

## ANALYTIC HAUSDORFF GAPS II: THE DENSITY ZERO IDEAL

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ABSTRACT. We prove two results about the quotient over the asymptotic density zero ideal. First, it is forcing equivalent to  $\mathcal{P}(\mathbb{N})/\text{Fin} * \mathcal{R}_c$ , where  $\mathcal{R}_c$  is the homogeneous probability measure algebra of character  $c$ . Second, if it has analytic Hausdorff gaps then they look considerably different from previously known gaps of this form.

We consider *density ideals*, ideals of the form  $\mathcal{Z}_\mu = \{A \mid \limsup_n \mu_n(A) = 0\}$  for a sequence  $\mu_m$  ( $m \in \mathbb{N}$ ) of probability measures concentrating on pairwise disjoint intervals  $I_m$  ( $m \in \mathbb{N}$ ) of  $\mathbb{N}$ . In Theorem 1.3 we prove that the regular open algebra of such quotient is isomorphic to the regular open algebra of  $\mathcal{P}(\mathbb{N})/\text{Fin} * \mathcal{R}_c$ . Study of quotients  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  as forcing notions has recently attracted a bit of attention ([1], [12], [8]).

In [19] it was proved that there are no analytic Hausdorff gaps over  $\text{Fin}$ . Todorćević actually proved that every pregap  $\mathcal{A}, \mathcal{B}$  over  $\text{Fin}$  such that  $\mathcal{A}$  is analytic and  $\mathcal{B}/\text{Fin}$  is  $\sigma$ -directed can be countably separated (and more). In [3, Theorem 5.7.1, Theorem 5.7.2 and Lemma 5.8.7] we have proved that  $\text{Fin}$  is the only analytic P-ideal that has this property: If  $\mathcal{I}$  is an analytic P-ideal that is not Rudin–Keisler isomorphic to  $\text{Fin}$ , then there is a gap  $\mathcal{A}, \mathcal{B}$  over  $\mathcal{I}$  such that  $\mathcal{A}$  and  $\mathcal{B}$  are Borel,  $\mathcal{B}/\mathcal{I}$  is  $\sigma$ -directed and  $\mathcal{A}$  is not countably separated from  $\mathcal{B}$ .

In [4] it was proved that there are analytic Hausdorff gaps over any dense  $F_\sigma$  P-ideal. Recall that

$$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_n |A \cap n|/n = 0\}$$

is the ideal of *asymptotic density zero* sets. In §2 we prove results on the structure of analytic Hausdorff gaps in its quotients, making some progress towards [3, Question 5.13.7] and [4, Question 8a and Question 10].

In Proposition 3.2 we show that if  $\mathcal{I}$  is a dense analytic P-ideal without analytic Hausdorff gaps in its quotient, then the restriction of  $\mathcal{I}$  to some positive set is a generalized density ideal. This gives a partial solution to the problem of characterizing those analytic P-ideals that do not have analytic Hausdorff gaps in their quotients ([3, Problem 5.13.5]; see also Question 4.1).

**Terminology.** Our terminology and notation follow [3]. Two families  $\mathcal{A}, \mathcal{B}$  in a quotient  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  form a *pregap* if  $A \cap B \in \mathcal{I}$  for all  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ . A pregap is *separated* (or *split*) by  $C \subseteq \mathbb{N}$  if for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  we have  $A \setminus C \in \mathcal{I}$  and  $B \cap C \in \mathcal{I}$ . If it is not separated by any  $C$ , then it is a *gap*.

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*Date:* February 16, 2005.

1991 *Mathematics Subject Classification.* 03E15, 03E05, 03C20, 06E05.

Filename: n2003d08-s.tex.

Partially supported by NSERC.

We also say that  $\mathcal{A}$  and  $\mathcal{B}$  form a gap over  $\mathcal{I}$ . A pregap is *countably separated* if there are sets  $C_n \subseteq \mathbb{N}$  ( $n \in \mathbb{N}$ ) such that for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  there is  $n$  such that  $A \setminus C_n \in \mathcal{I}$  and  $B \cap C_n \in \mathcal{I}$ . A gap is *Hausdorff* if both of its sides  $\mathcal{A}$  and  $\mathcal{B}$  are countably directed under inclusion modulo  $\mathcal{I}$ . A gap is *analytic* if  $\mathcal{A}$  and  $\mathcal{B}$  are analytic subsets of  $\mathcal{P}(\mathbb{N})$ , taken with its Cantor-set topology.

An ideal  $\mathcal{I}$  on  $\mathbb{N}$  is a *P-ideal* if for every sequence  $A_n$  of sets in  $\mathcal{I}$  there is  $A \in \mathcal{I}$  such that  $A_n \setminus A$  is finite for all  $n$ . An ideal  $\mathcal{I}$  is *dense* if every infinite  $A \subseteq \mathbb{N}$  has an infinite subset in  $\mathcal{I}$ .

