# A SIMPLE AF ALGEBRA NOT ISOMORPHIC TO ITS OPPOSITE

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*Dedicated to Menachem Magidor on the occasion of his 70th birthday.*

Abstract. We show that it is consistent with ZFC that there is a simple nuclear non-separable $C^*$-algebra which is not isomorphic to its opposite algebra. We can furthermore guarantee that this example is an inductive limit of unital copies of the Cuntz algebra $\mathcal{O}_2$, or of the CAR algebra.

**Significance statement.** The Hilbert space $\ell^2$ is the (usually infinite-dimensional) modification of our standard three-dimensional space. $C^*$-algebras are suitably closed algebras of linear operators on $\ell^2$. The algebras of complex $n \times n$ matrices are the simplest examples of $C^*$-algebras. The opposite of a $C^*$-algebra is the algebra in which the direction of the multiplication is reversed. Although every matrix algebra is isomorphic to its opposite, we construct an inductive limit of matrix algebras not isomorphic to its opposite. This is the first known example of a simple amenable $C^*$-algebra not isomorphic to its opposite. Our examples can have exactly $n$ inequivalent irreducible representations for any $n$, showing that Glimm's dichotomy can fail for simple nonseparable $C^*$-algebras.

## 1. Introduction

The opposite algebra of a $C^*$-algebra $A$ is the $C^*$-algebra whose underlying Banach space structure and involution are the same as that of $A$, but the product of $x$ and $y$ is defined as $yx$ rather than $xy$. It is denoted by $A^{\mathrm{op}}$. In [4] Connes constructed examples of factors not isomorphic to their opposites. Phillips used Connes' results in [18] to construct simple separable examples, and Phillips–Viola in [19] improved this to construct a simple separable exact example. In the nuclear setting, one can construct non-simple examples ([21, 17]), however the simple nuclear case remained open both in the separable and in the non-separable settings.

The separable case remains a difficult open problem. AF algebras are necessarily isomorphic to their opposites, due to Elliott's classification theorem, and our results show that this cannot be recast as a result purely of a local approximation property. There has been major progress in the Elliott classification program recently, but the

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state-of-the art classification theorems all assume the UCT. Notably, we do not know if there are Kirchberg algebras which are not isomorphic to their opposites. If such an algebra exists, then it would necessarily be a counterexample to the UCT. More generally, both the Elliott invariant and the Cuntz semigroup of any $C^*$-algebra $A$ are isomorphic to that of $A^{\mathrm{op}}$.

The additional axiom we add to ZFC is Jensen's $\diamondsuit_{\aleph_1}$, discussed below in Section 3, and our construction is motivated by the work of Akemann and Weaver from [3], where they use $\diamondsuit_{\aleph_1}$ to construct a counterexample to the Naimark problem. Our main theorem is:

**Theorem 1.1.** *Assume $\diamondsuit_{\aleph_1}$ holds. Then there exists a nuclear, simple, unital $C^*$-algebra $A$ not isomorphic to its opposite algebra.*

In fact, we obtain the following strengthening.

**Theorem 1.2.** *Assume $\diamondsuit_{\aleph_1}$ holds and $1 \leq n \leq \aleph_0$ is given. Then there exists a $C^*$-algebra $A$ with the following properties.*

1. *$A$ is nuclear, simple, unital and of density character $\aleph_1$.*
2. *$A$ is not isomorphic to its opposite algebra.*
3. *$A$ has exactly $n$ unitarily nonequivalent irreducible representations.*
4. *All automorphisms of $A$ are inner.*

*In addition, one can ensure that one of the following holds.*

5. *$A$ is an inductive limit of subalgebras isomorphic to the Cuntz algebra $\mathcal{O}_2$.*
6. *$A$ is an inductive limit of subalgebras isomorphic to full matrix algebras of the form $M_{2^n}(\mathbb{C})$.*

By Glimm's theorem (see the remark in the second paragraph from the end of page 586 of [9]), every separable and simple $C^*$-algebra with nonequivalent irreducible representations has $2^{\aleph_0}$ nonequivalent irreducible representations. Item (3) above shows that the failure of this dichotomy for nonseparable $C^*$-algebras is relatively consistent with ZFC.

The observation that the proof of [3] gives a nuclear counterexample to Naimark's problem is due to N. C. Phillips. We don't know whether a simple, nuclear $C^*$-algebra not isomorphic to its opposite can be constructed in ZFC, and whether a counterexample to Naimark's problem can be constructed in ZFC. Another problem raised by our proof of Theorem 1.2 is whether a counterexample to Naimark's problem can have an outer automorphism.

We use the following notation throughout. We count 0 as a natural number. If $\mathcal{Y} = \langle a_j : j \in \mathbb{N} \rangle$ is a sequence of elements in some set, we denote by $b^\frown\mathcal{Y}$ the sequence whose first element is $b$, and whose $j+1$ element is $a_j$.

## 2. Extending states

This section contains technical lemmas which will be used in the induction step of our construction. We first give a modification of a lemma of Kishimoto, Lemma 2.2, and a toy version, Lemma 2.1.

**Lemma 2.1.** *Let $A$ be a a non-type I, separable, simple, unital $C^*$-algebra. Let $C$ and $D$ be non-zero hereditary subalgebras of $A$, and let $\varepsilon > 0$. Let $n \geq 1$ and let $u_0, u_1, \ldots u_n$ be some elements in $A^+$. Then there exist positive elements $c \in C$ and $d \in D$ of norm 1 such that $\|cu_k d\| < \varepsilon$ for $k = 0, 1, \ldots, n$.*

*Proof.* We denote $A_\infty := l^\infty(\mathbb{N}, A)/C_0(\mathbb{N}, A)$, and we identify $A$ with the subalgebra given by constant sequences. As $A$ is not a continuous trace algebra, by [2, Theorem 2.4], the central sequence algebra $A_\infty \cap A'$ is nontrivial. Let $x \in A_\infty \cap A'$ be a self-adjoint element whose spectrum has more than one point. Since $A$ is simple, the $C^*$-algebra generated by $x$ and $A$ inside of $A_\infty$ is isomorphic to $C(\sigma(x)) \otimes A$, and therefore, if $y \in C^*(x)$ and $a \in A$ then $\|ya\| = \|y\|\|a\|$. Since $\sigma(x)$ has more than one point, we may pick $y, z \in C^*(x)_+$ with norm 1 such that $yz = 0$. Pick $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}, A)_+$ which lift $y$ and $z$, respectively. Fix elements $c_0 \in C_+$ and $d_0 \in D_+$ of norm 1. Then $\lim_{n\to\infty} \|c_0^{1/2} y_n c_0^{1/2}\| = \lim_{n\to\infty} \|d_0^{1/2} z_n d_0^{1/2}\| = 1$, and $\lim_{n\to\infty} \|c_0^{1/2} y_n c_0^{1/2} \cdot u_k \cdot d_0^{1/2} z_n d_0^{1/2}\| = \lim_{n\to\infty} \|c_0 y_n z_n u_k d_0\| = 0$. For all sufficiently large $n$, the elements $c = \frac{1}{\|c_0^{1/2} y_n c_0^{1/2}\|} \cdot c_0^{1/2} y_n c_0^{1/2}$ and $d = \frac{1}{\|d_0^{1/2} z_n d_0^{1/2}\|} \cdot d_0^{1/2} z_n d_0^{1/2}$ satisfy the requirements. □

