VON NEUMANN'S PROBLEM AND LARGE CARDINALS

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Abstract. It is a well known problem of Von Neumann whether the countable chain condition and weak distributivity of a complete Boolean algebra imply that it carries a strictly positive probability measure. It was shown recently by Balcar–Jech–Pazák and Velickovic that it is consistent with ZFC, modulo the consistency of a supercompact cardinal, that every ccc weakly distributive complete Boolean algebra carries a contiuous strictly positive submeasure, i.e., is a Maharam algebra. We use some ideas of Gitik and Shelah and implications from the inner model theory to show that some large cardinal assumptions are necessary for this result.

In 1937 von Neumann asked whether every ccc weakly distributive complete Boolean algebra is a measure algebra ([11]). We show that a positive answer to von Neumann's problem, if consistent at all, requires a large cardinal assumption. A complete Boolean algebra is a *Maharam algebra* if it carries a strictly positive continuous submeasure ([14]). Every measure algebra is a Maharam algebra, and every Maharam algebra has ccc and is weakly distributive (see e.g., [3]).

Theorem 1. *Assume every ccc weakly distributive complete Boolean algebra is a Maharam algebra. Then there is an inner model with a measurable cardinal κ such that $o(\kappa) = \kappa^{++}$.*

By results of [14] and [1], a consequence of the Proper Forcing Axiom implies that every ccc, weakly distributive, complete Boolean algebra is a Maharam algebra. Our result gives a lower bound for the consistency strength of this statement. The remaining part of von Neumann's problem, whether every Maharam algebra is a measure algebra, is known under the names of Maharam's Problem and Control Measure Problem (see [10], [8], [2, §393]).

Besides the known results (Lemma 4 below), the main component of Theorem 1 and Theorem 3 below is the following.

Theorem 2. *Assume there is a cardinal κ such that $\kappa^{\aleph_0} = \kappa$, $2^\kappa = \kappa^+$ and $\square_\kappa$ holds. Then there is a complete Boolean algebra $\mathcal{B}$ of size κ^+ such that $\mathcal{B}$ is not a Maharam algebra but every subalgebra of size $\leq \kappa$ is a measure algebra.*

In the model of [1] and [14] every weakly distributive complete Boolean algebra such that every completely countably generated subalgebra is a measure algebra must be a measure algebra. Together with the following result this completely answers a question of David Fremlin ([4, Problem AU(d)]).

Theorem 3. *Assume every weakly distributive complete Boolean algebra $\mathcal{B}$ such that every completely countably generated subalgebra is a measure algebra and $\mathcal{B}$ has*

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property K is a Maharam algebra. Then there is an inner model with a measurable cardinal κ such that $o(\kappa) = \kappa^{++}$.

Terminology. A subset of a Boolean algebra is an *antichain* if it consists of nonzero elements but the meet of any two of its members is zero and it is *linked* if the meet of any two of its elements is nonzero. A Boolean algebra is *ccc* if it does not have uncountable antichains. It has *property K* if every uncountable subset includes an uncountable linked subset. A complete Boolean algebra is *weakly distributive* if for every sequence $\mathcal{A}_n$ ($n \in \mathbb{N}$) of maximal antichains there is a maximal antichain $\mathcal{A}$ such that for every $a \in \mathcal{A}$ and $n \in \mathbb{N}$ the set $\{b \in \mathcal{A}_n : b \wedge a \neq 0\}$ is finite. A complete Boolean algebra is a *measure algebra* if it carries a σ-additive measure $\mu\colon \mathcal{B} \to [0,1]$ that is *strictly positive*: $\mu(a) = 0$ implies $a = 0$. A $\phi\colon \mathcal{B} \to [0,1]$ is a *submeasure* if $\phi(0) = 0$ and it is monotonic and subadditive ($\phi(a \cup b) \leq \phi(a) + \phi(b)$). It is *continuous* if $\phi(\wedge_n a_n) = \inf_n \phi(a_n)$ whenever $\{a_n\}$ is a decreasing sequence in $\mathcal{B}$.