A function  $\phi$  defined on the power-set of some set  $I$  is a *submeasure* if  $\phi(\emptyset) = 0$ , it is monotonic ( $A \subseteq B$  implies  $\phi(A) \leq \phi(B)$ ), and subadditive ( $\phi(A \cup B) \leq \phi(A) + \phi(B)$ ). We say that  $\phi$  is a submeasure on  $I$ . A submeasure on  $\mathcal{P}(\mathbb{N})$  is *lower semicontinuous* if for all  $A$  we have  $\phi(A) = \sup \phi(s)$ , where  $s$  ranges over all finite subsets of  $A$ . In this case  $\text{Exh}(\phi) = \{A \mid \limsup_n \phi(A \setminus n) = 0\}$  is an analytic P-ideal, and all analytic P-ideals are of this form ([16]).

If  $\mathbb{N} = \bigcup_n I_n$  is a partition into finite intervals and  $\phi_n$  is a submeasure on  $I_n$ , then

$$\mathcal{Z}_\phi = \{A \subseteq \mathbb{N} \mid \limsup_n \phi_n(A \cap I_n) = 0\}$$

is a typical *generalized density ideal* (see [3, §13]). These ideals are  $F_{\sigma\delta}$  subsets of  $\mathcal{P}(\mathbb{N})$  (when taken in its natural Cantor-set topology). Each  $\mathcal{Z}_\phi$  is a P-ideal, and it is *dense* if and only if  $\limsup_i \sup_n \phi_i(\{n\}) = 0$ .

If each  $\phi_n$  is a measure  $\nu_n$ , then  $\mathcal{Z}_\nu$  is a *density ideal*. It is an *EU-ideal* if it is dense and  $\nu_n(I_n) = 1$  for all  $n$ . This is not the original definition given in [14], but in [3, Theorem 1.13.3 (b)] the two conditions were proved to be equivalent. By [3, p. 48]  $\mathcal{Z}_0$  is an EU-ideal and a density ideal  $\mathcal{Z}_\nu$  is an EU-ideal if and only if  $\sup_n \nu_n(I_n) < \infty$ .

**Acknowledgments.** I would like to thank Juris Steprāns for many conversations, one of which resulted in Theorem 1.3, as well as for his kind permission to include his part of the proof of this theorem. I would also like to thank David Fremlin for pointing out that Lemma 1.2 does not necessarily hold for an arbitrary ultrafilter.

## 1. ULTRAPRODUCTS OF MEASURE ALGEBRAS

By  $[C]^\infty$  we denote the family of all infinite subsets of  $C$ . In this section  $C$  will always stand for an infinite subset of  $\mathbb{N}$ . For  $n \in \mathbb{N}$  let  $C/n = C \setminus (n+1)$ . A family  $\mathcal{F} \subseteq [\mathbb{N}]^\infty$  is *dense* if for every  $C \in [\mathbb{N}]^\infty$  we have  $\mathcal{F} \cap [C]^\infty \neq \emptyset$ . An ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  is *selective* if it intersects every dense analytic subset of  $[\mathbb{N}]^\infty$ . By the localized version of Silver's theorem due to Mathias ([15]) this is equivalent to the standard definition of a selective ultrafilter.

Lemma 1.2 is well-known. The use of a selective ultrafilter in the context of Loeb measure dates back to [11] and it was studied in [2].

**Lemma 1.1.** *Assume  $\mathbb{A}_n$  is a finite Boolean algebra with submeasure  $\phi_n$  and  $\mathcal{U}$  is a selective ultrafilter. On the ultraproduct  $\mathbb{A} = (\prod_n \mathbb{A}_n)/\mathcal{U}$  define*

$$\phi_{\mathcal{U}}([V]_{\mathcal{U}}) = \lim_{m \rightarrow \mathcal{U}} \phi_m(V_m).$$

*Then  $\phi_{\mathcal{U}}$  is a countably subadditive submeasure.*

*Proof.* Clearly  $\phi_{\mathcal{U}}$  is a well-defined submeasure on  $\mathbb{A}$ . We first prove  $\phi_{\mathcal{U}}$  is countably subadditive. Pick  $B^n \in \mathbb{A}$  ( $n \in \mathbb{N}$ ) so that  $\phi_{\mathcal{U}}(B^n \cap B^m) = 0$  for  $m \neq n$ . Write  $B^n = (B_i^n)/\mathcal{U}$  (where  $B_i^n \in \mathbb{A}_i$ ). The families

$$\mathcal{D} = \left\{ C \mid (\forall i \in C)(\forall j \in C/i) \left| \sum_{n < i} \nu_j(B_j^n) - \nu_j\left(\bigcup_{n < i} B_j^n\right) \right| < 2^{-2i} \right\}$$