**Lemma 2.2.** *Suppose $A$ is a non-type I, separable, simple, unital $C^*$-algebra and suppose $\alpha$ is an antiautomorphism of $A$ or an outer automorphism of $A$. Then for any nonzero hereditary $C^*$-subalgebra $B$ of $A$ and every unitary $u \in A$ we have*

$$\inf\{\|bu\alpha(b)\| : b \in B_+, \|b\| = 1\} = 0.$$

*Proof.* Since an automorphism of a simple $C^*$-algebra is outer if and only if its Connes spectrum is distinct from $\{1\}$, the case in which $\alpha$ is an outer automorphism is a special case of [13, Lemma 1.1].

Suppose $\alpha$ is an antiautomorphism and let $\alpha' := \mathrm{Ad}\, u \circ \alpha$. By [10, Theorem 1] we have $\inf\{\|b\alpha'(b)\| : b \in B_+, \|b\| = 1\} = 0$. But $\|b\alpha'(b)\| = \|bu\alpha(b)u^*\| = \|bu\alpha(b)\|$ and the conclusion follows. □

**Lemma 2.3.** *Suppose $A$ is a separable, simple, unital $C^*$-algebra. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are disjoint countable sets of unitarily nonequivalent pure states of $A$ and suppose $E$ is an equivalence relation on $\mathcal{Y}$. Then there exists a separable simple unital $C^*$-algebra $C$ with the following properties.*

1. *$A$ is a unital subalgebra of $C$.*
2. *Every $\psi \in \mathcal{Y}$ has a unique extension $\tilde{\psi}$ to a pure state of $C$.*
3. *If $\psi_0$ and $\psi_1$ are in $\mathcal{Y}$ then $\psi_0 \, E \, \psi_1$ if and only if $\tilde{\psi}_0$ and $\tilde{\psi}_1$ are unitarily equivalent pure states of $C$.*

(4) *Every* $\psi \in \mathcal{X}$ *has more than one extension to a pure state of* $C$.

*In addition, if* $A \cong \mathcal{O}_2$ *then one can arrange* $C \cong \mathcal{O}_2$.

*Proof.* We shall construct an automorphism $\beta$ of $A$ of infinite order such that the crossed product $C := A \rtimes_\beta \mathbb{Z}$ is as required. By [3, Theorem 2], a pure state $\varphi$ of $A$ has a unique extension to a pure state of $C$ if and only if $\varphi$ is nonequivalent to $\varphi \circ \beta^n$ for all $n \neq 0$. Since $A$ is non-type I and separable, by Glimm's theorem it has $2^{\aleph_0}$ nonequivalent pure states. We can therefore extend $\mathcal{Y}$ to ensure that every $E$-equivalence class is infinite and that there are infinitely many equivalence classes. We can similarly assume $\mathcal{X}$ is infinite. Let $\pi_j^k$, for $j \in \mathbb{Z}$, be an enumeration of GNS representations corresponding to states in the $k$-th $E$-equivalence class. Let $\sigma_j$, for $j \in \mathbb{N}$, be an enumeration of GNS representations corresponding to states in $\mathcal{X}$. All of these representations correspond to pure states and are therefore irreducible. By the extension of [14] proved in [3, p. 7523–7524] there exists an automorphism $\beta$ of $A$ such that

(5) $\pi_j^k$ is equivalent to $\pi_l^m \circ \beta$ if and only if $k = m$ and $j = l + 1$.

(6) $\sigma_j$ is equivalent to $\sigma_j \circ \beta$ for all $j$.

By [13, Theorem 3.1] the crossed product $C := A \rtimes_\beta \mathbb{Z}$ is simple. By [3, Theorem 2] it satisfies (1), (2), and (4).

To prove (3), fix $\psi_0$ and $\psi_1$ in $\mathcal{Y}$. If $\psi_0 \, E \, \psi_1$ then (6) implies that the unique pure state extensions of $\psi_0$ and $\psi_1$ to $C$ are equivalent. Now suppose $\psi_0$ and $\psi_1$ are not $E$-related. Then $\psi_0$ and $\psi_1 \circ \beta^n$ are inequivalent for all $n \in \mathbb{Z}$. To get a contradiction, suppose that the unique pure state extensions of $\psi_0$ and $\psi_1$ to $C$ are equivalent and let $v$ be a unitary in $C$ such that $\psi_0 = \psi_1 \circ \operatorname{Ad} v$. Let $u$ be the canonical unitary implementing $\beta$. Approximate $v$ up to $1/2$ by a finite linear combination $\sum_{n=-k}^{k} c_n u^n$, where $c_n \in A$. Choose decreasing sequences $a_j, b_j$, for $j \in \mathbb{N}$, of positive elements of norm 1 such that the $a_j$ excise $\psi_0$ and the $b_j$ excise $\psi_1$ ([1, Proposition 2.2]). Note that $\beta^n(b_j)$ excises $\psi_1 \circ \beta^{-n}$ for all $n$. By [3, Lemma 1], for all $x \in A$ we have $\|a_j x \beta^n(b_j)\| \to 0$ as $j \to \infty$. Thus, for $j$ large enough, we have $\|a_j c_n \beta^n(b_j)\| < 1/(4k+2)$ for all $-k \leq n \leq k$. Then $\|a_j v b_j v^*\| = \|a_j v b_j\| < 1$. On the other hand, the Cauchy–Schwarz inequality implies $\psi_0(a_j v b_j v^*) = \psi_0(v b_j v^*) = \psi_1(b_j) = 1$; contradiction.

