OMITTING TYPES IN LOGIC OF METRIC STRUCTURES

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Abstract. This paper is about omitting types in logic of metric structures introduced by Ben Yaacov, Berenstein, Henson and Usvyatsov. While a complete type is omissible in some model of a countable complete theory if and only if it is not principal, this is not true for the incomplete types by a result of Ben Yaacov. We prove that there is no simple test for determining whether a type is omissible in a model of a theory $\mathbf{T}$ in a countable language. More precisely, we find a theory in a countable language such that the set of types omissible in some of its models is a complete $\mathbf{\Sigma}^1_2$ set and a complete theory in a countable language such that the set of types omissible in some of its models is a complete $\mathbf{\Pi}^1_1$ set. Two more unexpected examples are given: (i) a complete theory $\mathbf{T}$ and a countable set of types such that each of its finite sets is jointly omissible in a model of $\mathbf{T}$, but the whole set is not and (ii) a complete theory and two types that are separately omissible, but not jointly omissible, in its models.

The Omitting Types Theorem is one of the most useful methods for constructing models of first-order theories with prescribed properties (see [28], [25], or any general text in model theory). It implies, among other facts, the following (here $S_n(\mathbf{T})$ denotes the space of complete n-types in theory $\mathbf{T}$).

(1) If $\mathbf{T}$ is a theory in a countable language, then the set of all n-types realized in every model of $\mathbf{T}$ is Borel in the logic topology on $S_n(\mathbf{T})$.
(2) If $\mathbf{T}$ is in addition complete, then any sequence $\mathbf{t}_n$, for $n \in \omega$, of types each of which can be omitted in a model of $\mathbf{T}$ can be simultaneously omitted in a model of $\mathbf{T}$.

Types $\mathbf{t}_n$ appearing in (2) are not required to be complete, but the theory $\mathbf{T}$ is.

While the standard omissibility criterion for a given type in some model of a given theory in classical logic applies regardless of whether the type is complete or not, situation in logic of metric structures is a bit more subtle.

The omitting types theorem in logic of metric structures ([2, §12] or [23, Lecture 4]) has following straightforward consequences (see Proposition 1.11 for a proof of (3) and Corollary 4.2 for a proof of (4)).

(3) If $\mathbf{T}$ is a theory in a countable language of logic of metric structures, then the set of all *complete* n-types realized in every model of $\mathbf{T}$ is Borel in the logic topology on $S_n(\mathbf{T})$.

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(4) If $\mathbf{T}$ is moreover complete, then any sequence $\mathbf{t}_n$, for $n \in \omega$, of *complete* types each of which can be omitted in a model of $\mathbf{T}$ can be simultaneously omitted in a model of $\mathbf{T}$.

Examples constructed by I. Ben-Yaacov ([1]) and T. Bice ([6]) demonstrate that omitting partial types in logic of metric structures is inherently more complicated than omitting complete types. Our results, expressed using descriptive set theory, show that the problem of omitting types in logic of metric structures is essentially intractable.

Theorem 1. (5) *There is a complete theory $\mathbf{T}$ in a countable language such that the set of all types omissible over a model of $\mathbf{T}$ is $\mathbf{\Pi}^1_1$-complete.*

(6) *There is a theory $\mathbf{T}$ in a countable language such that the set of all types omissible in a model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$-complete.*

(7) *There is a separable structure M in a countable language such that the set of all partial types omitted in M is a complete $\mathbf{\Pi}^1_1$ set.*

Proof. (5) is proved in Theorem 2.5, (6) is proved in Theorem 2.8, and (7) is Corollary 2.6. □

We also show that (2) fails in logic of metric structures (as customary in logic, ω denotes the least infinite ordinal identified with the set of natural numbers).

Theorem 2. *There are a complete theory $\mathbf{T}$ in a countable language and types $\mathbf{s}_n$, for $n \in \omega$, such that for every k there exists a model of $\mathbf{T}$ that omits all $\mathbf{s}_n$ for $n \leq k$ but no model of $\mathbf{T}$ simultaneously omits all $\mathbf{s}_n$.*

Proof. This is proved in §6. □

Theorem 2 should be compared to a consequence of [3, Corollary 4.7]: under certain additional conditions any countable set of types that are not omissible in a model of a complete separable theory $\mathbf{T}$ has a finite subset consisting of types that are not omissible in a model of $\mathbf{T}$.

The following gives an another striking example of a peculiar behaviour of types in the logic of metric structures.

Theorem 3. *There are a complete theory $\mathbf{T}$ in a countable language L and types $\mathbf{s}$ and $\mathbf{t}$ omissible in models of $\mathbf{T}$ such that no model of $\mathbf{T}$ simultaneously omits both $\mathbf{s}$ and $\mathbf{t}$.*

Proof. This is proved in §6. □

Following [2] we write $r \dot{-} s := \max(0, r - s)$.

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1. Preliminaries

We assume that the reader is acquainted with the logic of metric structures ([2], [23]). We strictly follow the outline of this logic given in [2]. In particular, all metric structures are required to have diameter 1 and all formulas are $[0,1]$-valued. All function and predicate symbols are equipped with a fixed modulus of uniform continuity. Every structure is a complete metric space in which interpretations of functional and relational symbols respect this modulus. It is a straightforward exercise to see that our results apply both to the modification of this logic adapted to operator algebras ([17]) and the unbounded variations of the logic of metric structures.

For a formula $\varphi(\bar{x})$, a structure of the same language M and a tuple $\bar{a}$ in M, the interpretation of $\varphi(\bar{x})$ at $\bar{a}$ in M is denoted $\varphi(\bar{a})^M$. We define the theory of a structure to be the set of sentences

$$\mathrm{Th}(M) := \{\varphi : \varphi^M = 0\}.$$

Since every constant scalar function is a formula, the evaluation functional $\varphi \mapsto \varphi^M$ is uniquely determined by its kernel. It is therefore possible, and often convenient, to consider this functional as the theory of M. This is similar to the treatment in [16, §2.2], with one (inconsequential) difference. Formulas in [16] are $\mathbb{R}$-valued and therefore form a vector space. In our presentation the formulas are $[0,1]$-valued and form an affine subspace of the said vector space.

We shall tacitly use completeness theorem for the logic of metric structures whenever convenient ([5]).

1.1. **Conditions and types.** A *closed condition* is an expression of the form $\varphi(\bar{x}) = 0$ for a formula $\varphi(\bar{x})$ and *type* is a set of closed conditions. (Open conditions will be defined after Lemma 1.1 below.) Type $\mathbf{t}$ is *realized* in a structure M if there is a tuple $\bar{a}$ in M of the appropriate sort such that $\varphi(\bar{a})^M = 0$ for all conditions $\varphi(\bar{x}) = 0$ in $\mathbf{t}$. Type $\mathbf{t}$ is *consistent with theory* $\mathbf{T}$ if it is realized in a model of $\mathbf{T}$. By a compactness argument, this is equivalent to every finite subset of $\mathbf{t}$ being approximately realizable in a model of $\mathbf{T}$. If free variables of every formula appearing in $\mathbf{t}$ are included in $\{x_0, \dots, x_{n-1}\}$ and n is minimal with this property, we say that $\mathbf{t}$ is an *n-type*.

A type is *complete* if it is equal to the set of all closed conditions satisfied by a tuple in some metric structure. A fragment of a complete type is an *incomplete* type.

Throughout the following discussion L, $\mathbf{T}$, and M denote a language, an L-theory, and an L-structure. For simplicity of notation we shall assume that the language L is single-sorted. Generalizations of our results to multi-sorted language are straightforward.

1.1.1. *Continuous functional calculus.* Every L-formula $\varphi(\bar{x})$ has a modulus of uniform continuity and the set R_φ of all possible values of φ in all L-structures is a compact subset of $[0,1]$ ([2]). If $\varphi(\bar{x})$ is a formula and $f\colon R_\varphi \to [0,1]$ is continuous then $f(\varphi(\bar{x}))$ is a formula.

We shall consider generalized conditions of the form $\varphi \in K$, where $K \subseteq [0,1]$. Two generalized conditions $\varphi(\bar{x}) \in K$ and $\psi(\bar{x}) \in M$ are *equivalent* if in every L-structure A for every $\bar{a}$ of the appropriate sort one has $\varphi(\bar{a})^A \in K$ if and only if $\psi(\bar{a})^A \in M$.

Lemma 1.1. *Consider $K \subseteq [0,1]$ and a generalized condition $\varphi(\bar{x}) \in K$.*

(1) *If K is closed then $\varphi(\bar{x}) \in K$ is equivalent to a closed condition.*
(2) *If K is open then $\varphi(\bar{x}) \in K$ is equivalent to a condition of the form $\varphi(\bar{x}) < 1$.*

Proof. Since $[0,1]$ is compact and metric, by the Tietze extension theorem every closed subset is a zero set. Therefore if K is closed then we have a continuous $f\colon R_\psi \to [0,1]$ such that $f^{-1}(\{0\}) = K$. If K is open then we can choose f so that $f^{-1}(\{1\}) = [0,1] \setminus K$. In either case $\varphi = f(\psi)$ is as required. $\square$

Lema 1.1 implies that every set of closed conditions is equivalent to a type, and we shall slightly abuse the language and refer to sets of closed conditions as types.

1.1.2. *Pairing types.* The reader will excuse us for making some easy observations for future reference.

Lemma 1.2. *If $\mathbf{t}$ and $\mathbf{s}$ are types over a consistent and complete theory then there are types $\mathbf{t} \wedge \mathbf{s}$ and $\mathbf{t} \vee \mathbf{s}$ such that for every $M \models \mathbf{T}$ we have that*

(1) *M omits $\mathbf{t} \vee \mathbf{s}$ if and only if it omits both $\mathbf{t}$ and $\mathbf{s}$,*
(2) *M omits $\mathbf{t} \wedge \mathbf{s}$ if and only if it omits at least one of $\mathbf{t}$ or $\mathbf{s}$.*

Proof. By renaming the free variables of $\mathbf{s}$ if necessary, we may assume that for some $m \geq 1$ and $k \geq 1$ types $\mathbf{t}$ and $\mathbf{s}$ are an m-type and a k-type, respectively, in disjoint sets of variables.

By adding dummy conditions to $\mathbf{t}$ and/or $\mathbf{s}$ we may assume that the types are of the same cardinality κ. Enumerate conditions in $\mathbf{t}$ as $\varphi_\xi(\bar{x}) = 0$ for

$\xi < \kappa$ and conditions in $\mathbf{s}$ as $\psi_\xi(\bar y) = 0$ for $\xi < \kappa$. For a finite $F \subseteq \kappa$ let

$$\theta_F(\bar x, \bar y) := \max_{\xi \in F}(\varphi_\xi(\bar x), \psi_\xi(\bar y)),$$
$$\zeta_F(\bar x, \bar y) := \min\{\max_{\xi \in F} \varphi_\xi(\bar x), \max_{\xi \in F} \psi_\xi(\bar y)\}.$$

Let

$$\mathbf{t} \wedge \mathbf{s} := \{\theta_F : F \subseteq \kappa, F \text{ finite}\},$$
$$\mathbf{t} \vee \mathbf{s} := \{\zeta_F : F \subseteq \kappa, F \text{ finite}\}.$$

Fix $M \models \mathbf{T}$ and an $m+k$-tuple $\bar a, \bar b$ in M. This tuple realizes $\mathbf{t} \wedge \mathbf{s}$ if and only if for all ξ we have $\varphi_\xi(\bar a) = 0 = \psi_\xi(\bar b)$. This is equivalent to $\bar a$ realizing $\mathbf{t}$ and $\bar b$ realizing $\mathbf{s}$. Since the $m+k$-tuple $\bar a, \bar b$ was arbitrary, M realizes $\mathbf{t} \wedge \mathbf{s}$ if and only if it realizes both $\mathbf{t}$ and $\mathbf{s}$.

On the other hand, tuple $\bar a, \bar b$ realizes $\mathbf{t} \vee \mathbf{s}$ if and only if for every finite $F \subseteq \kappa$ at least one of (i) $\max_{\xi \in F} \varphi_\xi(\bar a) = 0$ or (ii) $\max_{\xi \in F} \psi_\xi(\bar b) = 0$ holds. If for a cofinal set of finite subsets F of κ (i) applies then $\bar a$ realizes $\mathbf{t}$. Otherwise, there is a finite $F_0 \subseteq \kappa$ such that for all $F \supseteq F_0$ (ii) applies and $\bar b$ realizes $\mathbf{s}$. Since this tuple was arbitrary, M realizes $\mathbf{t} \vee \mathbf{s}$ if and only if it realizes at least one of the types $\mathbf{t}$ or $\mathbf{s}$. □

1.1.3. *Type* $\mathbf{t}_\omega$. Assume $\mathbf{T}$ is a theory and $\mathbf{t} = \{\varphi_j(\bar x) = 0 : j \in \omega\}$ is an n-type omissible in a model of $\mathbf{T}$. We shall assume $n = 1$ for simplicity. To $\mathbf{t}$ we associate the type $\mathbf{t}_\omega$ in infinitely many variables x_j, for $j \in \omega$, consisting of the following generalized conditions.