$$\mathcal{F}_n = \left\{ C \mid (\forall i \in C) |\nu_i(B_i^n) - \nu_{\mathcal{U}}(B^n)| < \frac{2^{-i}}{n} \right\}$$

are dense in  $[\mathbb{N}]^{\infty}$ . Since  $\mathcal{U}$  is selective, we can pick  $C \in \mathcal{U} \cap \mathcal{D}$  such that  $C/n \in \mathcal{F}_n$  for all  $n$ . Let  $m_i$  ( $i \in \mathbb{N}$ ) be an increasing enumeration of  $C$ . Define  $B = [B_k]_{\mathcal{U}}$  by  $B_{m_{i+1}} = \bigcup_{n < m_i} B_{m_{i+1}}^n$  and  $B_k = \emptyset$  for  $k \notin C$ . Then  $B \supseteq B^n$  for all  $n$ , and for all pairs  $i < j$  in  $C$  we have

$$\left| \phi_j(B_j) - \phi_j\left(\bigcup_{n < i} B^n\right) \right| < \left| \phi_j(B_j) - \phi_j\left(\bigcup_{n < i} B_j^n\right) \right| + 2^{-i} < 2^{-i+1},$$

hence  $\phi_{\mathcal{U}}(B) = \lim_{i \rightarrow \mathcal{U}} \phi_i(B_i) = \lim_n \phi_{\mathcal{U}}(\bigcup_{i=1}^n (B^n))$ .  $\square$

The finiteness of algebras  $\mathbb{A}_n$  can obviously be replaced by the appropriate completeness assumption. It is not difficult to see that the algebra  $\mathbb{A}/\text{Null}(\phi_{\mathcal{U}})$  does not have to be  $\sigma$ -complete in general.

**Lemma 1.2.** *Assume  $(\mathbb{A}_n, \nu_n)$  are probability measure algebras and  $\mathcal{U}$  is a selective ultrafilter. Then  $\nu_{\mathcal{U}}$  is a countably additive probability measure and the quotient  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$  is a measure algebra.*

*Proof.* Clearly  $\nu_{\mathcal{U}}$  is a finitely additive probability measure on  $\mathbb{A}$ , so  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$  is ccc. By Lemma 1.1,  $\nu_{\mathcal{U}}$  is countably subadditive. Being in addition finitely additive, it is countably additive.

Let  $B^n$  and  $B$  be as in the proof of Lemma 1.1. In order to prove  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$  is  $\sigma$ -complete, it will suffice to check that  $B$  is the supremum of  $B_i$ . For  $A \in \prod_{i=1}^{\infty} \mathbb{A}_n$  write  $\bar{A} = [A]_{\text{Null}(\nu_{\mathcal{U}})}$ . We need to check  $\bar{B} = \bigvee_n \bar{B}^n$  in  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$ . Indeed,  $\supseteq$  is immediate since  $\bar{B} \supseteq \bar{B}^n$  for all  $n$ . To prove the reverse inclusion, note that  $\bar{D} \subseteq \bar{B}$  and  $\bar{D} \neq \bar{B}$  implies  $\nu_{\mathcal{U}}(D) < \nu_{\mathcal{U}}(B)$ . Then if  $m$  is large enough so that  $\nu_{\mathcal{U}}(\bigcup_{n < m} B^n) > \nu_{\mathcal{U}}(D)$ , we have  $\bigcup_{n < m} \bar{B}^n \setminus \bar{D} \neq 0_{\mathbb{A}}$ . Since  $D$  was arbitrary, this implies  $\bar{B} = \bigvee_n \bar{B}^n$ .

By ccc-ness, the algebra is complete and therefore it is a measure algebra.  $\square$

Let  $\mathcal{R}_{\mathfrak{c}}$  denote the homogeneous probability measure algebra of Maharam character  $\mathfrak{c}$  (see e.g., [9]). The forcing terminology used in the proof of Theorem 1.3 is standard. Neither forcing nor this theorem will be used elsewhere in this note.

**Theorem 1.3.** *If  $\mathcal{Z}_{\nu}$  is an EU-ideal, then the regular open algebras of  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_{\nu}$  and  $\mathcal{P}(\mathbb{N})/\text{Fin} * \mathcal{R}_{\mathfrak{c}}$  are isomorphic.*

First we prove a lemma. An ideal  $\mathcal{I}$  is *proper* if  $\mathbb{N} \notin \mathcal{I}$ .

