ANOTHER CHARACTERIZATION OF NONMEAGER IDEALS

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I will prove a characterization of nonmeager hereditary subsets of $\mathcal{P}(\mathbb{N})$. It is an n-dimensional version of [2, Theorem 3.10.1], which is an extension of the well-known characterization of nonmeager ideals by Jalali–Naini ([3]) and Talagrand ([4]).

A subset $\mathcal{J}$ of $\mathcal{P}(\mathbb{N})$ is *hereditary* if for every $A \in \mathcal{J}$ and every $B \subseteq A$ we have $B \in \mathcal{J}$. By $[\mathbb{N}]^d$ I denote the set of all d-element subsets of $\mathbb{N}$. By $h''X$ I denote the image of X by d, $h''X = \{h(a) : a \in X\}$. A function $h\colon [\mathbb{N}]^d \to \mathbb{N}$ *has no infinite homogeneous sets* if h is not constant on $[X]^d$ for any $X \in [\mathbb{N}]^\omega$. If $s = \{n_0, n_1, \dots, n_{d-1}\}_< \in [\mathbb{N}]^d$ and $t \subseteq d$ then

$$s \restriction t = \{n_i : i \in t\}.$$

(Here $\{n_0, n_1, \dots, n_{d-1}\}_<$ is a d-element subset of $\mathbb{N}$ such that $n_0 < n_1 < \dots < n_{d-1}$.)

Theorem 1. *Assume $\mathcal{J}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$. Then the following are equivalent:*

(1) *$\mathcal{J}$ is nonmeager,*
(2) *for every $d \in \mathbb{N}$ and every $h\colon [\mathbb{N}]^d \to \mathbb{N}$ with no infinite homogeneous sets there is $X \in [\mathbb{N}]^\omega$ such that $h''[X]^d \in \mathcal{J}$.*

Proof. If $h\colon [\mathbb{N}]^1 \to \mathbb{N}$, then it can be identified with $h'\colon \mathbb{N} \to \mathbb{N}$. This h' is finite-to-one if and only if h has no infinite homogeneous sets. So by [2, Theorem 3.10.1], (1) is equivalent to the special case of (2) when $d = 1$ It therefore remains only to prove that (1) implies (2) for an arbitrary d.

Assume $\mathcal{J}$ is nonmeager and $h\colon [\mathbb{N}]^d \to \mathbb{N}$ has no infinite homogeneous sets. Let us first treat the case when h is one-to-one. For $i \in \mathbb{N}$ let

$$u_i = \{h(s) : s \in [\mathbb{N}]^d \text{ and } \max(s) = i\}.$$

These sets are pairwise disjoint and finite, therefore by [2, Theorem 3.10.1] there exists $X \in [\mathbb{N}]^\omega$ such that $A = \bigcup_{i\in X} u_i$ belongs to $\mathcal{J}$. Then $h''[X]^d \in \mathcal{J}$ because $h''[X]^d \subseteq \bigcup_{i\in X} u_i$.

We now prove the general case. Let $h\colon [\mathbb{N}]^d \to \mathbb{N}$ be a function that has no infinite homogeneous sets. By the Erdös–Rado canonical partition theorem ([1]), there are $X_0 \in [\mathbb{N}]^\omega$ and $t \subseteq d$ such that for all $s_1, s_2 \in [X_0]^d$ we have

$$s_1 \restriction t = s_2 \restriction t \text{ if and only if } h(s_1) = h(s_2).$$

Since h has no infinite homogeneous sets, $d' = |t| \geq 1$. Define $g\colon [X_0]^{d'} \to \mathbb{N}$ by

$$g(u) = h(s), \text{ where } s \in [X_0]^d \text{ is such that } s \restriction t = u.$$

Then g is well-defined and one-to-one, and for every infinite $Y \subseteq X_0$ we have $g''[Y]^d \subseteq h''[Y \setminus m]^d$ for some m. By the above, there is $X \in [X_0]^\omega$ such that $g''[X]^{d'} \in \mathcal{J}$. This concludes the proof. □

References

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