($\mathbf{t}_\omega 1$) $\varphi_j(x_n) \leq \frac{1}{n}$ for all $j < n$, and

($\mathbf{t}_\omega 2$) $d(x_j, x_{j+1}) \leq 2^{-j}$ for all $j \in \omega$.

We can think of $\mathbf{t}_\omega$ as an ω-type—an increasing union of n-types $\mathbf{t}_n$, where $\mathbf{t}_n$ is the restriction of $\mathbf{t}_\omega$ to the formulas involving only x_j, for $j < n$. The following is clear.

Lemma 1.3. *A model M realizes type $\mathbf{t}$ if and only if every (equivalently, some) dense subset D of its universe includes a sequence that realizes $\mathbf{t}_\omega$.* □

A slightly finer fact is true. If D is an arbitrary dense subset of the universe of M, consider $D^{<\omega}$ as a tree with respect to the end-extension. Let $T_{D,\mathbf{t}}$ be the family of all $\bar d \in D^{<\omega}$ such that, with n denoting the length of $\bar d$ (using von Neumann's convention that $n = \{0, 1, \dots, n-1\}$, this is the $n \in \omega$ such that $\mathrm{dom}(d) = n$), we have $M \models \mathbf{t}_n(\bar d)$. We use common terminology from descriptive set theory and say that a tree is *well-founded* if it has no infinite branches.

Lemma 1.4. *A model M omits a type $\mathbf{t}$ if and only if the tree $T_{D,\mathbf{t}}$ is well-founded for some (equivalently, for every) dense $D \subseteq M$.*

Proof. A proof virtually identical to the proof of Lemma 1.3 shows that a tuple $\bar a$ realizes $\mathbf{t}$ if and only if it is a limit of nodes on an infinite branch of $T_{D,\mathbf{t}}$. □

1.2. **Spaces.** For a fixed countable language L (see §1.2.2 below) we shall now review definitions of standard Borel spaces of formulas, structures, complete types, and incomplete types.

1.2.1. *The Borel space* $\hat{\mathcal{M}}(L)$ *of models.* Fix a countable language L. There are several ways to encode separable L-structures. We shall consider the space essentially introduced in [4, p. 2]. Although this space was denoted $\mathcal{M}(L)$ in [4], we use the notation $\hat{\mathcal{M}}(L)$ to avoid conflict with [12]. The space $\hat{\mathcal{M}}(L)$ is defined as follows.

For simplicity we consider the case when L has no predicate symbols. Let d_j, for $j \in \omega$, be a sequence of new constant symbols and let $L^+ = L \cup \{d_j : j \in \omega\}$. Let $\mathfrak{p}_j$, for $j \in \omega$, be an enumeration of all L^+-terms with no free varlables closed under the application of function symbols from L. Recall that in logic of metric structures every function symbol f is equipped with a fixed modulus of uniform continuity $\Delta_f \colon (0,1] \to (0,1]$ and that in every L-structure M it is required that the representation of f satisfies (using the max-metric on tuples in M)

$$d(\bar{x}, \bar{y}) < \Delta_f(\varepsilon) \text{ implies } d(f(\bar{x}), f(\bar{y})) \leq \varepsilon.$$

All relational symbols are also equipped with moduli of uniform continuity. By $\hat{\mathcal{M}}(L)$ we denote the space of all functions

$$\gamma \colon \omega^2 \to [0,1]$$

such that

(i) γ is a metric on ω,
(ii) γ respects the moduli of uniform continuity of all functions in L. In particular [2, (UC), p. 8] holds: if f is a function symbol with modulus of uniform continuity Δ_f and i, j, i' and j' are such that $\mathfrak{p}_{j'} = f(\mathfrak{p}_j)$ and $\mathfrak{p}_{i'} = f(\mathfrak{p}_i)$ then

$$\gamma(i,j) < \Delta_f(\varepsilon) \text{ implies } \gamma(i', j') \leq \varepsilon.$$

(An analogous condition holds for n-ary function symbols for $n \geq 2$.)

The set of $\gamma \in [0,1]^{\omega^2}$ satisfying (i) and (ii) is a closed subspace of the Hilbert cube, and $\hat{\mathcal{M}}(L)$ is equipped with the induced compact metric topology. To $\gamma \in \hat{\mathcal{M}}(L)$ we associate the structure $M_0(\gamma)$ with the universe ω and the metric defined by $d(i,j) := \gamma(i,j)$. Terms $\mathfrak{p}_j$, for $j \in \omega$ (and therefore function symbols) are interpreted in $M_0(\gamma)$ using (ii). This structure is not an L-structure because it is incomplete, and its metric completion $M(\gamma)$ is a separable L-structure. Every complete separable metric L-structure M is isometric to a completion of $M_0(\gamma)$ corresponding to some $\gamma \in \hat{\mathcal{M}}(L)$ and every such M has many nonisomorphic representations in $\hat{\mathcal{M}}(L)$. (Necessarily so, because the isomorphism relation of L-structures is frequently not classifiable by countable structures; see [24].)

One can modify $\hat{\mathcal{M}}(L)$ to accommodate the case when L has countably many predicate symbols. For an n-ary predicate symbol R let $\gamma_R \colon \omega^n \to$

$[0,1]$ correspond to the interpretation of R in $M(\gamma)$. If $R(j)$ and $k(j) \in \omega$, for $j \in \omega$, enumerate the relational symbols in L and their arities then $M(\gamma)$ is identified with a closed subset of the Hilbert cube $[0,1]^{\omega^2} \times \prod_{j<\omega} [0,1]^{\omega^{k(j)}}$ in the natural fashion.

Remark 1.5. The space $\hat{\mathcal{M}}(L)$ is similar to the space of separable C*-algebras $\hat{\Gamma}$ introduced in [18]. Although $\hat{\mathcal{M}}(L)$ is different from the Borel space of L-structures $\mathcal{M}(L)$ defined in [12], these two spaces are equivalent in the sense of [19, Definition 2.1] by [19, Proposition 2.6 and Proposition 2.7]. The proof of this fact is analogous to the proof given in [12, §3] for the case of C*-algebras.

A special case of the following lemma in the case of C*-algebras was proved in [18, Proposition 5.1]. The proof of the general case is virtually identical.

Lemma 1.6. *The function from $\hat{\mathcal{M}}(L)$ to the space of L-theories that associates the theory of M to M is Borel.* □

Lemma 1.7 below is a straightforward consequence of Lemma 1.6 (similar to the one in the appendix of [18], where it was proved that separable C*-algebras that tensorially absorb the Jiang–Su algebra $\mathcal{Z}$ form a Borel set.

Lemma 1.7. *Suppose L is a countable language. If $\mathbf{T}$ is an L-theory then the set of all $\gamma \in \hat{\mathcal{M}}(L)$ such that $M(\gamma) \models \mathbf{T}$ is Borel.* □

1.2.2. *The linear space of formulas.* For $n \in \omega$ let $\mathbb{F}_n(L)$ denote the set of all formulas whose free variables are among $\{x_0, \dots, x_{n-1}\}$. If L is clear from the context we shall write $\mathbb{F}_n$ instead of $\mathbb{F}_n(L)$. On the affine space $\mathbb{F}_n(L)$ consider the seminorm

$$\|\varphi\|_\infty := \sup_{M,\vec{a}} |\varphi(\vec{a})^M|$$

and the pseudometric

$$d_\infty(\varphi, \psi) := \|\varphi - \psi\|_\infty.$$

Here M ranges over all L-structures, $\vec{a}$ ranges over all n-tuples of elements in M, and $\varphi(\vec{a})^M$ is the interpretation of φ at $\vec{a}$ in M.

Let $\mathfrak{W}_n(L)$ denote the completion of the quotient space $\mathbb{F}_n(L)/\|\cdot\|_\infty$. While in [16, §6] formulas are $\mathbb{R}$-valued and the analogous space is a real Banach space, our $\mathfrak{W}_n(L)$ is an affine space. Given an L-theory $\mathbf{T}$ consider the seminorm

$$\|\varphi\|_{\mathbf{T}} := \sup_{M,\vec{a}} |\varphi(\vec{a})^M|,$$

where M ranges over the models of $\mathbf{T}$ and $\vec{a}$ ranges over n-tuples of elements of M. To $\|\cdot\|_{\mathbf{T}}$ we associate a pseudometric $d_{\mathbf{T}}$ and an affine space $\mathfrak{W}_n(\mathbf{T})$ defined analogously to d_∞ and $\mathfrak{W}_n(L)$ above.

As pointed out in [2], if L is countable then $\mathfrak{W}_n(L)$ is separable with respect to this pseudometric. In some situations one may implicitly consider theory as inseparable (no pun intended) from the language L. For example, if L has a function symbol associated to every complex number (as e.g. in the axiomatization of C*-algebras given in [17]) then L is uncountable. However, $\mathbf{T}$ may still imply that $\mathfrak{W}_0(L)$ is separable. In this case we say that L *is separable.* The separability of L is assumed throughout the present section (and most of the paper).

1.2.3. *Compact metrizable spaces of theories and types.* Let L be a countable language. Consider the affine spaces of formulas $\mathbb{F}_n(L)$ and $\mathfrak{W}_n(L)$ as defined in §1.2.2. Every L-structure M defines the evaluation linear functional on $\mathfrak{W}_0$ by

$$\varphi \mapsto \varphi^M.$$

Since $\mathfrak{W}_0(L)$ is normed by $\|\varphi\|_\infty = \sup_M |\varphi^M|$ (see §1.2.2), every evaluation functional has norm ≤ 1. Such functionals are *complete L-theories* (see §1). The space of all complete L-theories is denoted S_0^L and equipped with the topology of pointwise convergence (also known as the *logic topology* or the *weak*-topology*). By the compactness theorem for the logic of metric structures, S_0^L is compact (see [2, Theorem 5.8]). Since L is separable ${}^L_0$ is metrizable, although it does not have a canonical metric.

Suppose an incomplete L-theory $\mathbf{T}$ is fixed (possibly $\mathbf{T} = \emptyset$). Every complete extension $\mathbf{T}'$ of $\mathbf{T}$ defines a continuous affine functional on $\mathfrak{W}_0(\mathbf{T})$ and hence an element of S_0^L. The set of functionals associated to complete extensions of $\mathbf{T}$ in this manner is, by the compactness theorem for the logic of metric structures, a closed subset of S_0^L (see also [16, Lemma 6.1.1 and Lemma 6.1.2]). Since each functional in this set uniquely determines a complete extension of $\mathbf{T}$, we identify this compact metrizable space with the space of complete extensions of $\mathbf{T}$ and denote it $S_0^L(\mathbf{T})$. We write $S_0(\mathbf{T})$ whenever L is clear from the context.

An L-model M and an n-tuple a_i, for $i < n$, in M define by interpretation a linear functional $\mathrm{tp}_M(\bar{a})$ on $\mathfrak{W}_n(L)$ by

$$\mathrm{tp}_M(\bar{a})(\varphi(\bar{x})) = \varphi(\bar{a})^M.$$

The space $S_n(\mathbf{T})$ of all consistent complete n-types is compact in the logic topology, and we identify it with the space of (consistent and complete) n-types over $\mathbf{T}$.

The *set of realizations* of n-ary type $\mathbf{t}$ in model M is

$$\mathbf{t}(M) = \{\bar{b} \in M^n : M \models \mathbf{t}(\bar{b})\}.$$

Completeness of $\mathbf{T}$ implies that for every $M \models \mathbf{T}$ the set

$$\{\mathbf{t} \in S_n(\mathbf{T}) : \mathbf{t} \text{ is realized in } M\}$$

is dense in the logic topology on $S_n(\mathbf{T})$.

1.2.4. *A metric on the space of complete n-types over a complete theory* $\mathbf{T}$. Let L be a countable language, let $\mathbf{T}$ be a complete L-theory, and fix $n \geq 1$.

Following ([2, p. 44]) on the space $S_n(\mathbf{T})$ of complete n-types over $\mathbf{T}$ (§1.2.3) we define metric $d = d_{\mathbf{T},n}$ by (M^n is considered with the max-distance)

$$d(\mathbf{t},\mathbf{s}) = \inf\{d(\bar a,\bar b) : \text{ there exist } M \models T \text{ and } \bar a \text{ and } \bar b \text{ in } M \\ \text{such that } M \models \mathbf{t}(\bar a) \text{ and } M \models \mathbf{s}(\bar b)\}.$$

Since both types and $\mathbf{T}$ are complete, the triangle inequality is satisfied. Also, $d(\mathbf{t},\mathbf{s}) = 0$ if and only if $\mathbf{s}(M) = \mathbf{t}(M)$ for every $M \models \mathbf{T}$, in which case $\mathbf{s}$ and $\mathbf{t}$ correspond to the same functional on $\mathfrak{W}_n(\mathbf{T})$. This topology is stronger than the compact topology and these topologies roughly correspond to the norm and weak topologies in functional analysis (see however Lemma 1.8).