**Lemma 1.4.** *If  $\mathcal{Z}_{\phi}$  is a proper generalized density ideal, then  $\mathcal{P}(\mathbb{N})/\text{Fin}$  regularly embeds into  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_{\phi}$ .*

*Proof.* The assumption that  $\mathcal{Z}_{\phi}$  is proper is equivalent to  $\limsup_n \phi_n(I_n) > 0$ . We may assume  $\liminf_n \phi_n(I_n) > 0$ , by possibly joining some of the  $I_n$ s (see [3, §13]). Let  $h$  be a function that collapses  $I_n$  to  $n$ . We claim that  $[A]_{\text{Fin}} \mapsto$

$[h^{-1}(A)]_{\mathcal{Z}_0}$  is a regular embedding. (Here  $[A]_{\mathcal{I}}$  is the  $\mathcal{I}$ -equivalence class of  $A \subseteq \mathbb{N}$ .) It is clearly a homomorphism of Boolean algebras, and since  $\liminf_n \phi_n(I_n) > 0$  it is also an embedding. Fix a maximal antichain  $\mathcal{A}$  in  $\mathcal{P}(\mathbb{N})/\text{Fin}$ . We need to prove that  $\{h^{-1}(A) \mid A \in \mathcal{A}\}$  is maximal over  $\mathcal{Z}_\phi$ . For  $C \in \mathcal{Z}_\nu^+$  there is  $\varepsilon > 0$  such that the set  $\{n \mid \phi_n(C) > \varepsilon\}$  is infinite. By the maximality of  $\mathcal{A}$ , this set has an infinite intersection with some  $A \in \mathcal{A}$ , hence  $h^{-1}(A) \cap C \notin \mathcal{Z}_\phi$ .  $\square$

*Proof of Theorem 1.3.* We find a regular embedding of  $\mathcal{P}(\mathbb{N})/\text{Fin}$  into  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$  such that  $\mathcal{P}(\mathbb{N})/\text{Fin}$  forces that the quotient is an atomless measure algebra. The character of this algebra is not bigger than its size,  $\mathfrak{c}$ . This suffices since  $\mathcal{R}_\mathfrak{c}$  regularly embeds into  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$  by [10, Proposition 491P]. Let  $h: \mathbb{N} \rightarrow \mathbb{N}$  be a function that collapses  $I_n$  to  $n$ . By Lemma 1.4, the mapping  $A \mapsto h^{-1}(A)$  is a regular embedding of  $\mathcal{P}(\mathbb{N})/\text{Fin}$  into  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$ . Let  $G$  be the canonical name for some  $\mathcal{P}(\mathbb{N})/\text{Fin}$ -generic ultrafilter. Recall that  $\mathcal{P}(\mathbb{N})/\text{Fin}$  adds no reals and forces that  $G$  is selective ([15]).

It remains to check that  $\mathcal{P}(\mathbb{N})/\text{Fin}$  forces  $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu)/G$  is isomorphic to  $\mathcal{R}_\mathfrak{c}$ . We will be using the terminology of Lemma 1.4. First prove that  $\mathcal{P}(\mathbb{N})/\text{Fin}$  forces  $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu)/G$  and  $(\prod_n \mathcal{P}(I_n)/G)/\text{Null}(\nu_G)$  are isomorphic. Pick subsets  $B$  and  $C$  of  $\mathbb{N}$ . Identifying  $\mathcal{P}(\mathbb{N})$  with  $\prod_n \mathcal{P}(I_n)$ , and write  $B_n = B \cap I_n$  and  $C_n = C \cap I_n$ . Then  $B/G = C/G$  if and only if  $\lim_{n \rightarrow G} \nu_n(B_n \Delta C_n) = 0$  if and only if  $\nu_G([B]_G \Delta [C]_G) = 0$ . Since  $G$  is forced to be a selective ultrafilter, by Lemma 1.4, the conclusion follows.  $\square$

Under CH it is even true that all quotients over EU-ideals are pairwise isomorphic ([14], [5]). However, under Todorćević's OCA there are many pairwise nonisomorphic quotients over the ideals in this class (this was first proved by Just [13]; see also [3] and [7]).

## 2. GAPS OVER DENSITY IDEALS

In this section we prove a structure result on analytic Hausdorff gaps over density ideals. A pregap  $\mathcal{A}, \mathcal{B}$  in the quotient over  $\mathcal{Z}_\phi$  (given by  $I_n, \phi_n$  for  $n \in \mathbb{N}$ ) is *simple* if there are submeasures  $\sigma_m, \tau_m$  concentrating on  $I_m$  such that  $\mathcal{A} = \mathcal{Z}_\sigma$  and  $\mathcal{B} = \mathcal{Z}_\tau$ . If  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{A}', \mathcal{B}'$  are pregaps in the same algebra we say that  $\mathcal{A}, \mathcal{B}$  is *included* in  $\mathcal{A}', \mathcal{B}'$  if  $\mathcal{A} \subseteq \mathcal{A}'$  and  $\mathcal{B} \subseteq \mathcal{B}'$ . Fix a generalized density ideal  $\mathcal{Z}_\phi$  with witnesses  $\phi_n$  and  $I_n, n \in \mathbb{N}$ , throughout this section.

