\mathbb{N}}$.

Let $\mathcal{X}$ be the set of all pairs (S, a) such that

1. S is infinite,
2. S is contained in a maximal branch $B(S)$ in $2^{<\mathbb{N}}$
3. $a \in \mathcal{D}_S[\vec{J}]_{\leq 1}$ and $a \notin \mathcal{K}(H)\}$.

For $n \in \mathbb{N}$ define a graph G_n with vertex set $\mathcal{X}$ such that there is an edge between (S, a) and (T, b) iff all of the following hold:

(K1) $B(S) \neq B(T)$,

(K2) $(\forall i \in S \cap T)\|(a-b)P_{\{i\}}\| < 2^{-i}$.

(K3) $\|\Psi(a)\Psi(P_T) - \Psi(P_S)\Psi(b)\| > 2^{-n}$ or
$\|\Psi(P_T)\Psi(a) - \Psi(b)\Psi(P_S)\| > 2^{-n}$.

Fact

G_n has no uncountable cliques for any n.

Pf. Assume $\mathcal{X}$ is a clique of size $\aleph_1$. Let

$$M = \{t \in 2^{<\mathbb{N}} : (\exists (S_t, a_t) \in \mathcal{X}) t \in S_t\}$$

and

$$c = \sum_{t \in M} a_t P_{J_t}.$$

Then for all $(S, a) \in \mathcal{X}$ we have that $cP_S - a$ and $P_S c - a$ are both compact, and therefore $\Psi(c)\Psi(P_S) - \Psi(a)$ and $\Psi(P_S)\Psi(c) - \Psi(a)$ are both compact.
Hence for (S, a) and (T, b) in $\mathcal{X}$ we have

$$\Psi(a)\Psi(P_T) \approx \Psi(P_S)\Psi(c)\Psi(P_T) \approx \Psi(P_S)\Psi(b).$$

(Modulo some dull computations.)

Lemma

Assume $\mathcal{X} = \bigcup_j \mathcal{X}_j$*, each* $\mathcal{X}_j$ *is* G_n*-independent and* $\mathcal{D}_j$ *is countable and dense in* $\mathcal{X}_j$*. If* $B \in 2^{\mathbb{N}}$ *is distinct from* $B(S)$ *for all* $(S, a) \in \bigcup_j \mathcal{D}_j$*, then* $\bigcup_{k \in B} J_k \in \mathcal{J}^{10/n}$*.*

Proof.

This is my least favourite part of the proof; it takes forever and it is not very exciting. This remark just goes to show that I still don't understand it well enough. □

Open problems

$M(A)$ is the *multiplier algebra* of a C^*-algebra A.

Examples

If X is locally compact and $A = C_0(X)$ then $M(A) = C(\beta X)$.
If $A = \mathcal{K}(H)$ then $M(A) = \mathcal{B}(H)$.
If A is unital, then $M(A) = A$.

Question (G. Elliott)

What can be said about automorphisms of the corona algebras $M(A)/A$? In particular, what if A is a UHF (uniformly hyperfinite) algebra?

Question (P.W. Ng)

When does $M(A)/A \cong M(B)/B$ imply $A \cong B$?

Theorem (Brown)

If A and B are separable then $M(A) \cong M(B)$ iff $A \cong B$.

Theorem

Assume TA+MA. If $A = C_0(\xi)$ for a countable ordinal ξ then all automorphisms of $M(A)/A$ are trivial.

Each known automorphism of $\mathcal{C}(H)$ is 'pointwise inner'

$$(\forall a)(\exists u)\Phi(a) = uau^*.$$

Question

Is there an automorphism of $\mathcal{C}(H)$ that sends the unilateral shift to its adjoint?

Such an automorphism cannot be inner (Fredholm index!), hence TA implies the negative answer.

Other directions

Here is a short list of references illustrating recent (since 2007) applications of set theory to C*-algebras.

(1) N. Weaver, Set theory and C^*-algebras, Bull. Symb. Logic (2007), 13, 1–20. (A very nice survey of 'older' results.)
(2) I. Farah and E. Wofsey, Set theory and operator algebras, notes from the Appalachian set theory workshop. http://www.math.yorku.ca/∼ifarah/notes.html (Another survey geared towards set theorists with more details and some proofs.)
(3) I. Farah, Some problems about operator algebras with set-theoretic flavor, http://www.math.yorku.ca/∼ifarah/notes.html
(4) N.C. Phillips and N. Weaver, The Calkin algebra has outer automorphisms, Duke Math. Journal, (2007) 139, 185–202.
(5) I. Farah, All automorphisms of the Calkin algebra are inner, http://arxiv.org/abs/0705.3085v3

Papers (6) and (7) deal with relative commutants of a C*-algebra in its ultrapower.

(6) I. Farah, N.C. Phillips, and J. Steprāns, The commutant of $L(H)$ in its ultrapower may or may not be trivial, http://arxiv.org/abs/0808.3763v1

(7) I. Farah, The relative commutant of separable C*-algebras of real rank zero, arXiv:0809.2843v1 [math.OA]

The following two are applications of Hjorth's theory of turbulence to classification problems of von Neumann algebras and representations of a C*-algebra.

(8) A. Tørnquist and R. Sasyk, preprint, 2008

(9) D. Kerr, H. Li, and M. Pichot, Turbulence, representations, and trace-preserving actions arXiv:0808.1907 [math.DS]

The following two study masas in the Calkin algebra and nonseparable C*-algebras, respectively.

(10) S. Shelah and J. Steprāns, Masas in the Calkin algebra without the Continuum Hypothesis, preprint, 2008

(11) I. Farah and T. Katsura, Nonseparable UHF algebras, preprint, 2008+.