Finally, if $A \cong \mathcal{O}_2$, then $C = A \rtimes_\beta \mathbb{Z} \cong \mathcal{O}_2$. One way to see this is to note that by (5) above, no non-zero power of $\beta$ is inner, therefore by [16, Theorem 1] the automorphism $\beta$ has the Rokhlin property, hence by [11, Theorem 4.4] we have $C \cong C \otimes \mathcal{O}_2$, so by [12, Theorem 3.8] we have $C \cong \mathcal{O}_2$. □

The following is a strengthening of [13, Theorem 2.1].

**Lemma 2.4.** *Suppose* $A$ *is a non-type I, separable, simple, unital* $C^*$*-algebra, and suppose* $\alpha$ *is an antiautomorphism, or an outer automorphism. Then there exists a family* $\mathcal{W}$ *of* $2^{\aleph_0}$ *pure states of* $A$ *such that* $\varphi$ *is not unitarily equivalent to* $\varphi \circ \alpha$ *for every* $\varphi \in \mathcal{W}$.

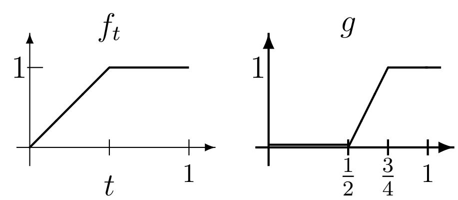
*Proof.* The proofs in the case when $\alpha$ is an outer automorphism and when $\alpha$ is an antiautomorphism differ very little and will be presented simultaneously.

Let $u_n$, for $n \in \mathbb{N}$, be an enumeration of a dense set of unitaries of $A$. By $\{0,1\}^{<\mathbb{N}}$ we denote the set of all finite sequences of $\{0,1\}$ ordered by the end-extension, denoted $s \sqsubset t$. The empty sequence $\langle\rangle$ is the minimal element of $\{0,1\}^{<\mathbb{N}}$, its immediate successors are 0 and 1, and the immediate successors of $s \in \{0,1\}^{<\mathbb{N}}$ are $s^\frown 0$ and $s^\frown 1$. The length of $s \in \{0,1\}^{<\mathbb{N}}$ is denoted $|s|$.

Given $\delta \in (0,1/2)$, we claim that there exist $a(s)$ and $e(s)$ in $A_+$, for $s \in \{0,1\}^{<\mathbb{N}}$ and $j = 0,1$ for $s \in \{0,1\}^{<\mathbb{N}}$, with the following properties:

(1) $\|a(s)\| = \|e(s)\| = 1$.
(2) $a(s)e(s^\frown j) = e(s^\frown j)$.
(3) $e(s)a(s) = a(s)$.
(4) $\|e(s^\frown 0)e(s^\frown 1)\| < \delta$.
(5) $\|e(s^\frown 0)u_k e(s^\frown 1)\| < \delta$ for all $k \leq |s|$.
(6) $\|a_s u_{|s|} \alpha(a_s)\| < \delta$.

The family $\{e(s), a(s)\}_{s\in\{0,1\}^{<\mathbb{N}}}$ will be constructed by recursion. Define $f_t, g\colon [0,1] \to [0,1]$ for $t \in (0,1)$ as follows.




Notice that $f_{1/2} \cdot g = g$, and $\|f_t - \mathrm{id}\| = 1 - t$. Fix $\varepsilon \in (0,1/2)$ such that whenever $x, y$ are positive contractions in some $C^*$-algebra and $z$ is any contraction such that $\|xzy\| < \varepsilon$ then $\|f_{1/2}(x)zf_{1/2}(y)\| < \delta$ and $\|g(x)zg(y)\| < \delta$. (This is done using polynomial approximations for $f_{1/2}$ and for $g$.)

Let $a(\langle\rangle) = 1$. Suppose $a(s)$ was chosen. By Lemma 2.1 applied to $n = |s| + 1$ and the unitaries $u_k$ for $k \leq n$, there exist $h_0, h_1 \in B(s)_+$ such that $\|h_0\| = \|h_1\| = 1$ and $\|h_0 u_k h_1\| < \varepsilon$. for all $k \leq |s|$. Let

$$e(s^\frown j) := f_{1/2}(h_j).$$

By Lemma 2.2 there exists $a(s^\frown j) \in \overline{g(h_j)Ag(h_j)}_+$ that satisfies $\|a(s^\frown j)\| = 1$ and $\|a(j)u_{|s|}\alpha(a(j))\| < \delta$. We may assume without loss of generality, that there exists a nonzero positive element $b(s^\frown j)$ with $a(s^\frown j)b(s^\frown j) = b(s^\frown j)$ (by replacing $a(s^\frown j)$ by $f_t(a(s^\frown j))$ for $t$ sufficiently close to 1 if need be).

The family $\{e(s), a(s)\}_{s\in\{0,1\}^{<\mathbb{N}}}$ satisfying (1)–(6) can now be constructed by using a standard bookkeeping device. Fix an enumeration $s_j$, for $j \in \mathbb{N}$, for $\{0,1\}^{<\mathbb{N}}$ such that $s_j \sqsubset s_k$ implies $j < k$ (e.g. let $\{s \in \{0,1\}^{<\mathbb{N}} : |s| = n\}$ be enumerated as $s_j$, for $2^{n-1} \leq j < 2^n$). By using the above, one can recursively find $e(s_j)$ and $a(s_j)$ for

$j \in \mathbb{N}$ in the hereditary subalgebra on which all the elements of the form $e(s)$ and $a(s)$, where $s \sqsubset s_j$, act as the identity.

Denote the set of all infinite sequences of $\{0,1\}$ by $\{0,1\}^{\mathbb{N}}$. For $h \in \{0,1\}^{\mathbb{N}}$ let $h \restriction n$ denote the initial segment of $h$ of length $n$, for $n \in \mathbb{N}$. For $h \in \{0,1\}^{\mathbb{N}}$ we have $h \restriction n \in \{0,1\}^{<\mathbb{N}}$ and

$$\mathcal{F}(h) := \{a(h \restriction n) : n \in \mathbb{N}\}$$

is a sequence of elements of $A_+$ of norm 1 such that

$$a(h \restriction n)a(h \restriction (n+1)) = a(h \restriction (n+1))$$

for all $n$. Hence

$$\{\zeta \in \mathcal{S}(A) : \zeta(a) = 1 \text{ for all } a \in \mathcal{F}(h)\}$$

is a face of $\mathcal{S}(A)$. Let $\zeta_h$ be an extreme point of this face; then $\zeta_h$ is a pure state of $A$ satisfying $\zeta_h(a(h \restriction n)) = 1$ for all $n$. By (3) we have $\zeta_h(e(h \restriction n)) = 1$ for all $n$ and thus, by the Cauchy–Schwarz inequality, we have $\zeta_h(e(h \restriction n)b) = \zeta_h(b)$ for all $b$ and for all $n$.