**Theorem 2.1.** *Every analytic Hausdorff pregap in the quotient over any  $\mathcal{Z}_\phi$  is included in a simple pregap.*

*Proof.* Both  $\mathcal{A}$  and  $\mathcal{B}$  are analytic P-ideals. By [16]  $\mathcal{A} = \text{Exh}(\sigma)$  and  $\mathcal{B} = \text{Exh}(\tau)$  for some lower semicontinuous submeasures  $\sigma$  and  $\tau$ . Define  $\sigma_m$  and  $\tau_m$  by

$$\begin{aligned}\sigma_m(C) &= \sigma(C \cap I_m) \\ \tau_m(C) &= \tau(C \cap I_m),\end{aligned}$$

and let  $\mathcal{A}' = \mathcal{Z}_\sigma, \mathcal{B}' = \mathcal{Z}_\tau$ .

**Claim 2.2.** *We have  $\mathcal{A}' \supseteq \mathcal{A}$  and  $\mathcal{B}' \supseteq \mathcal{B}$ .*

*Proof.* For  $\mathcal{A}' \supseteq \mathcal{A}$  it suffices to prove that  $\sup_m \sigma_m \leq \sigma$ . But this follows from  $\sigma_m \leq \sigma$  for all  $m$ . The proof that  $\mathcal{B}' \supseteq \mathcal{B}$  is analogous.  $\square$

**Claim 2.3.** *The families  $\mathcal{A}'$  and  $\mathcal{B}'$  are orthogonal over  $\mathcal{Z}_\phi$ .*

*Proof.* We need to check that  $\mathcal{A}' \cap \mathcal{B}' \subseteq \mathcal{Z}_\phi$ . Assume this fails, and fix  $X \in (\mathcal{A}' \cap \mathcal{B}') \setminus \mathcal{Z}_\phi$ . Let  $X_m = X \cap I_m$ . Since  $X \notin \mathcal{Z}_\phi$ , there is an  $\varepsilon > 0$  such that  $\phi_m(X_m) \geq \varepsilon$  for infinitely many  $m$ . We may assume this holds for all  $m$ . If we write

$$I_C = \bigcup_{m \in C} I_m, \quad X_C = X \cap I_C,$$

then for every infinite  $C \subseteq \mathbb{N}$  we have  $X_C \notin \mathcal{Z}_\phi$ . Since  $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{Z}_\phi$ , we have  $X_C \notin \mathcal{A} \cap \mathcal{B}$  for every such  $C$ . We may find an infinite  $C_0$  and  $Q \in \{\mathcal{A}, \mathcal{B}\}$  such that  $\{D \in [C_0]^{\aleph_0} : X_D \notin Q\}$  is dense in  $[C_0]^{\aleph_0}$  (dense in the forcing sense—every set has an infinite subset in this set). We may assume  $Q = \mathcal{A}$  and (since  $\mathcal{A}$  is hereditary) that  $X \cap I_D \notin \mathcal{A}$  for all  $D \in [C_0]^{\aleph_0}$ . For  $D \subseteq \mathbb{N}$  let

$$\alpha_D = \liminf_k \sigma(X_D \setminus k)$$

and note that  $\alpha_D > 0$  since  $X_D \notin \mathcal{A}$ . Since  $C \subseteq^* D$  implies  $\alpha_C \leq \alpha_D$  and  $([\mathbb{N}]^{\aleph_0}, \supseteq^*)$  is countably directed, for some  $C_1 \in [C_0]^{\aleph_0}$  we have  $\alpha_D = \alpha_{C_1} = \delta$  for all  $D \in [C_1]^{\aleph_0}$ . By the above  $\delta > 0$ .

But  $\sigma_m(X) = \sigma(X_m) \rightarrow 0$  as  $m \rightarrow \infty$ , so we can find  $C_2 \subseteq C_1$  such that  $\sum_{m \in C_2} \sigma(X_m) < \delta/2$ . Then  $\sigma(X_{C_2}) < \delta/2$ , a contradiction.  $\square$

By the above claims,  $\mathcal{A}'$  and  $\mathcal{B}'$  form a simple pregap that includes  $\mathcal{A}, \mathcal{B}$ . Clearly, if  $\mathcal{A}, \mathcal{B}$  is a gap then  $\mathcal{A}', \mathcal{B}'$  is a gap as well.  $\square$

By the following result, analytic Hausdorff gaps over EU-ideals (if they exist) must be rather different from known analytic Hausdorff gaps (see the proof of [4, Lemma 2]).