Given an n-type $\mathbf{t}$ over a theory $\mathbf{T}$ and a new n-tuple of constants $\bar c$, we let $\mathbf{T}_{\mathbf{t}/\bar c}$ denote the theory in the language $L \cup \{\bar c\}$ obtained by extending $\mathbf{T}$ with axioms asserting that $\bar c$ realizes $\mathbf{t}$. More precisely, one adds all conditions of the form $\varphi(\bar c) = 0$ to $\mathbf{T}$, where $\varphi(\bar x) = 0$ is a condition in $\mathbf{t}$. If $\mathbf{t}$ is complete then so is $\mathbf{T}_{\mathbf{t}/\bar c}$. One can iterate this definition and name realizations of more than one type, as in the following proof.

Lemma 1.8. *For every n and $\varepsilon \geq 0$ the set $\{(\mathbf{r},\mathbf{s}) : d(\mathbf{r},\mathbf{s}) > \varepsilon\} \subseteq S_n(\mathbf{T})^2$ is open in the logic topology.*

Proof. Fix types $\mathbf{t}$ and $\mathbf{s}$ such that $d(\mathbf{t},\mathbf{s}) > \varepsilon$. This is equivalent to stating that

$$\mathbf{T}_{\mathbf{t}/\bar c,\mathbf{s}/\bar d} \models d(\bar c,\bar d) > \varepsilon.$$

Then by compactness there exists a finite set of open conditions $\mathbf{T}_0$ in $\mathbf{T}_{\mathbf{t}/\bar c,\mathbf{s}/\bar d}$ such that $\mathbf{T}_0 \models d(\bar c,\bar d) > \varepsilon$. This defines a logic open neighbourhood U of $(\mathbf{t},\mathbf{s})$ in $S_n(\mathbf{T})$ such that $d(\mathbf{r},\mathbf{s}) > \varepsilon$ for all $(\mathbf{r},\mathbf{s}) \in U$. $\square$

1.2.5. *The compact metrizable space $S_n^-(\mathbf{T})$ of incomplete n-types over* $\mathbf{T}$. Fix a countable language L, $n \geq 1$ and a (not necessarily complete) L-theory $\mathbf{T}$. An n-type $\mathbf{t}$ in $\mathbf{T}$ is a countable set of conditions (§1.1), but we can also identify it with the set (considering a type as a set of conditions)

$$K_{\mathbf{t}} := \{\mathbf{s} \in S_n(\mathbf{T}) : \mathbf{t} \subseteq \mathbf{s}\}.$$

This set is closed (and therefore compact) in the logic topology. We claim that every nonempty compact $K \subseteq S_n(\mathbf{T})$ is equal to $K_{\mathbf{t}}$ for some $\mathbf{t}$. Fix K; then

$$\mathbf{t}_K := \bigcap_{\mathbf{s}\in K} \mathbf{s}$$

is a type that includes $\mathbf{T}$. If $\mathbf{s} \in S_n(\mathbf{T}) \setminus K$, then since K is closed there exists an open condition $\varphi(\bar x) < \varepsilon$ such that the closed condition $\varphi(\bar x) \leq \varepsilon/2$ belongs to $\mathbf{s}$ but not to any type in K. Therefore no type $\mathbf{s} \notin K$ extends $\mathbf{t}_K$ and we have

$$K = \{\mathbf{s} \in S_n(\mathbf{T}) : \mathbf{s} \supseteq \mathbf{t}_K\}.$$

We can therefore identify the space $S_n^-(\mathbf{T})$ of not necessarily complete types over $\mathbf{T}$ with the space of all closed subsets of $S_n(\mathbf{T})$ (the so-called exponential space) equipped with its compact metrizable topology given by the Hausdorff metric. (Note that the empty set corresponds to all inconsistent types extending $\mathbf{t}$.)

We identify $S_n(\mathbf{T})$ with a subset of $S_n^-(\mathbf{T})$ consisting of all singletons. This is a closed subset of $S_n^-(\mathbf{T})$ and the subspace topology agrees with the compact metrizable topology on $S_n(\mathbf{T})$. When discussing complexity of a set of types over $\mathbf{T}$ we consider the set of all types, complete and incomplete.

A straightforward proof of the following lemma is omitted (see Lemma 1.2 for the definitions of types $\mathbf{s} \vee \mathbf{t}$ and $\mathbf{s} \wedge \mathbf{t}$).

Lemma 1.9. *Suppose $\mathbf{T}$ is a theory in a countable language. For all k and m the type pairing maps*

$$\vee\colon S_m^-(\mathbf{T}) \times S_k^-(\mathbf{T}) \to S_{m+k}^-(\mathbf{T})$$
$$\wedge\colon S_m^-(\mathbf{T}) \times S_k^-(\mathbf{T}) \to S_{m+k}^-(\mathbf{T})$$

are continuous. □

1.2.6. *The compact metrizable space of pairs* $(\mathbf{T}, \mathbf{t})$. Fix a countable language L. For a fixed n the set of pairs $(\mathbf{T}, \mathbf{t})$ where $\mathbf{T}$ is a (not necessarily complete) L-theory and $\mathbf{t}$ is a complete type over $\mathbf{T}$ is endowed with a compact metrizable topology as follows. We identify each pair $(\mathbf{T}, \mathbf{t})$ with the complete theory $\mathbf{T}_{\mathbf{t}/\bar{c}}$ (see §1.2.4). Every complete theory in the language obtained by extending the language of $\mathbf{T}$ by adding constants $\bar{c}$ is equal to $\mathbf{T}_{\mathbf{t}/\bar{c}}$ for some pair $(\mathbf{T}, \mathbf{t})$. Therefore the set

$$\{(\mathbf{T}, \mathbf{t}) : \mathbf{T} \text{ is a not necessarily complete } L\text{-theory and } \mathbf{t} \in S_n(\mathbf{T})\}$$

is identified with the compact metrizable space $S_0^{L \cup \{\bar{c}\}}(\mathbf{T})$ as in §1.2.3.

1.3. **Omitting complete types.** Assume $\mathbf{T}$ is a complete theory in a countable language and $\mathbf{t}$ is a (not necessarily complete) n-type in the signature of $\mathbf{T}$. As in §1.2.5, we identify $\mathbf{t}$ with the set $K_{\mathbf{t}}$ of all complete types extending $\mathbf{t}$. An n-type (complete or not) $\mathbf{t}$ is *principal* (or *isolated*) if for every $\varepsilon > 0$ the set (using the metric on $S_n(\mathbf{T})$ defined in §1.2.4)

$$B_\varepsilon(\mathbf{t}) := \{\mathbf{s} \in S_n(\mathbf{T}) : \operatorname{dist}(\mathbf{s}, K_{\mathbf{t}}) < \varepsilon\}$$

is somewhere dense in the logic topology with respect to $\mathbf{T}$. This is equivalent to the definition of a principal type given in [2, Definition 12.2] by [2, Proposition 12.5]. The following is the omitting types theorem given in [2, Theorem 12.6] (note that in [2] all types were assumed to be complete).

Theorem 1.10. *Suppose $\mathbf{t}$ is a type over a theory $\mathbf{T}$ in a countable language.*

1. *If $\mathbf{t}$ is not principal, then $\mathbf{T}$ has a separable model omitting $\mathbf{T}$.*
2. *If $\mathbf{t}$ is principal and both $\mathbf{t}$ and $\mathbf{T}$ are complete then $\mathbf{t}$ is realized in every $M \models \mathbf{T}$.* □

This paper is about the case not covered by Theorem 1.10: principal, but not complete, types over a (possibly complete) theory in a countable language.

1.3.1. *The set of omissible complete types is Borel.* The set of n-types $\mathbf{t}$ omissible in a model of a theory $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$ (see Proposition 2.2). We consider the logic topology on the space of all complete theories in a fixed countable language L (§1.2.3). For $n \in \omega$ consider the space of pairs $(\mathbf{T}, \mathbf{t})$ where $\mathbf{T}$ is an L-theory and $\mathbf{t}$ is a complete n-type over $\mathbf{T}$ with respect to the logic topology defined in §1.2.6.

Proposition 1.11. *Fix $n \in \omega$.*

(1) *The set of all pairs $(\mathbf{T}, \mathbf{t})$ such that $\mathbf{T}$ is a complete theory and $\mathbf{t}$ is a complete n-type realized in every model of $\mathbf{T}$ is Borel.*
(2) *The set of all pairs $(\mathbf{T}, \mathbf{t})$ such that $\mathbf{T}$ is a theory and $\mathbf{t}$ is a not necessarily complete n-type realized in some model of $\mathbf{T}$ is closed.*

Proof. (1) By the Omitting Types Theorem ([2, §12], [23], or Corollary 4.2) type $\mathbf{t}$ has to be realized in every model of $\mathbf{T}$ if and only if it is principal (principal types were defined in §1.3).

Since the logic topology is second countable, expressing the fact that $B_\varepsilon(\mathbf{t})$ is nowhere dense requires only quantification over a countable set of open conditions. It therefore suffices to show that the set $\{\mathbf{s} : d(\mathbf{t}, \mathbf{s}) \geq \varepsilon\}$ is Borel, and this follows from Lemma 1.8.

(2) This is a consequence of the compactness theorem, [2, Theorem 5.8]. □

The completeness assumption on types in Proposition 1.11 (1) is necessary by Theorem 1, but see also Proposition 5.7.

1.4. **A test for elementary equivalence.** We include a general test for elementary equivalence used only in §6 below. A subset Y of a metric space is *ε-dense* if for every point $x \in X$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$.

Lemma 1.12. *Assume A and B are L-structures such that for every finite $L_0 \subseteq L$ and every $\varepsilon > 0$ there are ε-dense substructures A_0 and B_0 of L_0-reducts of A and B, respectively, which are isomorphic. Then A and B are elementarily equivalent.*

Proof. By [2, Proposition 6.9] every formula can be uniformly approximated by formulas in prenex form. It will therefore suffice to show that every formula in prenex form has the same value in A and B. Let φ be an L-sentence in prenex form,

$$\varphi := \sup_{x_0} \inf_{x_1} \dots \sup_{x_{2n-2}} \inf_{x_{2n-1}} \psi(\bar{x}),$$

where $\psi(\bar{x})$ is quantifier-free and let $L_0 \subseteq L$ be a finite subset consisting only of symbols that appear in ψ. For $\delta > 0$ fix $\varepsilon > 0$ small enough so that perturbing variables in $\bar{x}$ by $\leq \varepsilon$ does not change the value of $\psi(\bar{x})$ by more than $\delta/2$. Let A_0 and B_0 be isomorphic ε-dense substructures of reducts of A and B, respectively. Then

$$|\varphi^A - \varphi^{A_0}| \leq \delta/2 \qquad \text{and} \qquad |\varphi^B - \varphi^{B_0}| \leq \delta/2.$$

Since $A_0 \cong B_0$ we have $\varphi^{A_0} = \varphi^{A_0}$ and $|\varphi^A - \varphi^B| \leq \delta$. Since $\delta > 0$ was arbitrary, we conclude that $\varphi^A = \varphi^B$. Since φ was arbitrary A and B are elementarily equivalent. □

1.5. **A simple fact about definable sets.** We include a brief discussion of definable sets as a response to the anonymous referee's request. Let L be a language and let M be an L-structure. Following [2, Definition 9.1] (see also [16, §3.1]) we say that a predicate $R\colon M \to [0,1]$ is *definable* if it can be uniformly approximated by L-formulas. More precisely, for every $\varepsilon > 0$ there is an L-formula $\varphi(x)$ such that $\sup_{x\in M} |R(x) - \varphi(x)^M| \leq \varepsilon$. A closed subset Z of M is *definable* ([2, Definition 9.16], also [16, §3.2]) if the associated distance predicate

$$\operatorname{dist}(x, Z) := \inf_{z\in Z} d(x, z)$$

is definable. In the case when M is a multi-sorted structure and Z is a subset of one of its sorts, S, one requires only $\sup_{x\in S^M} |\operatorname{dist}(x, Z) - \varphi(x)| \leq \varepsilon$. By [2, Theorem 9.17] (see also [16, Theorem 3.2.2] for the treatment of sets definable in a not necessarily complete theory) a set is definable if and only if for every L-formula $\psi(\bar{y}, x)$ the predicates

$$\sup_{x\in Z} \psi(\bar{y}, x) \text{ and } \inf_{x\in Z} \psi(\bar{y}, x)$$

are definable. In other words, expanding the definition of an L-formula by allowing quantification over definable sets results in a conservative extension of the language L. (An extensive discussion of definability in multi-sorted and unbounded case, imaginaries, $\mathbf{T}^{\mathrm{eq}}$, and A^{eq} in the logic of metric structures can be found in [16, §3.1–3.3].) We state and prove the single-sorted, single variable version of an (undoubtedly well-known) lemma that will be used in §6.