We claim that the states $\zeta_h$ and $\zeta_{h'}$ are not unitarily equivalent if $h \neq h'$. Suppose otherwise. Then for some $j \in \mathbb{N}$ we have $\|\zeta_h - \zeta_{h'} \circ \operatorname{Ad} u_j\| < 1/2$. Fix $n \geq j$ large enough to have $h \restriction n \neq h' \restriction n$. By (5) we have $\|e(h \restriction n) \operatorname{Ad} u_j(e(h' \restriction n))\| < \delta < 1/2$, but $|\zeta_h(e(h \restriction n) \operatorname{Ad} u_j(e(h' \restriction n))| = |\zeta_h(\operatorname{Ad} u_j(e(h' \restriction n))| > 1/2$, a contradiction.

By the same argument and (6), $\zeta_h$ is not equivalent to $\zeta_h \circ \alpha$ for every $h \in \{0,1\}^{\mathbb{N}}$. We should note that whether $\alpha$ be an automorphism or an antiautomorphism, it preserves the order structure of $A$ and it is an affine homeomorpism of $\mathcal{S}(A)$ onto itself. Therefore $\zeta_h \circ \alpha$ is a pure state of $A$. □

The next few technical lemmas will be used to construct a UHF example.

**Definition 2.5.** Suppose $A$ is a separable UHF algebra. A family of pure states $\langle \varphi_n : n \in \mathbb{N}\rangle$ of $A$ will be called *separated product states* if there exist $\langle k(n) : n \in \mathbb{N}\rangle$, a map $\Phi$, subalgebras $A_n$, and projections $\langle p_{n,j} : n \in \mathbb{N}, j < n\rangle$ and $\langle q_n : n \in \mathbb{N}\rangle$ with the following properties.

(1) $k(n) \geq 1$, for $n \in \mathbb{N}$.
(2) $\Phi \colon A \to \bigotimes_n M_{k(n)}(\mathbb{C})$ is an isomorphism.
(3) $A_n := \bigotimes_{j<n} M_{k(j)}(\mathbb{C})$.
(4) $p_{n,j}$, for $0 \leq j < n$, are orthogonal rank 1 projections in $M_{k(n)}(\mathbb{C})$, for all $n$,
(5) $q_m \in A_m$ is a rank-1 projection, and
(6) $\varphi_m$ is the product state of $A_n \otimes \bigotimes_{j=m+1}^{\infty} M_{k(j)}(\mathbb{C})$ uniquely determined by the requirement that for all $l \geq 1$ we have

$$\varphi_m(q_m \otimes p_{m+1,m} \otimes p_{m+2,m} \otimes \cdots \otimes p_{m+l,m}) = 1.$$

**Lemma 2.6.** *Suppose $A$ is a UHF algebra and $\pi_n$ for $n \in \mathbb{N}$, are irreducible representations of $A$. Then the following are equivalent.*

(1) *$\langle \pi_n : n \in \mathbb{N}\rangle$ are pairwise nonequivalent irreducible representations of $A$,*

(2) *There are separated product states $\varphi_n$, for $n \in \mathbb{N}$, such that $\pi_n$ is the GNS representation corresponding to $\varphi_n$ for all $n$.*

*Proof.* Suppose $\varphi_j$, for $j \in \mathbb{N}$, are separated product states of a UHF algebra. For all $j \neq l$ and $n \in \mathbb{N}$ there exists a projection $p \in A_n' \cap A$ such that $\varphi_j(p) = 0$ and $\varphi_l(p) = 1$, and therefore [8, Theorem 3.4] implies that $\varphi_l$ is not unitarily equivalent to $\varphi_j$ for $j \neq l$.

Now suppose $\pi_j$, for $j \in \mathbb{N}$, are as in (2). Let $\psi_j$ be a pure state such that $\pi_j$ is the GNS representation corresponding to $\psi_j$ for $j \in \mathbb{N}$. Let $\varphi_j$, for $j \in \mathbb{N}$, be a sequence of separated pure states of $A$. By (1) these pure states are nonequivalent and by the extension of [14] proved in [3, p. 7523–7524] (or, since $A$ is UHF, by [7, Theorem 7.5]) there exists an automorphism $\beta$ of $A$ such that $\varphi_j = \psi_j \circ \beta$ for all $j \in \mathbb{N}$, as required. □

We need the following variant of Lemma 2.3 for the CAR algebra, $M_{2^\infty}$.

**Lemma 2.7.** *Suppose $A \cong M_{2^\infty}$. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are disjoint countable sets of unitarily nonequivalent pure states of $A$ and $E$ is an equivalence relation on $\mathcal{Y}$. Then there exists a separable simple unital $C^*$-algebra $C$ with the following properties.*

(1) *$C \cong M_{2^\infty}$.*
(2) *$A$ is a unital subalgebra of $C$.*
(3) *Every $\psi \in \mathcal{Y}$ has a unique extension $\tilde{\psi}$ to a pure state of $C$,*
(4) *If $\psi_0$ and $\psi_1$ are in $\mathcal{Y}$ then $\psi_0 \, E \, \psi_1$ if and only if $\tilde{\psi}_0$ and $\tilde{\psi}_1$ are unitarily equivalent pure states of $C$.*
(5) *Every $\psi \in \mathcal{X}$ has more than one extension to a pure state of $C$.*

*Proof.* We shall first provide a proof in case when $E$ is the identity relation on $\mathcal{Y}$. By Lemma 2.6 we may identify $A$ with $\bigotimes_n M_{k(n)}(\mathbb{C})$ witnessing that the pure states in $\mathcal{X} \cup \mathcal{Y}$ are separated. Since $A \cong M_{2^\infty}$, for every $n$ there exists $l(n) \in \mathbb{N}$ such that $k(n) = 2^{l(n)}$. We may assume that $k(n) > 2n$ for all $n$. In $M_{k(n)}(\mathbb{C})$ we have $n$ orthogonal rank 1 projections $p_{n,j}$, for $j \leq n$, each corresponding to a unique state in $\mathcal{X} \cup \mathcal{Y}$. Let $\mathcal{P}$ be a maximal family of orthogonal rank 1 projections in $M_{k(n)}$ including $\{p_{n,j} : j \leq n\}$. Since $k(n) > 2n$, we can find a permutation $\sigma$ of $\mathcal{P}$ such that

(6) $\sigma(p_{n,j}) = p_{n,j}$ if and only if $p_{n,j}$ corresponds to a pure state in $\mathcal{X}$,
(7) $\sigma(p_{n,j}) \neq p_{n,k}$ if $p_{n,j}$ and $p_{n,k}$ correspond to distinct pure states in $\mathcal{Y}$, and
(8) $\sigma^2 = \mathrm{id}_{\mathcal{P}}$.