**Theorem 2.4.** *Assume  $\mathcal{Z}_\nu$  is an EU-ideal and  $\mathcal{A}, \mathcal{B}$  is an analytic Hausdorff pregap in its quotient. Then every infinite  $Y \subseteq \mathbb{N}$  has an infinite subset  $X$  such that  $\mathcal{A}, \mathcal{B}$  is separated on  $\bigcup_{n \in X} I_n$ .*

*Proof.* Assume  $\mathcal{A}, \mathcal{B}$  is an analytic Hausdorff gap over  $\mathcal{Z}_\nu$ . By Theorem 2.1 we may assume  $\mathcal{A}, \mathcal{B}$  is a simple gap given by submeasures  $\sigma_m, \tau_m$  ( $m \in \mathbb{N}$ ). Since  $\mathcal{P}(\mathbb{N})/\text{Fin}$  adds a selective ultrafilter without adding reals, and therefore without splitting gaps, we may assume there exists a selective ultrafilter  $\mathcal{U}$  concentrating on  $Y$ . Let

$$\mathbb{A} = \left( \prod_n \mathcal{P}(I_n) \right) / \mathcal{U}$$

and define  $\nu_{\mathcal{U}}$  on  $\mathbb{A}$  as in Lemma 1.2. Identify  $D \subseteq \mathbb{N}$  with the element  $\langle D \cap I_n : n \in \mathbb{N} \rangle$  of  $\prod_n \mathcal{P}(I_n)$ , and let

$$\begin{aligned} \mathcal{A}_{\mathcal{U}} &= \{[A]_{\mathcal{U}} : A \in \mathcal{A}\} \\ \mathcal{B}_{\mathcal{U}} &= \{[B]_{\mathcal{U}} : B \in \mathcal{B}\}. \end{aligned}$$

These two families form a pregap in  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$ . By Lemma 1.2, the algebra  $\mathbb{A}/\text{Null}(\nu_{\mathcal{U}})$  is a measure algebra and therefore some  $[W]_{\mathcal{U}}$  splits the pregap. Let  $W_n = W \cap I_n$  and for each  $k$  define

$$X_k = \{n \mid \sigma_n(I_n \setminus W_n) \leq 1/k \text{ and } \tau_n(W_n) \leq 1/k\}.$$

Then  $X_k \in \mathcal{U}$  for all  $k$ , and since  $\mathcal{U}$  is selective we can find  $X \in \mathcal{U}$  such that  $X \setminus X_k$  is finite for all  $k$ . Then  $X$  is clearly as required.  $\square$

## 3. ON QUOTIENTS WITHOUT ANALYTIC HAUSDORFF GAPS

We prove that if an analytic P-ideal  $\mathcal{I}$  is dense and does not have analytic Hausdorff gaps in its quotient, then its restriction to some positive set is a generalized density ideal. This improves the main result of [4] since a dense generalized density ideal cannot be  $F_\sigma$ . To this effect we prove a slight strengthening of [4, Lemma 2].

**Lemma 3.1.** *Assume  $\mathcal{I} = \text{Exh}(\phi)$  is a dense analytic P-ideal and  $I_i$  ( $i \in \mathbb{N}$ ) are finite pairwise disjoint sets such that for some  $\varepsilon > 0$  and  $a > 0$  we have*

$$(1) \quad (\forall n)(\forall S \subseteq \bigcup_{i=n}^{\infty} I_i)(\forall i \geq n)\phi(I_i \setminus S) < \varepsilon \\ \Rightarrow (\exists B \subseteq S)(\forall i)\phi(B \cap I_i) < 1/n \wedge \phi(B) > a).$$

*Then there is an analytic Hausdorff gap over  $\mathcal{I}$ .*

*Proof.* By replacing  $\phi$  with  $\phi/a$  we may assume  $a = 1$ . Recursively find an increasing sequence  $n_k$  ( $k \in \mathbb{N}$ ) so that for every  $k$  we have (let  $J_k = [n_k, n_{k+1})$ )

$$(2) \quad (\forall S \subseteq \bigcup_{i \in J_k} I_i)(\forall i \in J_k)\phi(I_i \setminus S) < \varepsilon \\ \Rightarrow (\exists B \subseteq S)(\forall i)(\phi(B \cap I_i) < 1/k^2) \wedge \phi(B) > 1)$$

If  $n_1, \dots, n_k$  are as required, let  $T$  be the family of all pairs  $(S, p)$  so that  $p > n_k$ ,  $S \subseteq \bigcup_{i=n_k}^p I_i$ ,  $\phi(I_i \setminus S) < \varepsilon$  for all  $i \in [n_k, p]$ , but for every  $B \subseteq S$  such that  $(\forall i)\phi(B \cap I_i) < 1/k^2$  we have  $\phi(B) \leq 1$ . Order  $T$  by  $(S, p) \preceq (U, l)$  if and only if  $p \leq l$  and  $U \cap I_i = S \cap I_i$  for all  $i \leq p$ . Then  $T$  is a finitely branching tree.

An infinite branch of  $T$  would give some  $S$  contradicting the assumption (2), since  $\phi$  is lower semicontinuous. By König's lemma,  $n_{k+1} = \sup\{p + 2 \mid (\exists (S, p) \in T)\}$  is finite and satisfies (2). From this point we follow the proof of [4, Lemma 2] rather closely.