Recall that $\Box_\kappa$ is the statement that there exists a family C_α ($\alpha < \kappa^+$) such that (i) C_α is a closed and unbounded subset of α, (ii) $\mathrm{otp}(C_\alpha) \leq \kappa$ for all α and (iii) if $\beta \in C_\alpha$ is a limit point then $C_\beta = C_\alpha \cap \beta$. A cardinal κ is *strong limit* if $2^\lambda < \kappa$ for all $\lambda < \kappa$. Recall that $o(\kappa)$ denotes the Mitchell order of a measurable cardinal κ; see [9].

1. Proofs from Theorem 2

Lemma 4. *Assume there is no inner model with a measurable cardinal κ of Mitchell order $o(\kappa) \geq \kappa^{++}$. Then the assumptions of Theorem 2 hold for every singular strong limit cardinal λ of uncountable cofinality.*

Proof. If $\Box_\lambda$ fails for a strong limit singular λ, then the Axiom of Determinacy, AD, holds in $L(\mathbb{R})$ ([13]), hence there are inner models with Woodin cardinals. If the Singular Cardinal Hypothesis, SCH, fails at κ, then there is an inner model with a measurable of Mitchell order $o(\kappa) = \kappa^{++}$ ([5]). □

The lower bound given in Theorem 1 and Theorem 3 is not optimal; for example, by [5] one can improve it to 'for every ordinal α there exists an inner model with κ such that $o(\kappa) \geq \kappa^{++} + \alpha$.' On the other hand, starting from κ such that $o(\kappa) = \kappa^{+++}$, Merimovich forced a model in which SCH fails everywhere ([12]).

Proof of Theorem 1 from Theorem 2. Immediate from Lemma 4. □

Proof of Theorem 3 from Theorem 2. If the large cardinal assumption fails, then by Lemma 4 the assumptions of Theorem 2 are satisfied. Let $\mathcal{B}$ be a Boolean algebra as in Theorem 2. We first check that $\mathcal{B}$ has the property K. Every subset $\mathcal{A}$ of $\mathcal{B}$ of size $\aleph_1$ is contained in a subalgebra of size $\aleph_1$. This subalgebra is a measure algebra, and it therefore has the property K. Therefore $\mathcal{A}$ has a linked subset of size $\aleph_1$. Now we check that $\mathcal{B}$ is weakly distributive. Since $\mathcal{B}$ has the countable chain condition, every countable family of maximal antichains is contained in a countably completely generated subalgebra; it is a measure algebra. The weak distributivity of $\mathcal{B}$ follows. □

2. Proof of Theorem 2

Every Maharam algebra that is not a measure algebra contains a countably completely generated subalgebra that is not a measure algebra ([2, p. 584]). It will therefore suffice to prove that $\mathcal{B}$ is not a measure algebra.

Recall that if $S \subseteq \kappa^+$ is stationary then $\diamondsuit^*(S)$ is the statement: There are sets $\mathcal{A}_\xi \subseteq \mathcal{P}(\xi)$ ($\xi \in S$) such that $|\mathcal{A}_\xi| \leq \kappa$ for all ξ and for every $X \subseteq \kappa^+$ the set $\{\xi < \kappa^+ : X \cap \xi \in \mathcal{A}_\xi\}$ contains a club.

Lemma 5. *Assume $\kappa^{\aleph_0} = \kappa$ and $2^\kappa = \kappa^+$. Then*

1. *$\diamondsuit^*(\operatorname{cof}\omega)$ holds at κ^+.*
2. *$\diamondsuit(S)$ holds at every $S \subseteq \kappa^+$ stationary in cofinality ω.*