Let $u_n \in M_{k(n)}(\mathbb{C})$ be an order 2 unitary such that $\mathrm{Ad}\, u_n(q) = \sigma(q)$ for all $q \in \mathcal{P}$ and such that $\mathrm{Tr}(u_n) = 0$. (One can construct such a unitary by first considering a permutation matrix corresponding to $\sigma$, and noting that the number of 1's on the diagonal must be even; we then define $u_n$ to be a matrix obtained by starting out with this permutation matrix and replacing half of the 1's on the diagonal by $-1$'s.) Note that the automorphism $\beta := \bigotimes_n \mathrm{Ad}\, u_n$ also satisfies $\beta^2 = \mathrm{id}_A$.

Set $A_n$ as in Definition 2.5. Each $A_n$ is $\beta$-invariant and we have $A \rtimes_\beta \mathbb{Z}/2\mathbb{Z} = \overline{\bigcup_n A_n \rtimes_{\beta|_{A_n}} \mathbb{Z}/2\mathbb{Z}}$. Note that $A_n \rtimes_{\beta|_{A_n}} \mathbb{Z}/2\mathbb{Z} \cong A_n \oplus A_n$, and the inclusion

$$A_n \rtimes_{\beta|_{A_n}} \mathbb{Z}/2\mathbb{Z} \to A_{n+1} \rtimes_{\beta|_{A_{n+1}}} \mathbb{Z}/2\mathbb{Z} \cong (A_n \rtimes_{\beta|_{A_n}} \mathbb{Z}/2\mathbb{Z}) \otimes M_{k(n)}$$

is given by a direct sum of $k(n)/2$ copies of the identity map, and $k(n)/2$ copies of the map $a \oplus b \mapsto b \oplus a$. Thus, by considering the Bratteli diagram of this AF system, we see that $A \rtimes_\beta \mathbb{Z}/2\mathbb{Z} \cong M_{2^\infty}$.

By [3, Theorem 2] a pure state $\varphi$ of $A$ has a unique extension to a pure state of $C$ if and only if $\varphi$ and $\varphi \circ \beta$ are not unitarily equivalent. By the choice of $u_n$ and $\beta$, a pure state $\varphi \in \mathcal{X} \cup \mathcal{Y}$ has a unique extension to a pure state of $C$ if and only if $\varphi \in \mathcal{X}$. If $\varphi$ and $\psi$ are distinct and belong to $\mathcal{Y}$, then by (7) for every finite-dimensional subalgebra $B$ of $C$ there exists a projection $p \in B' \cap C$ (one can choose it of the form $q + \sigma(q)$ for $q$ which corresponds to $\psi$) such that $\tilde{\varphi}(p) = 0$ and $\tilde{\psi}(p) = 1$. Therefore [8, Theorem 3.4] implies that $\tilde{\varphi}$ is not unitarily equivalent to $\tilde{\psi}$.

We now consider the case when $E$ is a nontrivial equivalence relation on $\mathcal{Y}$. Enumerate the $i$-th $E$-equivalence class as $\langle \zeta^i_j : j < n \rangle$, for some $1 \leq n \leq \aleph_0$. In the above construction there is sufficient room for us to choose the symmetry $\sigma$ so the resulting automorphism $\beta$ satisfies $\zeta^i_0 \circ \beta = \zeta^i_1$ for all $i$. The resulting crossed product, $A_1$, is isomorphic to $M_{2^\infty}$, every $\zeta^i_j \in \mathcal{Y}$ has a unique extension $\tilde{\zeta}^i_j$ to a pure state of $A_1$, and $\tilde{\zeta}^i_j$ is equivalent to $\tilde{\zeta}^l_k$ if and only if $i = l$ and $\max(j, k) \leq 1$. We can now apply this construction to $A_1$, with $\mathcal{X} := \emptyset$, $\mathcal{Y} := \{\tilde{\zeta}^i_j : j \geq 1\}$ and $E$ defined by $\tilde{\zeta}^i_j E \tilde{\zeta}^l_k$ if and only if $i = l$ and $\min(j, k) \geq 1$ and obtain crossed product $A_2$. After at most $\aleph_0$ steps all $E$-equivalence classes will be taken care of. The inductive limit $C$ of $A_n$ is, by the classification of AF algebras, isomorphic to $M_{2^\infty}$ and it has all the required properties. □

The following lemma serves as the inductive step in our construction.

**Lemma 2.8.** *Suppose $A$ is a non-type I, separable, simple, unital $C^*$-algebra and let $\mathcal{Y}$ be a countable set of pure states of $A$. Let $\zeta$ be a pure state of $A$ which is not unitarily equivalent to any of the states in $\mathcal{Y}$. Suppose $\alpha$ is an antiautomorphism, or an outer automorphism, of $A$. Then there exist a separable simple unital $C^*$-algebra $C$ and a pure state $\psi$ of $C$ such that:*

1. *$A$ is a unital $C^*$-subalgebra of $C$.*
2. *Each $\varphi \in \mathcal{Y}$ has a unique extension to a pure state of $C$, and those unique extensions are pairwise unitarily inequivalent.*
3. *$\zeta$ has a unique extension to a pure state in $C$ which is unitarily equivalent to the extension of some pure state from $\mathcal{Y}$.*
4. *$\psi$ is the unique extension of some pure state in $\mathcal{Y}$.*
5. *$\alpha$ cannot be extended to an antiautomorphism or an automorphism of $C$.*

(6) *If a $C^*$-algebra $D$ has $C$ as a subalgebra and $\psi$ has a unique state extension to $D$ then $\alpha$ cannot be extended to an antiautomorphism or an automorphism of $D$.*

*In addition, if $A \cong \mathcal{O}_2$ then we can arrange $C \cong \mathcal{O}_2$, and if $A \cong M_{2^\infty}$ then we can arrange $C \cong M_{2^\infty}$.*

*Proof.* Again, the proofs in the case in which $\alpha$ is an outer automorphism and when $\alpha$ is an antiautomorphism differ very little and will be presented simultaneously. We note in passing that our assumptions imply that $A$ is nonabelian, hence an automorphism of $A$ cannot be extended to an antiautomorphism of $C$ and vice versa; however this is unimportant for the proof.