Lemma 1.13. *Suppose that M is a metric L-structure with a sort S and that $Z \subseteq S^M$ is the zero set of some formula $\varphi(x)$ such that 0 is an isolated point of the set $Y := \{\varphi(a) : a \in S^M\}$. Then both Z and $S \setminus Z$ are definable.*

Proof. Let $f\colon [0,1] \to [0,1]$ be any continuous function such that $f(0) = 0$ and $f(t) = 1$ for $t \in Y \setminus \{0\}$. Then $f(\varphi(a))^M$ is equal to 0 if $a \in Z$ and it is equal to 1 if $a \in S^M \setminus Z$. The formulas

$$\psi_0(x, z) := \inf_z \min(1, f(\varphi(z)) + d(x, z)),$$
$$\psi_1(x, z) := \inf_z \min(1, (1 - f(\varphi(z))) + d(x, z))$$

satisfy $\psi_0(a)^M = \operatorname{dist}(a, Z)$ and $\psi_1(a)^M = \operatorname{dist}(a, M \setminus Z)$ for all $a \in S^M$, showing that both Z and $S \setminus Z$ are definable. □

2. Complexity of spaces of types

From now on, *all types are assumed to be partial and consistent.* Let us briefly recall definitions of pointclasses in projective hierarchy. A subset of a Polish space X is $\mathbf{\Delta}^1_1$ if it is Borel (this is Souslin's Theorem, [26, 14.11]). The images of $\mathbf{\Delta}^1_1$ sets under Borel-measurable functions are $\mathbf{\Sigma}^1_1$ (or analytic) sets, and their complements are $\mathbf{\Pi}^1_1$ sets. For $n \geq 2$ one defines $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ subsets of X by recursion. Continuous images of $\mathbf{\Pi}^1_n$ sets are $\mathbf{\Sigma}^1_{n+1}$ sets and their complements are $\mathbf{\Pi}^1_{n+1}$ sets. This is a proper hierarchy and for every n the pointclasses $\mathbf{\Sigma}^1_{n+1} \setminus \mathbf{\Sigma}^1_n$, $\mathbf{\Sigma}^1_{n+1} \setminus \mathbf{\Pi}^1_{n+1}$, $\mathbf{\Pi}^1_{n+1} \setminus \mathbf{\Sigma}^1_{n+1}$, and $\mathbf{\Pi}^1_{n+1} \setminus \mathbf{\Pi}^1_n$ are nonempty. If Γ is a pointclass then a subset Z of a Polish space is Γ-*hard* if every set in Γ is a preimage of Z by a continuous function. If a Γ-hard set Z itself belongs to Γ then it is said to be Γ-*complete*. See [26] for more information.

Whenever we say that a type $\mathbf{t}$ is omissible in a model of $\mathbf{T}$ it is assumed to be consistent with $\mathbf{T}$.

2.1. Simple complexity results.

Recall that the space $S_n(\mathbf{T})$ of complete n-types over a complete theory $\mathbf{T}$ is a compact metric space (§1.2.3) and that the space $S^-_n(\mathbf{T})$ of not necessarily complete n-types over $\mathbf{T}$ is identified with the compact metrizable space of its closed subsets (§1.2.5).

Lemma 2.1. *Suppose L is a countable language, M is a separable L-structure, and $n \in \omega$.*

1. *The set of all complete n-types realized in M is $\mathbf{\Sigma}^1_1$.*
2. *The set of all not necessarily complete n-types realized in M is $\mathbf{\Sigma}^1_1$*

Proof. (1) Let $a(M, j)$, for $j \in \omega$, be an enumeration of a countable subset of the universe of M. Define $f\colon (\omega^\omega)^n \to S_n(\mathbf{T}) \cup \{*\}$ by setting $f(x(0), \dots, x(n-1))$ to be the type of the limit of $x(0)(j), \cdots x(n-1)(j)$, for $j \in \omega$, in M if this limit exists in M and to $*$ if the sequence $\{a(M, x(i)(j)) : j \in \omega\}$ is not Cauchy for some $i < n$. This function is Borel and the image of $(\omega^\omega)^n$ is the set of n-types realized in M.

(2) By (1), the set $A \subseteq S_n(\mathbf{T})$ of all complete n-types omitted in M is $\mathbf{\Pi}^1_1$. We need to show that the set $\{K \in S^-_n(\mathbf{T}) : K \subseteq A\}$ is also $\mathbf{\Pi}^1_1$. This is standard but we include an argument for the convenience of the reader. The set

$$Z = \{(x, K) \in S_n(\mathbf{T}) \times S^-_n(\mathbf{T}) : x \in K\}$$

is closed and $K \subseteq A$ if and only if $(\forall x)((x, K) \in Z \to x \in A)$, giving the required $\mathbf{\Pi}^1_1$ definition. □

The set of complete n-types realized in a model M of $\mathbf{T}$ is always dense in the logic topology on $S_n(\mathbf{T})$ and therefore if M does not realize all types then the set of complete types realized in M is not closed in logic topology. A model M of a complete theory $\mathbf{T}$ is *atomic* if the set of realizations of principal types is dense in M^n for every $n \geq 1$ (see [2, p. 79]). This is equivalent to every tuple of M^n having a principal type. Compare the following with Proposition 1.11.

Proposition 2.2. *Fix $n \geq 1$ and a countable language L.*

(1) *If $\mathbf{T}$ is a complete L-theory with an atomic model then the set of (not necessarily complete) n-types omissible in a model of $\mathbf{T}$ is $\mathbf{\Pi}^1_1$.*
(2) *If $\mathbf{T}$ is a (not necessarily complete) L-theory then the set of all (not necessarily complete) n-types omissible in a model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$.*

Proof. Suppose that $\mathbf{T}$ has an atomic model. Then it has a separable atomic model and this model is an elementary submodel of every other model of $\mathbf{T}$ (i.e. it is a *prime* model of $\mathbf{T}$). This implies that a type is omissible in a model of $\mathbf{T}$ if and only if it is omitted in its atomic model. Together with (2) of Lemma 2.1 this implies (1).

(2) By the proof of Lemma 2.1 the set

$$\{(M, \mathbf{t}) : M \models \mathbf{T}, \mathbf{t} \in S^-_n(\mathbf{T}), \text{ and } M \text{ omits } \mathbf{t}\}$$

is $\mathbf{\Pi}^1_1$ and the set of all types omitted in some model of $\mathbf{T}$ is clearly its continuous image. □

Proposition 2.2 (1) is sharp by Corollary 2.6 below. The following related result was inspired by [6].

Proposition 2.3. *There are a countable language L and a separable L-structure M such that the set of quantifier-free unary types realized in M is a complete $\mathbf{\Sigma}^1_1$ set.*

Proof. Language L has only one unary predicate symbol f, interpreted as a 1-Lipshitz function. Consider the Baire space ω^ω with a complete separable metric. We fix a homeomorphism of ω^ω with $[0,1] \setminus \mathbb{Q}$ (e.g. send an irrational to the associated continuous fraction). Fix a closed $X \subseteq \omega^\omega \times \omega^\omega$. Consider X with the max-metric induced from ω^ω and interpret f as the projection to the x-axis. Then f is 1-Lipshitz.

The only atomic formulas in L are $f(x)$ and $d(x, y)$. If $\varphi(x)$ is a quantifier-free formula with only one free variable, then the only atomic subformulas of φ are of the form $f(x)$ and $d(x, x)$. The latter is identically equal to 0, and therefore the quantifier-free type of an element of M is completely determined by its projection to the x-axis (with ω^ω identified with a subset of $[0, 1]$). Choosing X so that its projection is a complete analytic set (combine [26, §25.A] and [26, §27.B]) completes the proof. □

2.2. **Theory of the Baire space.** Let $L_{\mathcal{N}}$ be a language with a single sort D_1. The intended interpretation of D_1 is $\omega^{<\omega} \sqcup \omega^\omega$. Language $L_{\mathcal{N}}$ is equipped with the following.

(1) Constant symbols for all elements of $\omega^{<\omega}$ (we shall identify $t \in \omega^{<\omega}$ with the corresponding constant).
(2) Unary function symbols f_k for $k \in \omega$.

Interpretations of each f_k is required to be 1-Lipshitz. Theory $\mathbf{T}_{\mathcal{N}}$ is the theory of the $L_{\mathcal{N}}$-model $\mathcal{N}$ whose universe is $\omega^{<\omega} \sqcup \omega^\omega$, described as follows.

The length of $s \in \omega^{<\omega}$ will be denoted by $|s|$, hence $\mathrm{dom}(s) = |s|$. The metric on $\omega^{<\omega} \sqcup \omega^\omega$ is the standard Baire space metric (with $\Delta(s,t) = \min(|s|, |t|, \{n : s(n) \neq t(n)\}$ for $s \neq t$)

$$d(s,t) = 1/(\Delta(s,t) + 1).$$

If $s \sqsubseteq t$ then $d(s,t) = 1/(|s| + 1)$. For $s \in \omega^{<\omega}$ and $k \leq |s|$ we denote the initial segment of s with length k by $s \restriction k$. For $k \in \omega$ the function f_k is interpreted as

$$f_k(s) := \begin{cases} s \restriction k & \text{if } k \leq |s| \\ s & \text{if } k > |s|. \end{cases}$$

Clearly $\omega^{<\omega}$ is a dense subset of D_1^M which is closed under all f_k.

Since $\mathcal{N}$ has a subset consisting of elements that are interpretations of constant symbols, it is an atomic model of $\mathbf{T}_{\mathcal{N}}$ and every $N \models \mathbf{T}_{\mathcal{N}}$ has an elementary submodel isometrically isomorphic to $\mathcal{N}$.

2.3. **Well-foundedness and the language $L^-_{\mathcal{N}}$.** Let $L^-_{\mathcal{N}}$ be the reduct of $L_{\mathcal{N}}$ with sort D_1 and functions f_k for $k \in \omega$, but without constant symbols for the elements of $\omega^{<\omega}$. Suppose T is a tree of height ω. It is isomorphic to a tree of finite sequences of elements of a large enough set with respect to the end-extension ordering. An $L^-_{\mathcal{N}}$ structure is associated to T as in §2.2. One defines distance $d(s,t) := 1/(\Delta(s,t) + 1)$ for distinct elements s and t in T, and the interpretation of f_k is defined so that $f_k(s) = s$ if $|s| \leq k$ and $f_k(s) = s \restriction k$ if $|s| > k$. The d-completion of T is an $L^-_{\mathcal{N}}$-structure whose domain is naturally identified with the union of T and the set of its infinite branches. Let $\mathbf{T}_T$ be the theory of this $L^-_{\mathcal{N}}$-structure.

Every $N \models \mathbf{T}_T$ has an F_σ subset

$$T_N := \{a \in D_1^N : f_k(a) = a \text{ for some } k\}.$$

This is a dense subset of N since $d(a, f_n(a)) \leq 1/(n + 1)$ for all a. With the ordering defined by $a \sqsubseteq b$ if and only if $a = f_k(b)$ for some k, T_N is a tree of height ω. Moreover, the elements of $N \setminus T_N$ are in a natural bijective correspondence to the branches of this tree, because $f_k(x) = f_k(y)$ for all k implies $d(x,y) = 0$ and therefore $x = y$.

Any $N \models \mathbf{T}_T$ has at most two kinds of elements, *nodes* and *branches*. Ranges of interpretations of functions f_m, for $m \in \omega$, in N comprise *nodes* of N. If $f_m(a) = a$ then $d(a,b) < 1/m$ implies $a = b$, and therefore nodes form a discrete subset of N. The *height* of a node $a \in N$, denoted $|a|$, is the least m such that $f_m(a) = a$. By the elementarity, N has the unique node of height 0, denoted $\langle\rangle$.

All non-nodes of N satisfy $d(f_m(a), a) = 1/(m + 1)$ for all m. These are the *branches* of N. If $N \models \mathbf{T}_T$ then T_N is a tree and we naturally extend the notation $|t|$ and $\sqsubseteq$ to elements of T_N. This applies to expansions of models of $\mathbf{T}_{\mathcal{N}}$ and $\mathbf{T}_T$ used in §2 and §6.

2.4. **Type $\mathbf{s}_0$ and the standard model.** We shall describe a theory $\mathbf{T}_{\mathcal{N},h}$ and a unary type $\mathbf{s}_0$ in the expanded language $L_{\mathcal{N},h} := L_{\mathcal{N}} \cup \{h\}$ (where h is a new unary function symbol) such that the only model of $\mathbf{T}_{\mathcal{N},h}$ omitting $\mathbf{s}_0$ is the unique expansion $\mathcal{N}_h$ of $\mathcal{N}$ to a model of $\mathbf{T}_{\mathcal{N},h}$.