Proof. Clause (1) is a special case of [7, Lemma 2.1(1)]. Clause (2) follows from (1) by a standard Kunen-type argument. Fix $S \subseteq \kappa^+$ stationary in cofinality ω and let $\mathcal{A}_\alpha$ be a sequence as in (1); we may assume that $\mathcal{A}_\alpha$ captures subsets of $\kappa^+ \times \kappa$, so that $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha \times \kappa)$ and for $X \subseteq \kappa^+ \times \kappa$ we have $X \cap (\alpha \times \kappa) \in \mathcal{A}_\alpha$ for club many $\alpha < \kappa^+$. Let $\mathcal{A}_\alpha = \{C^\eta_\alpha : \eta < \kappa\}$. If there is $\eta < \kappa$ such that $B^\eta_\alpha = \{\xi : (\xi, \eta) \in C^\eta_\alpha\}$ ($\alpha \in S$) is a $\diamondsuit$-sequence, then we are done. Let us assume there is no such η. Then for each $\eta < \kappa$ there is $X^\eta \subseteq \kappa^+$ such that for each $\alpha \in S$ we have $X^\eta \cap \alpha \neq B^\eta_\alpha$. Let $X = \{(\xi, \eta) : \xi \in X^\eta \, \eta < \kappa\}$. Since $X \cap (\alpha \times \kappa) \in \mathcal{A}_\alpha$ for club many α, we can find $\alpha \in S$ such that $X \cap (\alpha \times \kappa) \in \mathcal{A}_\alpha$. If η is such that $X \cap (\alpha \times \kappa) = C^\eta_\alpha$, then $X_\eta = B^\eta_\alpha$, contradicting our choice. □

A stationary subset S of λ is *nonreflecting* if for every limit $\alpha < \lambda$ there is a club $C \subseteq \lambda$ such that $C \cap S = \emptyset$.

Lemma 6. *Assume $\square_\kappa$ holds. Then there is a $\square_\kappa$ sequence D_α ($\alpha < \kappa^+$) and a nonreflecting stationary $S \subseteq \kappa$ in cofinality ω such that $D_\alpha \cap S = \emptyset$ for all α.*

Proof. Fix a $\square_\kappa$-sequence C_ξ ($\xi < \kappa^+$). Let $S_0 = \{\xi < \kappa^+ : \operatorname{cof}(\xi) = \omega\}$, and let $f \colon S_0 \to \kappa$ be $f(\xi) = \operatorname{otp}(C_\xi)$. Thus $f(C_\xi) \leq \kappa$ for all ξ and there is a stationary $S \subseteq S_0$ such that f is constant on S. Let γ be the constant value of f on S. In order to see that S is nonreflecting, note that if $\xi < \kappa^+$, then $\operatorname{otp}(C_\xi \cap \alpha) = \gamma$ for at most one $\alpha \in C_\xi$, hence an end-segment of C_ξ is a club in ξ disjoint from S.

Now modify each C_ξ as follows. If $\operatorname{otp}(C_\xi) > \gamma$ then remove its first $\gamma + 1$ elements. If $\operatorname{otp}(C_\xi) \leq \gamma$ then leave C_ξ unchanged. We need to check that thus obtained D_ξ ($\xi < \kappa^+$) is a $\square_\kappa$-sequence. Fix α and a limit $\beta \in D_\alpha$. Since $\beta \in D_\alpha$, $\operatorname{otp}(C_\beta) = \operatorname{otp}(C_\alpha \cap \beta) > \gamma$. If $\eta \in C_\beta$ is such that $\operatorname{otp}(C_\beta \cap \eta) = \gamma + 1$, then $D_\beta = C_\beta \setminus \eta$ and $D_\alpha = C_\alpha \setminus \eta$, hence $D_\beta = D_\alpha \cap \beta$. □

Proof of Theorem 2. The argument follows a construction from [6, §2]. We recursively construct Boolean algebras $\mathcal{B}_\alpha$ ($\alpha < \kappa^+$) so that $\mathcal{B}_\alpha$ is a complete subalgebra of $\mathcal{B}_\beta$ whenever $\alpha < \beta$. Each $\mathcal{B}_\alpha$ ($\alpha < \kappa^+$) will be a homogeneous probability measure algebra of Maharam character $|\alpha|$.

These algebras will be represented as follows. Let $\mathcal{A}_\alpha$ be the *Baire algebra*: namely the σ-algebra generated by clopen sets in $\{0,1\}^\alpha$. Then μ_α will be a measure on $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ will be the quotient $\mathcal{A}_\alpha / \operatorname{Null}(\mu_\alpha)$. Via identifying elements of $\{0,1\}^\alpha$ with eventually zero sequences in $\{0,1\}^{\kappa^+}$ we will identify $\mathcal{A}_\alpha$ with a subalgebra of $\mathcal{A}_{\kappa^+}$. We will prove that for all $\beta < \alpha$

(1) $\operatorname{Null}(\mu_\alpha) \cap \mathcal{A}_\beta = \operatorname{Null}(\mu_\beta)$.