Since the given set $\mathcal{Y}$ of pure states is countable, by Lemma 2.4, we can choose a pure state $\psi_0$ such that for any $\varphi \in \mathcal{Y} \cup \{\zeta\}$, neither $\psi_0$ nor $\psi_1 := \psi_0 \circ \alpha$ is unitarily equivalent to $\varphi$. Let $\mathcal{Y}' := \mathcal{Y} \cup \{\zeta, \psi_0\}$, and define an equivalence relation $E$ on $\mathcal{Y}'$ such that $\zeta \, E \, \varphi$ and $\psi_0 \, E \, \varphi$ for some $\varphi \in \mathcal{Y}$, and all other elements of $\mathcal{Y}'$ are equivalent via $E$ only to themselves. We then apply Lemma 2.3 or Lemma 2.7 to $\mathcal{X} = \{\psi_1\}$ and $\mathcal{Y}'$ to obtain a $C^*$-algebra $C$ (with $C \cong A$ if $A$ is $M_{2^\infty}$ or $\mathcal{O}_2$) such that $\psi_0$, $\zeta$ and all $\varphi \in \mathcal{Y}$ have unique pure state extensions to $C$, $\psi_1$ has multiple state extensions to $C$, and the unique extensions of $\psi_0$ and $\zeta$ are equivalent to the unique extension of some $\varphi \in \mathcal{Y}$; the latter state is $\psi$ as in (6).

Suppose $D$ is a $C^*$-algebra that has $C$ as a $C^*$-subalgebra, and assume that $\alpha$ extends to $\tilde{\alpha}$ which is an automorphism or an antiautomorphism of $D$. If $\psi$ has a unique state extension $\tilde{\psi}$ to $D$, then $\tilde{\psi} \circ \tilde{\alpha}$ is the unique extension of $\psi_1$ to $D$. As $\psi_1$ has multiple state extensions to $C$ this is a contradiction, and therefore (6) holds. □

## 3. Diamond and the construction

A subset $\mathcal{C}$ of $\aleph_1$ is called *closed and unbounded* (*club*) if for every $\eta < \aleph_1$ there exists $\xi \in \mathcal{C}$ such that $\xi > \eta$, and for every countable $X \subseteq \mathcal{C}$ we have $\sup(X) \in \mathcal{C}$ (see [15, §III.6]). A subset $\mathcal{S}$ of $\aleph_1$ is *stationary* if it intersects every club nontrivially. Since the intersection of two clubs (and even countably many clubs) is a club, the intersection of a stationary set with a club is again stationary. We shall use von Neumann's definition of an ordinal as the set of all smaller ordinals.

Jensen's $\diamondsuit_{\aleph_1}$ asserts that there exists a family of sets $S_\xi$, for $\xi < \aleph_1$, such that

(1) $S_\xi \subseteq \xi$ for all $\xi < \aleph_1$, and
(2) for every $X \subseteq \aleph_1$ the set $\{\xi : X \cap \xi = S_\xi\}$ is stationary.

This combinatorial principle is true in Gödel's constructible universe $L$ (see e.g. [15, §III.7.13]) and is therefore relatively consistent with ZFC. A much easier fact is that it implies the Continuum Hypothesis (see e.g. [15, III.7.2]).

Although $\diamondsuit_{\aleph_1}$ captures subsets of $\aleph_1$, it is well-known among logicians that $\diamondsuit_{\aleph_1}$ implies its self-strengthening which captures countable (or separable) subsets of any algebraic structure in countable signature of cardinality $\aleph_1$. This extends to metric structures. Since we could not find a reference for this fact in the literature, we work out the details in case of $C^*$-algebras equipped with some additional structure.

Suppose $A$ is a $C^*$-algebra with a given sequence of states $\mathcal{Y} = \langle \varphi_j : j \in \mathbb{N}\rangle$ and a linear isometry $\alpha\colon A \to A$. (We are interested in the case when $\alpha$ is an automorphism or an antiautomorphism.) Suppose we are given a dense subset of $A$, $\mathbb{A} := \{a_\xi : \xi < \theta\}$, indexed by an ordinal $\theta$. In addition suppose that $\mathbb{A}$ is closed under $+, \cdot, ^*$, $\alpha$, and multiplication by the complex rationals, $\mathbb{Q} + i\mathbb{Q}$. Consider the following subsets of $\theta^k$, for $1 \le k \le 3$ and of $\theta \times \mathbb{Q}$:

(1) $\mathbb{A}(+) := \{(\xi, \eta, \mu) \in \theta^3 : a_\xi + a_\eta = a_\mu\}$,
(2) $\mathbb{A}(\cdot) := \{(\xi, \eta, \mu) \in \theta^3 : a_\xi a_\eta = a_\mu\}$,
(3) $\mathbb{A}(^*) := \{(\xi, \eta) \in \theta^2 : a_\xi^* = a_\eta\}$,
(4) $\mathbb{A}(\|\cdot\|) := \{(\xi, r) \in \theta \times \mathbb{Q}_+ : \|a_\xi\| \ge r\}$,
(5) $\mathbb{A}(\mathbb{C}) := \{(\xi, \eta) \in \theta^2 : a_\xi = i a_\eta\}$,
(6) $\mathbb{A}(\varphi_j) := \{(\xi, r) \in \theta \times \mathbb{Q} : \varphi_j(a_\xi^* a_\xi) \ge r\}$, for $j \in \mathbb{N}$,
(7) $\mathbb{A}(\alpha) := \{(\xi, \eta) \in \theta^2 : \alpha(a_\xi) = a_\eta\}$.

This countable family of sets uniquely determines a countable normed algebra over $\mathbb{Q} + i\mathbb{Q}$ whose completion is isomorphic to $A$. It also uniquely determines both $\alpha$ and the sequence $\mathcal{Y}$. We say that the structure $(A, \mathbb{A}, \alpha, \varphi : \varphi \in \mathcal{Y})$ is *coded* by $\mathfrak{X} := \langle \mathbb{A}(\bullet) : \bullet \in \{+, \cdot, ^*, \|\cdot\|, \mathbb{C}, \alpha, \varphi : \varphi \in \mathcal{Y}\}\rangle$ and construe the latter as a subset of

$$\mathbb{X}(\theta) := \theta^3 \sqcup \theta^3 \sqcup \theta^2 \sqcup \theta \times \mathbb{Q} \sqcup \theta^2 \sqcup \theta \times \mathbb{Q} \times \mathcal{Y} \sqcup \theta^2.$$

Clearly $\mathbb{X}(\theta)$ and $\theta$ have the same cardinality for any infinite $\theta$.