Fix an enumeration s_n, for $n \in \omega$, of $\omega^{<\omega}$. Let $\mathcal{N}_h$ be the expansion of $\mathcal{N}$ to $L_{\mathcal{N},h}$ obtained by interpreting h as follows (here $\langle n \rangle$ denotes the element of $\omega^{<\omega}$ of length 1 whose only digit is n).

$$h(x) := \begin{cases} s_n, & \text{if } x = \langle n \rangle \text{ for some } n \\ f_1(x), & \text{otherwise.} \end{cases}$$

Then h is 2-Lipshitz because $d(x,y) \leq 1/2$ and $x \neq y$ implies $h(x) = h(y)$. Let $\mathbf{T}_{\mathcal{N},h} := \mathrm{Th}(\mathcal{N}_h)$ and

$$\psi := \sup_x \inf_y d(x, h(y)) + d(f_1(y), y).$$

The range of $h \restriction \{y : f_1(y) = y\}$ is $\omega^{<\omega}$. Since $\omega^{<\omega}$ is a dense subset of $\mathcal{N}$ we have $\psi^{\mathcal{N}_h} = 0$. By elementarity, in every model M of $\mathbf{T}_{\mathcal{N},h}$ the set $\{h^M(x) : f_1(x) = x\}$ is dense in M.

Lemma 2.4. *There exists a partial* 1*-type* $\mathbf{s}_0$ *over* $\mathbf{T}_{\mathcal{N},h}$ *such that* M *omits* $\mathbf{s}_0$ *if and only if* $M \cong \mathcal{N}_h$. *for every* $M \models \mathbf{T}_{\mathcal{N},h}$.

Proof. Let $\mathbf{s}_0(x)$ be the type consisting of conditions

$$d(f_1(x), \langle n \rangle) = 1$$

for all $n \in \omega$. Then every finite subset of $\mathbf{s}_0$ is realized in $\mathcal{N}_h$ by a large enough $\langle m \rangle$. On the other hand, $\mathcal{N}_h$ clearly omits $\mathbf{s}_0$. Fix $M \models \mathbf{T}_{\mathcal{N},h}$ which omits $\mathbf{s}_0$. Then $\{y \in M : f_1(y) = y\} = \{\langle n \rangle : n \in \omega\}$. It follows that $\mathcal{N}_h$ is dense in M and M and $\mathcal{N}_h$ are isometrically isomorphic. $\square$

2.5. **$\mathbf{\Pi}^1_1$-completeness.** Fix a complete theory $\mathbf{T}$ in a countable language. The set of (not necessarily complete) n-types omissible in a model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$, by Proposition 2.2.

Theorem 2.5. *There is a complete theory* $\mathbf{T}_2$ *in a countable language* L_2 *such that the space of all 2-types* $\mathbf{t}$ *omissible in a model of* $\mathbf{T}_2$ *is* $\mathbf{\Pi}^1_1$*-complete.*

Proof. We define an expansion L_2 of $L_{\mathcal{N}}$ (§2.2) to be a two-sorted language with the sorts D_1 and D_2. If M is an L_2-structure then D_1^M is an $L_{\mathcal{N}}$-structure.

A *subtree* of $\omega^{<\omega}$ is a (possibly empty) subset a of $\omega^{<\omega}$ such that $s \in a$ implies $t \in a$ for all $t \sqsubseteq s$ in $\omega^{<\omega}$.

The intended interpretation of D_2 is the space $\mathcal{T}$ of all subtrees of $\omega^{<\omega}$. The language L_2 is equipped with the following

(3) Constant symbols S_n, for $n \in \omega$, for all finite-width subtrees of $\omega^{<\omega}$ all of whose branches have eventually zero value.
(4) A binary predicate symbol Elm of sort $D_1 \times D_2$.

The interpretation of Elm is required to be 1-Lipshitz. Theory $\mathbf{T}_2$ is the theory of the L_2-model $\mathcal{N}_2$ described as follows. The universe of $\mathcal{N}_2$ is the set $\omega^{<\omega} \sqcup \omega^\omega \sqcup \mathcal{T}$ and $\mathcal{N}_2$ is an extension of the $L_{\mathcal{N}}$-model $\mathcal{N}$ as described in §2.2.

The metric on $\mathcal{T}$ is defined via (let $\delta(a,b) := \min\{k : a \cap k^{\leq k} \neq b \cap k^{\leq k}\}$)

$$d(a,b) := 1/\delta(a,b).$$

Since the set of all subtrees of $\omega^{<\omega}$ is a closed set in the Cantor-set topology of $\mathcal{P}(\omega^{<\omega})$, $D_2^{\mathcal{N}_2}$ is a compact metric space and metric d is easily seen to be compatible with this topology. Interpretations of constant symbols from (3) form a countable dense set.

We introduce an auxiliary function $\ell \colon \omega^{<\omega} \to \omega$ via

$$\ell(t) := \max(\{|t|\} \cup \operatorname{range}(t))$$

and define Elm on $\omega^{<\omega} \times \mathcal{T}$ via

$$\operatorname{Elm}(t,S) := \begin{cases} 0, & \text{if } t \in S, \\ 1/(\ell(t)+1), & \text{if } t \notin S. \end{cases}$$

The predicate Elm is Lipshitz on $\omega^{<\omega} \times D_2^{\mathcal{N}_2}$ because $d(S,T) \leq 1/k$ and $d(s,t) \leq 1/k$ implies that $\operatorname{Elm}(s,S) \leq 1/m$ iff $\operatorname{Elm}(t,T) \leq 1/m$ for all $m \leq k$. We can therefore continuously extend Elm to $\omega^\omega \times \mathcal{T}$. Then we have $\operatorname{Elm}(x,S) = 0$ for all $S \in \mathcal{T}$ and all branches x of S.

Let $\mathbf{T}_2 := \operatorname{Th}(\mathcal{N}_2)$. Since the interpretations of constants in L form a dense subset of $\mathcal{N}_2$, it is an atomic model of $\mathbf{T}_2$. Therefore a type is omissible in a model of $\mathbf{T}_2$ if and only if it is omitted in $\mathcal{N}_2$.

Let $S \in \mathcal{T}$. We let $\mathbf{t}^S$ be the partial type in variables x, y of the sort $D_1 \times D_2$ consisting of the following conditions:

(5) $\operatorname{Elm}(f_k(x), y) = 0$ for all k.
(6) $\operatorname{Elm}(t,y) = 0$ if $t \in S$ and $\operatorname{Elm}(t,y) = 1/(\ell(t)+1)$ if $t \notin S$, for all constants $t \in \omega^{<\omega}$.
(7) $d(S_n, y) = \varepsilon_n$, where $\varepsilon_n = d(S_n, S)$, for all n.
(8) $|d(f_k(x), x) - 1/(k+1)| = 0$.

Suppose b, c is a realization of $\mathbf{t}^S$ in a model M of $\mathbf{T}_2$. Then $D_1^M \models \mathbf{T}_{\mathcal{N}}$ and T_M is a tree of height ω (see §2.3). By (6) and (7) c is a subtree of T_M such that $c \cap \omega^{<\omega} = S$. By (8) b is a branch and by (5) it is a branch of the tree c.

The map $\mathcal{T} \ni S \mapsto \mathbf{t}^S \in S_2^-(\mathbf{T}_2)$ (see §1.2.5) is clearly continuous (each of the spaces is considered with respect to its compact metrizable topology). Since the set of well-founded trees in $\mathcal{T}$ is $\mathbf{\Pi}_1^1$-complete ([26, 32.B]), it only remains to check that $\mathbf{t}^S$ is omissible in a model of $\mathbf{T}_2$ if and only if S is well-founded. Since the standard model $\mathcal{N}_2$ of $\mathbf{T}_2$ is the atomic model of $\mathbf{T}_2$, this is equivalent to $\mathbf{t}^S$ being omissible in $\mathcal{N}_2$.

If S is well-founded then $\mathcal{N}_2$ omits $\mathbf{t}^S$. This is because if (b,a) realizes $\mathbf{t}^S$ then $a = S$, and therefore $b \in \omega^\omega$ has to be a 'true' branch of S. If S is ill-founded and b is its branch, then $\mathcal{N}_2$ realizes $\mathbf{t}^S$ by (b, S).

Since by (1) of Proposition 2.2 the set of types omissible in a model of $\mathbf{T}$ is $\mathbf{\Pi}^1_1$, this completes the proof. □

Corollary 2.6. *There exists a separable model $\mathcal{N}_2$ such that the set of types omitted in $\mathcal{N}_2$ is $\mathbf{\Pi}^1_1$-complete.*

Proof. Model $\mathcal{N}_2$ used in Theorem 2.5 is an atomic model of its theory, and therefore a type is omissible in a model of $\mathbf{T}_2$ if and only if $\mathcal{N}_2$ omits it. □

2.6. **$\mathbf{\Sigma}^1_2$-completeness.** Denote the compact metrizable space of all subtrees of $\omega^{<\omega} \times \omega^{<\omega}$ by $\mathcal{T}^2$. For $R \in \mathcal{T}^2$ and $x \in \omega^\omega$ let

$$R_x := \{s \in \omega^{<\omega} : (s, x \restriction |s|) \in R\}.$$

The following fact is well-known but we could not find a reference in the literature.

Lemma 2.7. *The subspace Z of all $R \in \mathcal{T}^2$ such that for some x the tree R_x is well-founded is a complete $\mathbf{\Sigma}^1_2$ set.*

Proof. This set is clearly $\mathbf{\Sigma}^1_2$. Since every uncountable Polish space is Borel-isomorphic to ω^ω, it suffices to show that every $\mathbf{\Sigma}^1_2$ subset of ω^ω is a continuous preimage of Z. Suppose A is a $\mathbf{\Sigma}^1_2$ subset of ω^ω. Fix a closed subset F of $(\omega^\omega)^3$ such that $A = \{x : (\exists y \in \omega^\omega)(\forall z \in \omega^\omega)(x, y, z) \notin F\}$ and let $T := \{(x \restriction m, y \restriction m, z \restriction m) : (x, y, z) \in F, m \in \omega\}$. Define $R \colon \omega^\omega \to \mathcal{T}^2$ by

$$R(x) := \{(s, t) : |s| = |t| \text{ and } (x \restriction |s|, s, t) \in T\}.$$

This is a continuous map, and $x \in A$ if and only if the tree $R(x)_y$ is well-founded for some $y \in \omega^\omega$. Therefore $A = R^{-1}(Z)$. Since A was an arbitrary $\mathbf{\Sigma}^1_2$ subset of ω^ω this completes the proof. □

The following theorem is logically incomparable with Theorem 2.5 since the theory $\mathbf{T}_3$ is not complete.

Theorem 2.8. *There is a theory $\mathbf{T}_3$ in a countable language L_3 such that the space of all 2-types omissible in a model of $\mathbf{T}_3$ is $\mathbf{\Sigma}^1_2$-complete.*

Proof. We define a joint expansion L_3 of L_2 as in the proof of Theorem 2.5 and $L_{\mathcal{N},h}$ as in §2.4. It is a three-sorted language with sorts D_1, D_2, and D_3, with D_1 and D_2 interpreted as before, with the exception of predicate Elm. The intended interpretation of D_3 is $\mathcal{T}^2$.

In addition to symbols imported from L_2 and h, in L_3 we have the following.

(9) A constant symbol c of sort D_1.
(10) Constant symbols R_n, for $n \in \omega$, of sort D_3 for all finite-width subtrees of $\omega^{<\omega} \times \omega^{<\omega}$ all of whose branches have eventually zero value.
(11) Predicate Elm of the sort $D_1 \times D_1 \times D_3$.

Let L_3^- be the language L_3 without the constant symbol c. Theory $\mathbf{T}_3$ is the theory of the L_3^--model $\mathcal{N}_3$ described as follows. Its universe is equal to $\omega^{<\omega} \sqcup \omega^\omega \sqcup \mathcal{T} \sqcup \mathcal{T}^2$ and it includes model $\mathcal{N}_2$ as defined in the proof of Theorem 2.5 and has predicate h as defined in 2.4.

The metric d on $\mathcal{T}^2$ defined as

$$d(R,S) = \inf\{1/k : R \cap (k^{\leq k})^2 \neq S \cap (k^{\leq k})^2\}$$

turns $\mathcal{T}^2$ into a compact metric space.

Define Elm on $\omega^{<\omega} \times \omega^{<\omega} \times \mathcal{T}^2$ via

$$\mathrm{Elm}(s,t,R) = \begin{cases} 0, & \text{if } (s,t) \in R, \\ 1/(\max(\ell(s),\ell(t))+1), & \text{if } (s,t) \notin R. \end{cases}$$

This predicate is Lipshitz since $d(S,T) \leq 1/k$, $d(s_1,t_1) \leq 1/k$ and $d(s_2,t_2) \leq 1/k$ together imply that $\mathrm{Elm}(s_1,s_2,S) = 1/m$ iff $\mathrm{Elm}(t_1,t_2,T) = 1/m$ for all $m \leq k$. We can therefore continuously extend Elm to $\omega^\omega \times \mathcal{T}^2$. By the continuity we have $\mathrm{Elm}(x,y,S) = 0$ for all $x,y \in \omega^\omega$ and all $S \in \mathcal{T}$.