Condition (1) will assure that $\mathcal{B}_\alpha$ is a (complete) subalgebra of $\mathcal{B}_\beta$ for $\alpha < \beta$. For $\alpha < \beta$ we also let $\mu_{[\alpha,\beta)}$ be the restriction of μ_β to $\{0,1\}^{[\alpha,\beta)}$ (identified with the subalgebra of $\mathcal{A}_\beta$ consisting of all functions whose restriction to $\{0,1\}^\alpha$ is identically 0) and $\mathcal{B}_{[\alpha,\beta)}$ be the corresponding measure algebra.

By Lemma 6 there is a nonreflecting stationary set $S \subseteq \kappa^+$ in cofinality ω and a $\square_\kappa$-sequence $\{C_\alpha\}$ such that $C_\alpha \cap S = \emptyset$ for all α. By Lemma 5 we can fix a $\diamondsuit(S)$-sequence D_α $(\alpha \in S)$. Since $\kappa^{\mathfrak{c}} = \kappa$, by a standard coding argument we can assume that for every measure ν on $\mathcal{A}_{\kappa^+}$ there are stationary many $\alpha < \kappa^+$ such that D_α codes $\nu \restriction \bigcup_{\xi<\alpha} \mathcal{A}_\xi$.

Now we describe the construction. Assume $\mathcal{B}_\xi$ and μ_ξ $(\xi < \alpha)$ have been defined. If $\alpha = \beta + 1$ for some β, let μ_α be the product of μ_β and the uniform probability measure on $\{0,1\}$ at the βth coordinate. This proves that each $\mathcal{B}_\alpha$ that is a measure algebra will be a homogeneous measure algebra.

At limit stages α of our recursive construction we shall do as follows. We shall require

(2) If $\alpha \notin S$ is a limit ordinal, then μ_α is the product measure of $\mu_{[d_\xi,d_{\xi+1})}$ $(\xi < \delta)$ for $C_\alpha = \{d_\xi : \xi < \delta\}$.

(3) If $\alpha \in S$ then μ_α is the product measure of ν_n $(n \in \mathbb{N})$ for some sequence $d_n \to \alpha$ $(d_0 = 0)$ disjoint from S and measures ν_n on $\mathcal{A}_{[d_n,d_{n+1})}$ such that $\mathrm{Null}(\nu_n) = \mathrm{Null}(\mu_{[d_n,d_{n+1})})$ for all n.

Before describing the construction, let us verify that μ_α defined in this way satisfies (1) for every $\beta < \alpha$. This is not immediate since the values that different measures give to the same set can be different. The proof proceeds by induction on α.

Assume $\alpha \notin S$. Fix $\gamma < \alpha$ such that (1) holds for all $\beta < \gamma$. Let $\beta = \min(C_\alpha \setminus \gamma)$. It suffices to prove (1) for the pair α, β. If $\beta = \min C_\alpha$ then μ_β is a factor of μ_α, and (1) follows. If β is a limit in C_α, then by the definition of measures and the coherence of $\square$-sequence μ_β is a factor of μ_α, and (1) follows. Now assume β is a successor in C_α. Let $\eta = \sup(C_\alpha \cap \beta)$. Then $\mu_\eta \times \mu_{[\eta,\beta)}$ is a factor of μ_α and the conclusion follows since (1) holds for β, η.

Now assume $\alpha \in S$. Let $\beta < \alpha$ and d_n, ν_n $(n \in \mathbb{N})$ be sequences used in the definition of μ_α. By induction, the null ideal of $\prod_{i=1}^{n+1} \nu_i$ coincides with the null ideal of $\mu_{d_{n+1}}$ for all n, even though the values of these measures may differ.

If $\bar{n}$ is the minimal such that $\beta < d_{\bar{n}}$, then (1) for μ_β and $\mu_{d_{\bar{n}}}$ holds by the inductive hypothesis, and the conclusion follows.