A nested transfinite sequence $A_\xi$, for $\xi < \aleph_1$, of $C^*$-algebras is said to be *continuous* if for every limit ordinal $\eta < \aleph_1$ we have $A_\eta = \overline{\bigcup_{\xi<\eta} A_\xi}$.

**Lemma 3.1.** $\diamondsuit_{\aleph_1}$ *implies that there exists a family* $\{T_\xi\}_{\xi<\aleph_1}$ *such that:*

(1) $T_\xi \subseteq \mathbb{X}(\xi)$ *for all* $\xi < \aleph_1$,
(2) *for every continuous nested family* $\{A_\xi\}_{\xi<\aleph_1}$ *of separable $C^*$-algebras, for any enumeration* $\{a_\xi | \xi < \aleph_1\}$ *of* $A = \varinjlim A_\xi$, *for any countable set* $\mathcal{Y}$ *of pure states of $A$ and for any linear isometry $\alpha$ of $A$ onto $A$, the set of all $\theta < \aleph_1$ such that*
  (a) $\varphi \restriction A_\theta$ *is pure for all* $\varphi \in \mathcal{Y}$,
  (b) $\alpha(A_\theta) = A_\theta$, *and*
  (c) $T_\theta$ *codes the structure* $(A_\theta, \{a_\xi : \xi < \theta\}, \alpha \restriction A_\theta, \varphi \restriction A_\theta : \varphi \in \mathcal{Y})$

*is stationary.*

*Proof.* Fix a bijection $f\colon \aleph_1 \to \mathbb{X}(\aleph_1)$. Writing $f[X] := \{f(x) : x \in X\}$, define $g\colon \aleph_1 \to \aleph_1$ by $g(\xi) := \min\{\eta : f[\xi] \subseteq \mathbb{X}(\eta), f^{-1}[\mathbb{X}(\xi)] \subseteq \eta\}$. (Since every countable subset of $\aleph_1$ is bounded, $g$ is well-defined.) The set of fixed points of $g$, $\mathcal{C} := \{\theta < \aleph_1 : g[\theta] = \theta\}$, is a club ([15, Lemma III.6.13]) and $\mathcal{C} \subseteq \{\theta < \aleph_1 : f[\theta] = \mathbb{X}(\theta)\}$. Let $\{S_\xi\}_{\xi<\aleph_1}$ be a family of sets as in the definition of $\diamondsuit_{\aleph_1}$. We claim that $T_\xi := f[S_\xi]$, for

$\xi \in \mathcal{C}$, and $T_\xi := \emptyset$, for $\xi \notin \mathcal{C}$, are as required. (Many of the $T_\xi$ don't code anything resembling a $C^*$-algebra, but this is of no concern for us.)

Suppose $A = \varinjlim A_\xi$, $\mathcal{Y}$, $\alpha$, and $\{a_\xi : \xi < \aleph_1\}$ are as in (2). Set $\mathbb{A}_\theta := \{a_\xi : \xi < \theta\}$. Note that the set

$$\mathcal{C}_0 := \{\theta < \aleph_1 : \mathbb{A}_\theta \text{ is a dense } \mathbb{Q} + i\mathbb{Q} \text{ subalgebra of } A_\theta\}$$

is a club. Since the intersection of countably many clubs is a club, [3, Lemma 4] implies that

$$\mathcal{C}_1 := \{\theta \in \mathcal{C}_0 : \varphi_j \restriction A_\theta \text{ is pure for all } j \in \mathbb{N} \text{ and } \alpha[A_\theta] = A_\theta\}$$

is also a club. Let $\mathfrak{X} \subseteq \mathbb{X}(\aleph_1)$ be the code of $(A, \mathbb{A}, \alpha, \varphi : \varphi \in \mathcal{Y})$ and with $f$ used to define $T_\xi$, let $X := f^{-1}(\mathfrak{X})$. By $\diamondsuit_{\aleph_1}$, the set $\{\theta : X \cap \theta = S_\theta\}$ is stationary, and therefore so is its intersection with $\mathcal{C}_1$. But $\{\theta : X \cap \theta = S_\theta\} \cap \mathcal{C}_1$ is precisely the set of ordinals $\theta$ which satisfy (2), as required. □

*Proof of Theorem 1.2.* We construct a continuous nested sequence $\{A_\eta : \eta < \aleph_1\}$ of simple, separable unital and nuclear $C^*$-algebras and inequivalent pure states $\varphi^j_\eta$, for $j < n$, of $A_\eta$, such that $\varphi^j_\eta$ and $\varphi^j_\xi$ agree on $A_\xi$ if $\xi < \eta$. Since $\diamondsuit_{\aleph_1}$ implies the Continuum Hypothesis, each $A_\eta$ as well as $\bigcup_{\eta<\aleph_1} A_\eta$ will be of cardinality $\aleph_1$. We shall choose an enumeration $A_\eta = \{b^\xi_\eta : \xi < \aleph_1\}$ for every $\eta$ and a countable dense subset $\mathbb{A}_\eta = \{a^\eta_\xi : \xi < \eta\}$ of $A_\eta$ for every limit ordinal $\eta$ such that

(1) $\mathbb{A}_\eta$ is closed under $+, \cdot, ^*$, and multiplication by the complex rationals, $\mathbb{Q}+i\mathbb{Q}$,
(2) $a^\zeta_\xi = a^\eta_\xi$ if $\xi < \zeta < \eta$ and $\zeta$ and $\eta$ are limit ordinals,
(3) $\{b^\xi_\zeta : \max\{\xi, \zeta\} < \eta\} \subseteq \mathbb{A}_\eta$.

We begin with $A_0 = \mathcal{O}_2$ or $A_0 = M_{2^\infty}$ and any fixed (finite or infinite) sequence $\langle \varphi^j_0 : j < n\rangle$ of inequivalent pure states of $A_0$.