Theory $\mathbf{T}_3 := \mathrm{Th}(\mathcal{N}_3)$ is not a complete L_3-theory. since it provides no information on the interpretation of the constant symbol c. For $R \in \mathcal{T}^2$ let $\mathbf{t}^R(x,y)$ be a type in the sort $D_1 \times D_3$ consisting of the following conditions.

(12) $d(R_n, y) = r$, where $r := d(R_n, R)$, for all $n \in \omega$.
(13) $\mathrm{Elm}(f_k(x), f_k(c), y) = 0$ for all $k \in \omega$.
(14) $|d(f_k(x), x) - 1/(k+1)| = 0$.

If (b,S) realizes $\mathbf{t}^R$ in $M \models \mathbf{T}_3$, then $R = S$ by (12), $f_k(b) \neq b$ for all k by (14), and b is a branch of R_c by (13). Let $\mathbf{s} := \mathbf{s}_0$ defined in §2.4

We claim that $\mathbf{t}^R$ and $\mathbf{s}$ are simultaneously omissible if and only if there exists a real $a \in \omega^\omega$ such that R_a is well-founded. If there is such a real, then the model of $\mathbf{T}_3$ obtained by interpreting c as a omits both $\mathbf{t}^R$ and $\mathbf{s}$. Now assume there is no such real and let N be a model of $\mathbf{T}_3$ in which $\mathbf{s}$ is omitted. By Lemma 2.4, the reduct of N to $L_{\mathcal{N}}$ is isometrically isomorphic to $\mathcal{N}_h$. If a is the interpretation of c in N, then $c \in \omega^\omega$. Therefore the tree R_c is ill-founded, and $\mathbf{t}^R$ is realized in N by (b,R) where b is any branch of R_c.

The function $\mathcal{T}^2 \ni R \mapsto \mathbf{t}^R \in S_1^-(\mathbf{T}_3)$ (see §1.2.5) is clearly continuous. Therefore Lemma 1.9 implies that the function $\mathcal{T}^2 \ni R \mapsto \mathbf{t}^R \vee \mathbf{s}_0 \in S_2^-(\mathbf{T}_3)$ is continuous. We therefore have a continuous map from $\mathcal{T}^2$ into $S_2^-(\mathbf{T}_3)$ such that the preimage of the set of omissible types is, by Lemma 2.7, a $\mathbf{\Sigma}_2^1$-complete set, and this concludes the proof. □

3. Forcing and omitting types

Our study of generic models is motivated by potential applications to operator algebras (see [14], [16], also §5 and §7). Related results were obtained in [3] and [10], similarly inspired by Keisler's classic [28]. Both of these papers study a version of Keisler's forcing adapted to the infinitary version of the logic of metric structures. In the first-order logic a type (complete or partial) is omitted in a generic model if and only if it is omissible. In the logic of metric structures this remains true for complete types (by [2, Theorem 12.6] or Theorem 3.3 below) but not for partial types (Corollary 6.10). There are several good sources for the metric model-theoretic forcing ([8], [10], [3], [22, Appendix A], [16, §6], and [15]). Since the present paper is a companion to [16] meant to be self-contained and accessible to non-logicians, we include some of the basics for the reader's convenience. The forcing construction described below is also known as the *Henkin construction.*

3.1. **The forcing notion $\mathbb{P}_{\mathbf{T},\Sigma}$.** Fix a (not necessarily complete) theory $\mathbf{T}$ in a (not necessarily separable) language L and a set of L-formulas Σ with the following closure properties.

(Σ1) Σ includes all quantifier-free formulas,

(Σ2) Σ is closed under taking subformulas and change of variables,

(Σ3) if $k \in \omega$, $\varphi_i(\bar{x})$, for $0 \leq i < k$, are in Σ, and $f\colon [0,1]^k \to [0,1]$ is a continuous function, then $f(\varphi_0(\bar{x}), \dots, \varphi_{k-1}(\bar{x}))$ is in Σ.

Two most interesting cases are when Σ is the set of all quantifier-free formulas and when Σ is the set of all formulas.

We postulate a simplifying assumption that L has a single sort with a single domain of quantification. If this is not the case, the forcing can be modified by adding an infinite supply of constants (like d_j, for $j \in \omega$ below) for every domain of quantification. For example, in the case of C*-algebras, tracial von Neumann algebras, or other Banach algebras one adds constants d_j^n, for $j \in \omega$, for elements of the n-ball for every $n \geq 1$; we omit the straightforward details (see [16, §6] for the case of C*-algebras).

Let d_j, for $j \in \omega$, be a sequence of new constant symbols and let

$$L^+ := L \cup \{d_j : j \in \omega\}.$$

If $F = (f(0), \dots, f(n-1))$ is an n-tuple of natural numbers, then we write

$$\bar{d}_F := (d_{f(i)} : i < n).$$

Let

$$\Sigma^+ := \{\varphi(\bar{d}_F) : \varphi(\bar{x}) \in \Sigma, \bar{d}_F \text{ is of the same length as } \bar{x}\}.$$

For every L-formula $\varphi(\bar{x})$ and every tuple F of the appropriate length, $\varphi(\bar{d}_F)$ is an L^+-sentence. Conversely, every L^+-sentence is of this form.

An open condition (see §1.1.1) $\varphi(\bar{d}) < \varepsilon$ is *satisfied* in model M if there exists $\bar{a}$ in M of the appropriate length such that $\varphi(\bar{a})^M < \varepsilon$. A condition is *consistent with* $\mathbf{T}$ if it is satisfied in some model of $\mathbf{T}$. Suppose φ and ψ are L^+-sentences such that every d_i that occurs in ψ occurs in φ. We write

$$\mathbf{T} \cup \{\varphi < \varepsilon\} \models \psi < \delta$$

if in every $M \models \mathbf{T}$ and every interpretation $\bar{a}$ of constants d_i occurring in φ in M one has that $\varphi(\bar{a})^M < \varepsilon$ implies $\psi(\bar{a})^M < \delta$.

A *condition* in $\mathbb{P}_{\mathbf{T},\Sigma}$ is a triplet

$$p = (\psi^p, F^p, \varepsilon^p)$$

(we shall write (ψ, F, ε) whenever p is clear from the context) where ψ is an n-ary formula in Σ, F is an n-tuple of natural numbers, $\varepsilon > 0$, and $\psi(\bar{d}_F) < \varepsilon$ is a condition consistent with $\mathbf{T}$. We shall write $\bar{d}^p$ instead of $\bar{d}_{F^p}$. The poset $\mathbb{P}_{\mathbf{T},\Sigma}$ is ordered by

$$p \geq q \qquad \text{if} \qquad F^p \subseteq F^q \text{ and } \mathbf{T} \cup \{\psi^q(\bar{d}^q) < \varepsilon^q\} \models \psi^p(\bar{d}^p) < \varepsilon^p.$$

If $p \geq q$ then we say that q *extends* p or that q is *stronger than* p. By Lemma 1.1 every condition is equivalent to some p such that $\varepsilon^p = 1$. Conditions p and q are *incompatible*, $p \perp q$, if no condition extends both p and q. Conditions p and q are *compatible*, $p \not\perp q$, if some condition extends both p and q.

In the terminology of [25] and [15], if Σ consists of all quantifier-free formulas then $\mathbb{P}_{\mathbf{T},\Sigma}$ is the *Robinson forcing*, or the *finite forcing*. If Σ consists of all formulas, then $\mathbb{P}_{\mathbf{T},\Sigma}$ is the *infinite forcing*. In the latter case, we shall write $\mathbb{P}_{\mathbf{T}}$ for $\mathbb{P}_{\mathbf{T},\Sigma}$.

We identify condition $p = (\psi^p, F^p, \varepsilon^p)$ in $\mathbb{P}_{\mathbf{T},\Sigma}$ with the open condition $\psi^p(\bar{d}^p) < \varepsilon^p$ and use notations $\mathbf{T} + p$ and $\mathbf{T} \cup \{\psi(\bar{d}^p) < \varepsilon\}$ interchangeably.

A recap of standard forcing terminology ([29], [31]) is in order. Subset G of $\mathbb{P}_{\mathbf{T},\Sigma}$ is a *filter* if every two elements of G have a common extension in G and $q \in G$ and $p \geq q$ implies $p \in G$. A subset $\mathbf{D}$ of $\mathbb{P}_{\mathbf{T},\Sigma}$ is *dense* if every $q \in \mathbb{P}_{\mathbf{T},\Sigma}$ has an extension in $\mathbf{D}$. It is *dense below* some $p \in \mathbb{P}_{\mathbf{T},\Sigma}$ if every $q \leq p$ has an extension in $\mathbf{D}$. If $\mathbf{F}$ is a family of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ then a filter G is $\mathbf{F}$-*generic* if $G \cap \mathbf{D} \neq \emptyset$ for all $\mathbf{D} \in \mathbf{F}$.

Lemma 3.1. *If $\mathbf{F}$ is a countable family of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ then there exists a $\mathbf{F}$-generic filter.*

Proof. Enumerate sets in $\mathbf{F}$ by ω and choose a decreasing sequence p_n, for $n \in \omega$, so that p_n belongs to the nth set in $\mathbf{F}$. Then

$$G := \{q \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists n) p_n \leq q\}$$

is an $\mathbb{F}$-generic filter. □

Lemma 3.2. *For $\varphi(\bar{d}_F) \in \Sigma^+$ and $k \geq 1$ the set*

$$\mathbf{D}_{\varphi(\bar{d}_F),k} = \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists r \in \mathbb{R}) \mathbf{T} + p \models |\varphi(\bar{d}_F) - r| < 1/k\}.$$

is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$.

Proof. Fix a condition $p \in \mathbb{P}_{\mathbf{T},\Sigma}$ and let $n := n^p$. Let $M \models \mathbf{T}$ be such that $\bar{a} \in M^n$ satisfies p. Thus we have $M \models \psi^p(\bar{a}) < \varepsilon^p$. With $r := \varphi(\bar{a})^M$, the condition

$$q := (\max(\psi^p, |\varphi(\bar{d}_F) - r|), F^p, \min(\varepsilon^p, 1/k))$$

is satisfied in M by $\bar{a}$ and it extends p. □

If L is separable then G meets all dense sets of the form $\mathbf{D}_{\varphi(\bar{d}_F),\varepsilon}$ if and only if it meets all dense sets of the form $\mathbf{D}_{\varphi_j(\bar{d}_F),1/k}$ where $\varphi_j(\bar{x})$, for $j \in \omega$, is a set of formulas dense in the pseudometric $d_{\mathbf{T}}$ defined in §1.2.2 and $k \in \omega$.

A formula is an $\forall\exists$-*formula* if it is of the form

$$\sup_{\bar{x}} \inf_{\bar{y}} \psi(\bar{x},\bar{y},\bar{z})$$

where ψ is quantifier-free. A theory $\mathbf{T}$ is $\forall\exists$-*axiomatizable* if it is axiomatizable by a set of $\forall\exists$-sentences. We emphasize that in the following proposition theory $\mathbf{T}$ is not assumed to be complete and that the language L is not assumed to be separable.

Theorem 3.3. *Suppose $\mathbf{T}$ is a theory in a language L and Σ is a set of L-formulas satisfying* (Σ1)–(Σ3)*. Then there is a family $\mathbf{F}$ of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ with the following property. If G is an $\mathbf{F}$-generic filter, then there exists a unique L^+-structure M_G satisfying the following.*

1. *The interpretations of $\{d_j : j \in \omega\}$ form a dense subset of M_G.*
2. *Every condition $p \in G$ is satisfied in M_G.*
3. *If Σ is the set of all L-formulas then $M_G \models \mathbf{T}$.*
4. *If $\mathbf{T}$ is $\forall\exists$-axiomatizable then $M_G \models \mathbf{T}$.*

If L is separable, then an $\mathbb{F}$-generic filter G and model M_G with the above properties exist.

The proof of Theorem 3.3 is broken up into a few lemmas. Family $\mathbf{F}$ shall include all sets $\mathbf{D}_{\varphi(\bar{d}_F),\varepsilon}$ defined above as well as five other families of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ similarly indexed by L^+-formulas $\varphi(\bar{s})$ in Σ, natural numbers, and tuples of constants in L^+. A proof that each of these sets is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$ is, being very similar to the proof of the density of $\mathbf{D}_{\varphi(\bar{d}_F),\varepsilon}$, omitted.