For  $A \subseteq \mathbb{N}$  and  $n \in \mathbb{N}$  define submeasures  $\alpha_n(A) = |\{j \in J_n : A \cap I_j \neq \emptyset\}|$  and  $\beta_n(A) = \sup_{j \in J_n} \phi(A \cap I_j)$ , then let

$$\alpha(A) = \sum_{n=1}^{\infty} \frac{\alpha_n(A)}{n} \quad \text{and} \quad \beta(A) = \sup_{n \in \mathbb{N}} n \cdot \beta_n(A)$$

Both  $\alpha$  and  $\beta$  are lower semicontinuous. We will prove that  $\mathcal{A} = \text{Exh}(\alpha)$  and  $\mathcal{B} = \text{Exh}(\beta)$  form an analytic Hausdorff gap. Since both are clearly analytic P-ideals, we need only prove that  $\mathcal{A}$  and  $\mathcal{B}$  are  $\text{Exh}(\phi)$ -orthogonal and that they are not separated by a single set over  $\text{Exh}(\phi)$ .

In order to prove  $\mathcal{A}$  and  $\mathcal{B}$  are  $\text{Exh}(\phi)$ -orthogonal note that for  $A, B \subseteq \mathbb{N}$  we have

$$\phi(A \cap B) \leq \sum_{n=1}^{\infty} \phi(A \cap B \cap \bigcup_{i \in J_n} I_i) \leq \sum_{n=1}^{\infty} \frac{\alpha_n(A)}{n} \cdot (n \cdot \beta_n(B)) \leq \alpha(A) \cdot \beta(B).$$

If  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ , then  $\alpha(A) < \infty$  and  $\lim_{m \rightarrow \infty} \beta(B \setminus \bigcup_{n=1}^m J_n) = 0$ , thus by the above  $\lim_{l \rightarrow \infty} \phi((A \cap B) \setminus [1, l]) = 0$ , and  $A \cap B \in \text{Exh}(\phi)$ , as required.

Assume  $\mathcal{A}$  and  $\mathcal{B}$  are separated over  $\text{Exh}(\phi)$  by  $C \subseteq \mathbb{N}$ . Then  $A \setminus C \in \text{Exh}(\phi)$  and  $B \cap C \in \text{Exh}(\phi)$  for all  $A \in \mathcal{A}$  and all  $B \in \mathcal{B}$ . We claim that

$$(3) \quad \lim_{n \rightarrow \infty} \sup_{m \geq n, j \in J_m} \phi(I_j \setminus C) = 0.$$

Otherwise, we may find an infinite  $X \subseteq \mathbb{N}$ ,  $\varepsilon > 0$ , and a 'choice function'  $f \in \prod_{n \in X} J_n$  such that

$$\phi(I_{f(n)} \setminus C) > \varepsilon$$

for all  $n \in X$ . We may furthermore shrink  $X$  so that  $\sum_{n \in X} 1/n < \infty$ . Let  $A = \bigcup_{n \in X} I_{f(n)} \setminus C$ ; then  $\alpha(A) \leq \sum_{n \in X} 1/n < \infty$ , thus  $A \in \mathcal{A}$ . Note that  $A \cap C = \emptyset$ . However, for  $n \in X$  we have  $\phi(A \cap \bigcup_{i \in J_n} I_i) \geq \phi(A \cap I_{f(n)}) \geq \varepsilon$ , therefore  $A \notin \text{Exh}(\phi)$ , contradicting the assumption on  $C$ .

By (3) for all but finitely many  $n$  we have  $\sup_{j \in J_n} \phi(I_j \setminus C) < \varepsilon$ . By (2), for each such  $n$  there is  $B_n \subseteq C \cap \bigcup_{i \in J_n} I_i$  such that  $\phi(B_n \cap I_i) < 1/n^2$  and  $\phi(B_n) \geq 1$ . Then  $B = \bigcup_{n \in Y} B_n$  satisfies  $B \subseteq C$  and  $n \cdot \beta_n(B) \leq 1/n$ . Therefore  $B \in \mathcal{B}$ , yet  $B \notin \text{Exh}(\phi)$ , a contradiction. This completes the proof of the lemma.  $\square$

**Proposition 3.2.** *If  $\mathcal{I}$  is an analytic  $P$ -ideal whose quotient does not have analytic Hausdorff gaps, then the restriction of  $\mathcal{I}$  to some positive set is a generalized density ideal.*