We proceed to define μ_α $(\alpha < \kappa^+)$. The successor case was defined above, and the case $\alpha \notin S$ is determined by (2). It remains to define μ_α and $\mathcal{B}_\alpha$ in the case when $\alpha \in S$. Let $\mathcal{A}_\alpha^-$ be the algebra of all sets in $\mathcal{A}_\alpha$ with supports bounded in α. Consider the following query regarding the element D_α of the diamond sequence.

> D_α codes a σ-additive measure ν on $\mathcal{A}_\alpha^-$ such that its null ideal coincides with the union of null ideals of μ_β $(\beta < \alpha)$ and such that there are a strictly increasing sequence $\{d_n\}$ of ordinals converging to α and a sequence $X_n \in \mathcal{A}_{[d_n,d_{n+1})}$ $(n < \omega)$ stochastically independent with respect to ν.

If this fails, we construct μ_α and $\mathcal{B}_\alpha$ by picking any strictly increasing sequence $d_n \to \alpha$ (with $d_0 = 0$) that avoids S, letting $\nu_n = \mu_{[d_n,d_{n+1})}$, and using (3).

Otherwise, by (the proof of) Maharam's theorem ([2, 331I]) in each $\mathcal{B}_{[d_n,d_{n+1})}$ find an independent family that includes X_n and completely generates this algebra. Modify the restriction of ν to $\mathcal{A}_{[d_n,d_{n+1})}$ via

$$\nu_n(X_n) = 1 - \frac{1}{\pi^2 n^2}$$

and $\nu_n(Y) = \nu(Y) = 1/2$ for all other Y in this stochastically independent family. Then ν_n extends to a measure on $\mathcal{A}_n$ with the same null ideal as the restriction of ν (and therefore the same null ideal as $\mu_{[d_n,d_{n+1})}$). We denote this extension by ν_n. Now use ν_n and $\{d_n\}$ to define μ_α as in (3).

This describes the construction of $\mathcal{B}_\alpha, \mu_\alpha$ $(\alpha < \kappa^+)$. The null ideals of the measures μ_α cohere by (1). Let $\mathcal{I} = \bigcup_{\alpha<\kappa^+} \mathrm{Null}(\mu_\alpha)$ and $\mathcal{B}_{\kappa^+} = \mathcal{A}_{\kappa^+}/\mathcal{I}$. We claim this algebra satisfies the requirements of the theorem. Every complete subalgebra of smaller size is contained in some measure algebra $\mathcal{B}_\alpha$ $(\alpha < \kappa^+)$ and is therefore a measure algebra itself. We need to check $\mathcal{B}_{\kappa^+}$ is not a measure algebra. Assume it is; then its strictly positive measure lifts to a strictly positive measure ν on $\mathcal{A}_{\kappa^+}$. Let $f\colon \kappa^+ \to \kappa^+$ be a function such that $\mathcal{A}_{[\beta,f(\beta))}$ contains a set stochastically independent from $\mathcal{A}_\beta$. Such a function exists since $\mathcal{B}_{\kappa^+}$ is a homogeneous measure algebra of character greater than the character of $\mathcal{B}_\beta$ (see [2, §331]). Let $\alpha \in S$ be an ordinal closed under f and such that D_α codes the restriction of ν to $\mathcal{A}_\alpha^-$. Then there exists a sequence $d_{n+1} \geq f(d_n)$ converging to α and stochastically independent $X_n \in \mathcal{A}_{[d_n,d_{n+1})}$ $(n \in \omega)$; fix the ones used to define μ_α. Then $\mu_\alpha(X_n) = 1 - \frac{1}{\pi^2 n^2}$ and $\mu_\alpha(\bigwedge_n X_n) = \prod_{n=1}^\infty \left(1 - \frac{1}{\pi^2 n^2}\right) = \sin(1) > 0$, but $\nu(\bigwedge_n X_n) = \prod_{n=1}^\infty \frac{1}{2} = 0$. Therefore ν is not a strictly positive measure on $\mathcal{B}_{\kappa^+}$. □

We do not know whether $\square_\kappa$ is a sufficient assumption for the conclusion of Theorem 2. Proving this would considerably improve the lower bound given in Theorem 1 and Theorem 3. It would also provide an example of smaller cardinality. Note that the algebra constructed from an anti-large cardinal assumption in this note has size at least $\beth_{\omega_1}^+$.

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