The first family, indexed by an L^+-formula $\varphi(\bar{d}_F, x)$ such that φ belongs to Σ with a single free variable x, $r \in \mathbb{Q}$, and $m \geq 1$ is defined by

$$\mathbf{C}_{\varphi(\bar{d}_F,x),r,m} := \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : \mathbf{T} + p \models \inf_x \varphi(\bar{d}, x) > r - 1/m \\ \text{ or } (\exists j)\mathbf{T} + p \models \varphi(\bar{d}_F, d_j) < r\}.$$

The second family is indexed by i, j and $k > 0$ in ω (‘d’ stands for ‘distance’)

$$\mathbf{D}_{d,i,j,k} = \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists r)\mathbf{T} + p \models |d(d_i, d_j) - r| < 1/k\}.$$

The third and fourth families are indexed by predicate symbols P or functional symbols g in L, sets $F \subseteq \omega$ whose cardinality is equal to the arity of P (or of g, respectively), and $k > 0$ in ω

$$\mathbf{D}_{P,F,k} := \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists r \in [0,1])\mathbf{T} + p \models |P(\bar{d}_F) - r| < 1/k\}$$
$$\mathbf{D}_{g,F,k} := \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists l \in \omega)\mathbf{T} + p \models |f(\bar{d}_F) - d_l| < 1/k\}.$$

The next family is indexed by $p \in \mathbb{P}_{\mathbf{T},\Sigma}$ such that $\psi^p = \inf_x \varphi(\bar{d}_F, x)$ for some φ in Σ:

$$\mathbf{D}_{\inf,p} := \{q : q \perp p \text{ or } (\exists j \in \omega)\mathbf{T} + q \models \varphi(\bar{d}_F, d_j) < \varepsilon^p\}.$$

If Σ consists of quantifier-free formulas and $\mathbf{T}$ is $\forall\exists$-axiomatizable then the fifth family of dense sets is indexed by quantifier-free formulas $\varphi(\bar{x}, \bar{y})$ (recall that all quantifier-free formulas belong to Σ) such that $\sup_{\bar x}\inf_{\bar y}\varphi(\bar x,\bar y)=0$ is an axiom of $\mathbf{T}$, finite $F\subseteq\omega$ of the appropriate cardinality, and $k>0$:

$$\mathbf{E}_{\varphi(\bar d_F,x),k}:=\{q\in\mathbb{P}_{\mathbf{T},\Sigma}:(\exists F')\mathbf{T}+q\models\varphi(\bar d_F,\bar d_{F'})<1/k\}.$$

Lemma 3.4. *Each of the sets* $\mathbf{C}_{\varphi(\bar d_F,x),r,m}$, $\mathbf{D}_{d,i,j,k}$, $\mathbf{D}_{P,F,k}$, $\mathbf{D}_{g,F,k}$, $\mathbf{D}_{\inf,p}$, *and* $\mathbf{E}_{\varphi(\bar d_F,x),k}$ *is dense in* $\mathbb{P}_{\mathbf{T},\Sigma}$ *for every choice of parameters, as long as* φ *belongs to* Σ.

Proof. We prove that every set of the form $\mathbf{E}_{\varphi(\bar d_F,x),k}$ is dense. Proofs of the other cases are very similar, and therefore omitted.

Suppose $p\in\mathbb{P}_{\mathbf{T},\Sigma}$ and with $n:=n^p$ find $M\models\mathbf{T}$ and $\bar a\in M^n$ that satisfies p. Since $\sup_{\bar x}\inf_{\bar y}\varphi(\bar x,\bar y)^M=0$, there exists $\bar b\in M^m$ (where m is the length of tuple $\bar y$) such that

$$\varphi(\bar a,\bar b)^M<1/k.$$

Let F' be an m-tuple in ω such that for all $i<j<m$ and all $l<n$ we have $f'(j)\neq f(l)$ and $f'(i)\neq f'(j)$. The open condition

$$q:=(\max(\psi^p,\varphi(\bar d_F,\bar d_{F'})),\min(\varepsilon^p,1/k))$$

is satisfied in M by $\bar a,\bar b$ and it extends p. Since p was arbitrary, this proves that $\mathbf{E}_{\varphi(\bar d_F,x),k}$ is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$. $\square$

All parameters of dense sets in Lemma 3.4 range over countable sets, with the exception of formulas φ and conditions p. Suppose L is countable. Then for every $n\in\omega$ we can choose a countable dense subset $\mathbf{D}_n$ of $\mathbb{F}_n(L)$ defined in §1.2.2. Then

$$\begin{aligned}\mathbb{F}_{\mathbf{T},\Sigma}:=\{&\mathbf{D}_{\varphi(\bar d_F),1/k},\mathbf{C}_{\varphi(\bar d_F,x),r,m},\mathbf{D}_{d,i,j,k},\mathbf{D}_{P,F,k},\mathbf{D}_{g,F,k},\mathbf{D}_{\inf,p},\mathbf{E}_{\varphi(\bar d_F,x),k}:\\&\{\varphi,\psi\}\subseteq\bigcup_n\mathbf{D}_n,\{i,j,k\}\subseteq\omega,F\in\omega^{<\omega},g\in L,P\in L\}\end{aligned}$$

is a countable family of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$.

Lemma 3.5. *Suppose* $\mathbf{T}$ *is a theory in a language* L *and* Σ *is a set of* L*-formulas satisfying* (Σ1)–(Σ3). *If* $G\subseteq\mathbb{P}_{\mathbf{T},\Sigma}$ *is an* $\mathbb{F}_{\mathbf{T},\Sigma}$*-generic filter then there is a unique* L^+*-structure* M_G *with the interpretations of* $\{d_j:j\in\omega\}$ *as a dense subset and such that for all* $p\in\mathbb{P}_{\mathbf{T},\Sigma}$ *the following holds:*

($*_p$) *If* $r\geq 0$ *then* $(|\psi^p-r|,F^p,\varepsilon^p)\in G$ *if and only if*

$$M_G\models|\psi^p(\mathbf{d}^p)-r|<\varepsilon^p.$$

Proof. Let $M_G^0:=\{\mathbf{d}_j:j\in\omega\}$. Since G intersects $\mathbf{D}_{d,i,j,k}$ for all i,j, and k,

$$d(\mathbf{d}_i,\mathbf{d}_j):=r\Leftrightarrow(\forall k\geq 1)(\exists p\in G)\mathbf{T}+p\models|d(d_i,d_j)-r|<1/k$$

defines a metric on M_G^0. Since G intersects $\mathbf{D}_{P,F,k}$ for every n-ary predicate P in L and every $n\in\omega$, for $F\in\omega^n$ we can define

$$P(\mathbf{d}_F):=r\Leftrightarrow(\forall k\geq 1)(\exists p\in G)\mathbf{T}+p\models|P(\mathbf{d}_F)-r|<1/k.$$

Let the universe of M_G be the metric completion of M_G^0. Since interpretation of every relational symbol P on M_G^0 respects the uniform continuity modulus associated with P, it has a unique extension to a predicate on M_G.

For every function symbol g in L filter G intersects all $\mathbf{D}_{g,F,k}$. Therefore for every $\bar{d}_F$ there exists a sequence $d_{n(j)}$, for $j \in \omega$, in M_G^0 such that

$$g(\mathbf{d}_F)^{M_G} := \lim_j \mathbf{d}_{n(j)}$$

is well-defined. This interpretation of g respects the modulus of uniform continuity associated with g and can therefore be extended to a function on M_G. This completes the definition of M_G.

We shall prove $(*_p)$ by induction on complexity of the formula ψ^p.

The case when ψ^p is an atomic formula is immediate from the definition. Assume that $(*_q)$ holds for all q such that ψ^q is a proper subformulas of ψ^p.

Suppose $\psi^p = f(\psi_0(\bar{d}_F), \dots, \psi_{n-1}(\bar{d}_F))$ for $n \geq 1$, $F \in \omega^{<\omega}$, continuous function f, and formulas ψ_j, for $j < n$. We shall prove the direct implication in $(*_p)$ in the case when $r = 0$. Suppose $p \in G$ and let

$$r_j := \psi_j(\mathbf{d}_F)^{M_G}$$

for $j < n$. Since f is uniformly continuous on bounded sets, we can fix a large enough $k \in \omega$ so that $|s_j - r_j| < 1/k$ for all $j < n$ implies $|f(\bar{s}) - f(\bar{r})| < \varepsilon^p$. Since G is $\mathbb{F}_{\mathbf{T},\Sigma}$-generic, there are $q_j \in G$ such that $q_j \in \mathbf{D}_{\psi_j(\bar{d}_F),1/k}$, for all $j < n$. By the inductive hypothesis we have $M \models |\psi_j(\mathbf{d}_F) - r_j| < 1/k$ for all $j < n$. Therefore $M \models \psi^p(\mathbf{d}_F) < \varepsilon^p$, thus proving the direct implication in $(*_p)$ in case when $r = 0$.

If $r \neq 0$ then the above argument applied when f is replaced with $f - r$ proves the direct implication in $(*_p)$. The converse is automatic since $M_G \models \psi^p(\mathbf{d}^p) = r$ for exactly one $r \geq 0$.

Suppose ψ^p is of the form $\inf_x \varphi(\bar{d}_F, x)$ for some φ for which the claim has been proven. As before, we first prove the direct implication in $(*_p)$ in case when $r = 0$.

Suppose $p \in G$ and fix $q \in G \cap \mathbf{D}_{\inf,p}$. Then q and p are compatible because G is a filter, and therefore $\mathbf{T} + q \models \varphi(\bar{d}_F, d_j) < \varepsilon^p$ for some $j \in \omega$. Again since G is a filter, the condition $\varphi(\bar{d}_F, d_j) < \varepsilon^p$ belongs to G. By applying the inductive hypothesis to this condition and φ we have $M_G \models \varphi(\mathbf{d}_F, d_j) < \varepsilon^p$ and therefore $M_G \models \inf_x \varphi(\mathbf{d}_F, x) < \varepsilon^p$. Since G was an arbitrary $\mathbb{F}$-generic filter containing p, the direct implication in $(*_p)$ holds with $r = 0$. If $r \neq 0$ then the above argument applied when $\varphi(d_F, x)$ is replaced with $|\varphi(d_F, x) - r|$ proves the direct implication in $(*_p)$. As before, the converse is automatic.

Since $\sup_x \varphi = 1 - \inf_x(1 - \varphi)$, this covers all cases and concludes the inductive proof of $(*_p)$ for all p.

It only remains to prove the uniqueness. But if M is an L^+-structure that satisfies all conditions in G and the set of interpretations $\{d_i^M : i \in \omega\}$ is dense in M then M is clearly isometrically isomorphic to M_G. $\square$

Proof of Theorem 3.3. Given L, $\mathbf{T}$, and Σ, set $\mathbb{F} := \mathbb{F}_{\mathbf{T},\Sigma}$ is a family of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ which by Lemma 3.5 has the property that every $\mathbb{F}$-generic filter defines a unique L^+-structure M_G such that (1) and (2) hold.

(3) If Σ consists of all L-formulas, then (2) implies that $M_G \models \mathbf{T}$.

(4) Assume $\mathbf{T}$ is $\forall\exists$-axiomatizable and recall that (Σ1) implies Σ includes all quantifier-free formulas. For every quantifier-free formula $\varphi(\bar{x}, \bar{y})$ such that $\sup_{\bar{x}} \inf_{\bar{y}} \varphi(\bar{x}, \bar{y}) = 0$ is an axiom of $\mathbf{T}$, for every F of the appropriate cardinality, and every $k > 0$ the set $\mathbf{E}_{\varphi(\bar{d}_F, x),k}$ is met by the generic filter G. A simple argument now shows that M_G satisfies all $\forall\exists$-axioms of $\mathbf{T}$ and (4) follows.

Now suppose L is countable. Then $\mathbb{F}_{\mathbf{T},\Sigma}$ is also countable and an $\mathbb{F}$-generic filter G exists by Lemma 3.1. □

Remark 3.6. The proof of Theorem 3.3 used the fact that for every sentence $\varphi(\bar{d}_F)$ in Σ^+ and $\varepsilon > 0$ the set $\mathbf{D}_{\varphi(\bar{d}_F),\varepsilon}$ of conditions that decide the φ^{M_G} up to ε is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$. A deeper fact is worth mentioning. Even if $\varphi(\bar{d}_F)$ is an L^+-sentence that does not belong to Σ^+ then for every $\varepsilon > 0$ the set of conditions p in $\mathbb{P}_{\mathbf{T},\Sigma}$ that decide the value of $\varphi(\bar{d}_F)^{M_G}$ for every sufficiently generic G (possibly more than merely $\mathbb{F}$-generic, but a countable family of dense sets still suffices) up to ε is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$. This is a consequence of Cohen's Truth Lemma, [29, Lemma IV.2.24].

Principal types were defined in §1.3.