If $\theta$ is a limit ordinal then we let $A_\theta := \lim_{\xi<\theta} A_\xi$ and let $\varphi^j_\theta$ be the unique state extending all $\varphi^j_\xi$ for $\xi < \theta$ for $j < n$; this state is necessarily pure. If in addition $\theta$ is a limit of limit ordinals, then $\mathbb{A}_\theta$ is already uniquely determined and conditions (2) and (3) for $\zeta < \eta < \theta$ imply the corresponding conditions for $\eta < \theta$. If $\theta$ is a limit ordinal, but not a limit of limit ordinals, then the supremum of limit ordinals $< \theta$ is the largest limit ordinal below $\theta$; we denote it by $\eta$. Then the set $\{\xi : \eta \leq \xi < \theta\}$ is infinite. Since $A_\theta$ is separable and the set on the left-hand side of (3) is countable, $\mathbb{A}_\theta$ can be defined so that it satisfies the requirements.

Now suppose $\theta$ is a successor ordinal, say $\theta = \xi + 1$. To proceed from $A_\xi$ to $A_{\xi+1}$, we first check whether there exists an outer automorphism or an antiautomorphism $\alpha$ of $A_\xi$, pure state $\psi$ of $A_\xi$, and (if $n$ is finite) an extension of $\langle \varphi^j_\xi : j < n\rangle$ to an infinite sequence $\mathcal{W}$ such that $(A_\xi, \mathbb{A}_\xi, \psi^\frown\mathcal{W}, \alpha)$ is coded by $T_\xi$. If so, let $A_{\xi+1}$ be the $C^*$-algebra $C$ given by Lemma 2.8 in which the unique extension of $\psi$ is unitarily equivalent to a unique extension of some $\varphi^j_\xi$. Let $\varphi^j_{\xi+1}$ be the unique extension of $\varphi^j_\xi$, for $j < n$. If $T_\xi$ does not code such $(A_\xi, \mathbb{A}_\xi, \psi^\frown\mathcal{W}, \alpha)$, let $A_{\xi+1} := A_\xi$. This describes the construction.

Let $A$ be the inductive limit of this nested sequence. It is nuclear, simple and unital, being the inductive limit of simple nuclear $C^*$-algebras with unital connecting maps. Using (2) we can write $a_\xi := a_\xi^\zeta$ for $\zeta$ being any limit ordinal greater than $\xi$. Since $A = \bigcup_\xi A_\xi$ by (3) we have $A = \{a_\xi : \xi < \aleph_1\}$.

The sequence of pure state extensions $\varphi_\theta^j$ defines $n$ inequivalent pure states $\varphi^j$, for $j < n$, of $A$. These states have the property that $\varphi^j$ is a unique extension of $\varphi_\theta^j$ to $A$, for every $\theta < \aleph_1$. If $n$ is finite let $\mathcal{W}$ be any infinite sequence of pure states of $A$ extending $\langle \varphi^j : j < n\rangle$.

Suppose $A_0 \cong \mathcal{O}_2$ and $A_\xi \cong \mathcal{O}_2$ for all $\xi < \theta$. If $\theta = \xi + 1$ then $A_\theta \cong \mathcal{O}_2$ since it was obtained by using Lemma 2.8. If $\theta$ is a limit ordinal then [20, Corollary 5.1.5] implies $A_\theta \cong \mathcal{O}_2$. Therefore by induction $A_\xi \cong \mathcal{O}_2$ for all $\xi < \aleph_1$. Likewise, if $A_\xi \cong M_{2^\infty}$ for all $\xi < \theta$ then $A_\theta \cong M_{2^\infty}$ by the classification of AF algebras (noting that the inclusion maps all induce an isomorphism on the $K_0$ groups). Since $A$ has density character $\aleph_1$, it is an inductive limit of full matrix algebras by [6, Theorem 1.3 (1)].

Suppose that $A$ has an antiautomorphism or an outer automorphism $\alpha$ and let $\varphi$ be any pure state of $A$. Then there exists $\theta < \aleph_1$ such that $(A_\theta, \mathbb{A}_\theta, \varphi^\frown\mathcal{W}, \alpha \restriction A_\theta)$ was coded by $T_\theta$ at stage $\theta$. Hence $A_{\theta+1}$ was produced by using Lemma 2.8 and there exists $j < n$ such that $\alpha \restriction A_\theta$ cannot be extended to an antiautomorphism or an outer automorphism of any $C^*$-algebra which contains $A_{\xi+1}$ and to which $\varphi_{\xi+1}^j$ has a unique state extension. By construction this state has a unique extension to $A_\eta$ for all $\eta \geq \xi + 1$ and therefore it has a unique extension to $A$. But $\alpha$ clearly extends $\alpha \restriction A_\theta$; contradiction.

We already know that $A$ has at least $n$ inequivalent pure states. Let $\psi$ be any pure state of $A$. With $\alpha = \mathrm{id}_A$, there exists $\theta < \aleph_1$ such that $(A_\theta, \mathbb{A}_\theta, \alpha \restriction A_\theta, \varphi \restriction A_\theta)$ was coded by $T_\theta$ at stage $\theta$. Hence $A_{\theta+1}$ was produced by using Lemma 2.8 and $\varphi \restriction A_\theta$ has a unique extension to $A_{\theta+1}$ equivalent to $\varphi_{\theta+1}^j$ for some $j < n$. Since $\varphi^j$ is the unique extension of the latter to a state of $A$, we conclude that $\psi$ is equivalent to $\varphi^j$. Since $\psi$ was arbitrary, we conclude that every pure state of $A$ is equivalent to some $\varphi^j$, for $j < n$, and therefore $A$ has exactly $n$ inequivalent pure states. □

**Remark 3.2.** *The AF algebra we constructed is not isomorphic to an (uncountable) infinite tensor power of copies of $M_2$ (or $M_n$). To see that, notice that an infinite tensor product of matrix algebras is the complexification of a real $C^*$-algebra (namely, the corresponding infinite tensor product of $M_2(\mathbb{R})$). A complexification of a real $C^*$-algebra is always isomorphic to its opposite (any real $C^*$-algebra is isomorphic to its opposite via the $*$ map, which is $\mathbb{R}$-linear, which one can then complexify).*

**Remark 3.3.** *Our construction is $C^*$-algebraic in nature. It does, however, raise the analogous question for von-Neumann algebras: is there a hyperfinite factor (with non-separable predual) which is not isomorphic to its opposite? More concretely, our*

*AF example has unique trace. Let $M$ be the weak closure of its image under the GNS representation. Is $M$ isomorphic to its opposite? A peculiar hyperfinite $II_1$ factor with no nontrivial central sequences was constructed using the Continuum Hypothesis in* [5].

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