*Proof.* By [16] fix a lower semicontinuous submeasure  $\phi$  such that  $\mathcal{I} = \text{Exh}(\phi)$ . Fix a partition of  $\mathbb{N}$  into intervals  $I_i$  ( $i \in \mathbb{N}$ ) so that  $\inf_i \phi(I_i) \geq 1$ . The conditions of Lemma 3.1 fail when  $a = \varepsilon = 1/m$  for every  $m \in \mathbb{N}$ . Hence we may assume that for every  $m \in \mathbb{N}$  there are  $n = f(m)$  and  $S \subseteq \bigcup_{i=f(m)}^\infty I_i$  such that  $(\forall i \geq f(m)) \phi(I_i \setminus S) < 1/f(m)$  and if  $B \subseteq S$  is such that  $\phi(B \cap I_i) < 1/f(m)$  for all  $i$ , then  $\phi(B) \leq 1/m$ . Fix  $\delta > 0$  so that  $\delta < \inf_i \phi(I_i)$ . We may assume  $f(m) \geq m/\delta$  for all  $m$ . For  $k \in \mathbb{N}$  pick  $S_k \subseteq \bigcup_{i=f(2^k)}^\infty I_i$  so that  $\phi(I_i \setminus S_k) < 2^{-k}\delta$  for all  $i \geq f(2^k)$  and

$$(\forall B \subseteq S_k)(\forall i) \phi(B \cap I_i) < 1/f(2^k) \Rightarrow \phi(B) < 2^{-k}.$$

Let  $S'_k = S_k \cup \bigcup_{i=1}^{f(2^k)-1} I_i$  and  $S = \bigcap_{k=1}^\infty S'_k$ . Then  $\phi(I_i \setminus S) < \delta$  for all  $i$ , therefore  $S$  is  $\mathcal{I}$ -positive.

We claim that  $\{A \subseteq S \mid A \in \mathcal{I}\} = \{A \subseteq S \mid \limsup_i \phi(A \cap I_i) = 0\}$ , and therefore the restriction of  $\mathcal{I}$  to  $S$  is a generalized density ideal.

It will suffice to prove that if  $\phi(A \cap I_i)$  approaches zero then  $A \in \mathcal{I}$ . Fix  $m \in \mathbb{N}$ . Find  $k$  such that  $\phi(A \cap I_i) < 1/f(2^m)$  for all  $i \geq k$ . Then  $\phi(A \setminus \bigcup_{i=1}^k I_i) < 2^{-m}$ , and therefore  $A \in \text{Exh}(\phi)$ .  $\square$

#### 4. CONCLUDING REMARKS

The question whether there are analytic Hausdorff gaps over  $\mathcal{Z}_0$  remains open. We record two of its equivalent reformulations. For terminology see [3].

**Proposition 1.** *Let  $\mathcal{I}$  be an analytic ideal. The following are equivalent.*

- (a) *There are analytic Hausdorff gaps over  $\mathcal{I}$ .*
- (b) *Every Baire monomorphism of the quotient over  $\mathcal{I}$  into an analytic quotient preserves all Hausdorff gaps.*
- (c) *Assuming OCA and MA, every monomorphism of the quotient over  $\mathcal{I}$  into an analytic quotient preserves all Hausdorff gaps.*

*Proof.* Each one of (b) and (c) is equivalent to (a) by [3, Proposition 5.9.1 and Proposition 5.9.4]. These equivalences are also implicit in [20].  $\square$

Let us repeat [4, Question 9] (see [4, Lemma 2] for a partial answer).

**Question 4.1.** *Assume a dense analytic  $P$ -ideal is equal to  $\text{Exh}(\phi)$  for a lower semicontinuous submeasure satisfying  $\phi(\mathbb{N}) = \infty$ . Is there an analytic Hausdorff gap in its quotient?*

A simple argument using the ideas from [5, Proposition 3.3 (1) and (2)] shows that if  $\mathcal{Z}_0 = \text{Exh}(\phi)$  for a lower semicontinuous  $\phi$  then  $\phi(\mathbb{N}) < \infty$ .

Theorem 1.3 implies that  $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$  is a proper forcing notion. The question of properness of quotients  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  as forcing notions, initiated by Balcar, has recently attracted considerable attention. Balcar, Hernandez Hernandez and Hrušák ([1]) proved that  $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$  is proper and adds only Cohen reals. (Here  $\text{NWD}(\mathbb{Q})$  stands for the  $F_{\sigma\delta}$  ideal of all nowhere dense subsets of the rationals.) Motivated by [5], Steprāns ([17]) has defined a family of  $2^{\aleph_0}$  coanalytic ideals whose quotients are pairwise nonequivalent proper forcing notions, each one being an iteration of a Sacks-like forcing and  $\mathcal{P}(\mathbb{N})/\text{Fin}$ . Hrušák and Zapletal ([12]) proved theorems relating forcings  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  with more familiar forcings of the form  $\text{Borel}/J$  for a  $\sigma$ -ideal  $J$ . They have also constructed an analytic P-ideal  $\mathcal{I}$  such that the forcing  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  is not proper, answering a question from an earlier version of this paper.

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