Definition 3.7. Suppose Σ is a set of L-formulas which satisfies (Σ1)–(Σ3) and $\mathbf{t}(\bar{x})$ is an n-ary type for some $n \geq 1$. An n-ary type $\mathbf{t}(\bar{x})$ is Σ-*non-principal* if there exists $k \geq 1$ such that for every $F \in \omega^n$ the set (using the max-distance on M^n)

$$\mathbf{D}_{\mathbf{t}(\bar{x}),F,k} := \{q \in \mathbb{P}_{\mathbf{T},\Sigma} : M \models \mathbf{T} \text{ implies } \inf\{\operatorname{dist}(\bar{a}, \mathbf{t}(M)) : M \models \psi^q(\bar{a}) < \varepsilon^q\} \geq 1/k\}.$$

is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$.

Theorem 3.8. *Suppose* $\mathbf{T}$ *is a complete* L*-theory and* Σ *is a set of* L*-formulas satisfying* (Σ1)–(Σ3)*. If* $\mathbf{t}$ *is a* Σ*-non-principal type then there is a family* $\mathbf{F}_{\mathbf{t}}$ *of dense subsets of* $\mathbb{P}_{\mathbf{T},\Sigma}$ *such that if* G *is* $\mathbf{F}_{\mathbf{t}}$*-generic then* M_G *omits* $\mathbf{t}$*. If* L *is separable then* $\mathbb{F}_{\mathbf{t}}$ *can be chosen countable.*

Proof. Suppose $\mathbf{t}(\bar{x})$ is a Σ-non-principal n-type. Fix $k \geq 1$ such that $\mathbf{D}_{\mathbf{t},F,k}$ is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$ for all $F \in \omega^n$. With $\mathbb{F}_{\mathbf{T},\Sigma}$ as defined before Lemma 3.5, let

$$\mathbb{F}_{\mathbf{t}} := \mathbb{F}_{\mathbf{T},\Sigma} \cup \{\mathbf{D}_{\mathbf{t},F,k} : F \in \omega^n\}.$$

If G is $\mathbb{F}_{\mathbf{t}}$-generic, then (with $\mathbf{d}_F$ and M_G^0 as in the proof of Lemma 3.5) we have $\operatorname{dist}(\mathbf{d}_F, \mathbf{t}(M_G)) \geq 1/k$ for all $F \in \omega^n$. Since $\{\mathbf{d}_F : F \in \omega^n\} = (M_G^0)^n$ is dense in M_G, the latter model omits $\mathbf{t}$.

The last claim follows from the fact that $\mathbb{F}_{\mathbf{T},\Sigma}$ is countable if L is. □

The following is well-known ([2, §12] or [23, Lecture 4]; see also Corollary 4.2).

Corollary 3.9. *Suppose* $\mathbf{T}$ *is a complete theory in a countable language and* $\mathbf{t}_n(\bar{x})$ *is a complete type over* $\mathbf{T}$ *for every* $n \in \omega$. *Then the following are equivalent.*

(1) *Each* $\mathbf{t}_n$ *is not principal.*
(2) *Each* $\mathbf{t}_n$ *is omissible in a model of* $\mathbf{T}$.
(3) *There exists a family of dense subsets* $\mathbb{F}$ *of* $\mathbb{P}_{\mathbf{T}}$ *such that for every* $\mathbb{F}$*-generic filter* G *model* M_G *is a model of* $\mathbf{T}$ *which omits all* $\mathbf{t}_n$.

Proof. The equivalence of (1) and (2) is Lemma 1.10, (3) clearly implies (2), and the converse is Theorem 3.8. □

3.2. **Forcing with 'certifying structures'.** Let $\mathbf{T}$ be a not necessarily complete theory, let Σ be a set of formulas in the language of $\mathbf{T}$ satisfying closure properties (Σ1)–(Σ3) as in §3.1 and let $\mathfrak{M}$ be a nonempty set of models of $\mathbf{T}$. Forcing $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is defined as follows. Its conditions are triples

$$p = (\psi^p, F^p, \varepsilon^p)$$

(we shall write (ψ, F, ε) whenever p is clear from the context) where ψ is an n-ary formula, F is an n-tuple of natural numbers, $\varepsilon > 0$, and $\psi(\bar{d}_F) < \varepsilon$ is a condition satisfied in some model in $\mathfrak{M}$. We shall write $\bar{d}^p$ instead of $\bar{d}_{F^p}$. We let $p \geq q$ if the following holds.

> $F^p \subseteq F^q$ and for every $M \in \mathfrak{M}$ and $\bar{a}$ in M of the appropriate length, if $\psi^q(\bar{a})^M < \varepsilon^q$ then $\psi^p(\bar{a}) < \varepsilon^p$.

If $p \geq q$ we say that q *extends* p or that q is *stronger than* p. If $\mathbf{T}$ is a complete theory, then every condition consistent with $\mathbf{T}$ is realized in every model of $\mathbf{T}$ and $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is isomorphic to $\mathbb{P}_{\mathbf{T},\Sigma}$ if $\mathfrak{M}$ is any nonempty set of models of $\mathbf{T}$.

A proof of Theorem 3.10 below is analogous to the proof of Theorem 3.3 and is therefore omitted. As before, theory $\mathbf{T}$ is not assumed to be complete and that the language L is not assumed to be separable.

Theorem 3.10. *Suppose* $\mathbf{T}$ *is a theory in a language* L *and that either* Σ *consists of all* L*-formulas or that* $\mathbf{T}$ *is* $\forall\exists$*-axiomatizable and* Σ *includes all quantifier-free formulas. Then there is a family* $\mathbf{F}$ *of dense subsets of* $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ *with the following property. If* G *is an* $\mathbf{F}$*-generic filter, then there exists a unique* L^+*-structure* M_G *which is a model of* $\mathbf{T}$ *and has the interpretations of* $\{d_j : j \in \omega\}$ *as a dense subset.* □

4. Forcing

In the present section we discuss the relation with the set-theoretic forcing, in which one constructs a generic extension of a model of ZFC. Some acquaintance with the method of forcing is required (e.g. [29] or [31]). Readers not interested in forcing may want to skip ahead to §5.

4.1. **Cohen forcing.** Recall that two forcing notions $\mathbb{P}_0$ and $\mathbb{P}_1$ are *forcing equivalent* if there is a poset $\mathbb{P}_3$ and order-preserving maps $f_j \colon \mathbb{P}_j \to \mathbb{P}_3$ such that $f_j[\mathbb{P}_j]$ is dense in $\mathbb{P}_3$ for $j < 2$ ([29, Definition IV.4.25]). Forcing equivalence of $\mathbb{P}_0$ and $\mathbb{P}_1$ is equivalent to asserting that for every V-generic filter G_j in $\mathbb{P}_j$ there exists a filter $G_{1-j} \subseteq \mathbb{P}_{1-j}$ in $V[G_j]$ such that $V[G_j] = V[G_{1-j}]$, for $j < 2$ ([29, Lemma IV.4.6]).

The poset for adding a Cohen real has nonempty rational intervals in $[0,1]$ as conditions, ordered by the inclusion. It is forcing-equivalent to every forcing with a countable dense set and no minimal elements (this is immediate from e.g. [29, Lemma IV.4.26]).

Lemma 4.1. *If* $\mathbf{T}$ *is a theory in a countable language and* $\mathfrak{M}$ *is a class of its models then each of* $\mathbb{P}_{\mathbf{T},\Sigma}$ *and* $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ *has a countable dense subset, and is therefore equivalent to the standard forcing for adding a Cohen real.*

Proof. By the separability of L, for every n there exists a countable set of formulas $\mathbf{D}_n$ which is $d_{\mathbf{T}}$-dense in $\mathbb{F}_n(\mathbf{T})$. We shall produce a countable dense set E of conditions in $\mathbb{P}_{\mathbf{T},\Sigma}$ by using a standard continuous functional calculus trick (§1.1.1).

For each $m \in \omega$ let $f_m(t) := \max(t - 1/m, 0)$. Consider the set $\mathbf{C}$ of all conditions in $\mathbb{P}_{\mathbf{T},\Sigma}$ of the form $f_m(\varphi(\bar{d})) < 1/n$, for m, n in ω and $\varphi \in \mathbf{D}$. If $\varphi(\bar{d}) < \varepsilon$ is a condition then there exist $M \models \mathbf{T}$ and a tuple $\bar{a}$ in M such that $M \models \varphi(\bar{a}) = r < \varepsilon$. Therefore $f_n(\varphi(\bar{d})) < \varepsilon$ is in $\mathbb{P}_{\mathbf{T},\Sigma}$ for all $n > 1/(\varepsilon - r)$.

We claim that the set $\mathbf{C}$ of all conditions of this sort is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$. Since this set is countable this will conclude the proof. Take a condition $\psi(\bar{d}) < \varepsilon$. Fix $n > 1/\varepsilon$ and let $\varphi(\bar{x}) \in \mathbf{D}$ be such that $d_\infty(\varphi, \psi) < 1/(2n)$. For a large enough $m > 2n$ we have that $f_m(\varphi(\bar{d})) < 1/m$ is a condition in $\mathbf{C}$ stronger than $\psi(\bar{d}) < \varepsilon$.

A proof in the case of $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is analogous. □

The *covering number*, cov(`meager`), for the ideal of meager (i.e. first category) subsets of $\mathbb{R}$ is the minimal cardinality of a family of meager sets required to cover the real line ([29, Definition III.1.2 and Definition III.1.6]). Since every separable, completely metrizable space X with no isolated points has a dense G_δ subset homeomorphic to the Baire space, cov(`meager`) is equal to the minimal number of first category subsets of X required to cover X.

Corollary 4.2. *If* $\mathbf{T}$ *is a complete theory in a countable language,* $\kappa <$ cov(`meager`)*, and* $\mathbf{t}_\gamma$ *for* $\gamma < \kappa$ *is a set of complete non-principal types over* $\mathbf{T}$*, then* $\mathbf{T}$ *has a separable model that omits all* $\mathbf{t}_\gamma$*.*

Proof. By Theorem 3.8 to every $\mathbf{t}_\gamma$ we associate a dense $\mathcal{D}_\gamma \subseteq \mathbb{P}_{\mathbf{T}}$ such that if G intersects $\mathcal{D}_\gamma$ and all dense sets in the family $\mathbb{F}$ from Theorem 3.3, then M_G is a model of $\mathbf{T}$ which omits each $\mathbf{t}_\gamma$.

By Lemma 4.1 the space of all filters in $\mathbb{P}_{\mathbf{T},\Sigma}$ is homeomorphic to the Cantor set, hence it is compact, metrizable and with no isolated points. Therefore cov(`meager`) is equal to the minimal number of dense subsets of

$\mathbb{P}_{\mathbf{T},\Sigma}$ not met by a single filter. We can therefore choose G to meet all dense sets in $\mathbb{F} \cup \{\mathcal{D}_\gamma : \gamma < \kappa\}$ and the generic model M_G is as required. $\square$

4.2. **Absoluteness.** In §4.3 we consider some properties of M_G. In order to put these properties in the proper context, we include a brief discussion of absoluteness for the convenience of the reader. A statement $\varphi(\bar{x})$ of ZFC is *absolute* if for any two transitive models $V \subseteq W$ of a sufficiently large fragment of ZFC and for all parameters $\bar{a}$ in V of the appropriate length one has

$$\varphi(\bar{a})^V \Leftrightarrow \varphi(\bar{a})^W$$

(see [29, §II.4]). If $V \subseteq W$ are models of a large enough fragment of ZFC then a complete metric structure N in V is identified with its completion $\tilde{N}$ in the larger model W. A property Θ of metric structures is *absolute* if for any two transitive models $V \subseteq W$ of a sufficiently large fragment of ZFC, for every metric structure N in V we have $\Theta(N)^V \Leftrightarrow \Theta(\tilde{N})^W$. A definition of a subset $P(x, N)$ of a metric structure N (such as the set of types realized in a model), is *absolute* if for every $a \in N^V$ we have $P(a, N)^V$ if and only if $P(a, \tilde{N})^W$.

Given a language L in the logic of metric structures and a string of characters φ, the assertion 'φ is an L-formula' is absolute. This essentially follows from [29, Lemma II.4.14, Theorem II.4.15, Corollary II.4.17] and the discussion in between, where the analogous statement for the first-order logic was proved. The semantics of L—i.e. the definition of the interpretation of a formula φ in a metric structure—is also proved to be absolute by a routine induction on the complexity of φ.

Suppose $V \subseteq W$ are transitive models of a sufficiently large fragment of ZFC and L is a language in V. The linear space of L-formulas in V (§1.2.2) is in general a proper subspace of the space of L-formulas in W. It is however always dense with respect to the metric defined in §1 (this is a consequence of the Stone–Weierstrass theorem, since all formulas built by using polynomial functions with rational coefficients belong to V), and therefore the space of L-formulas in W is the completion of the space of L-formulas in V. An L-theory is identified with a continuous functional on the space of L-formulas. Therefore every L-theory $\mathbf{T}$ in V has a unique extension to an L-theory in W, also denoted $\mathbf{T}$. By identifying a k-type with a functional on the space of L-formulas with free variables included in x_j, for $j < k$, we similarly identify a type over $\mathbf{T}$ in V with its unique extension to a type over $\mathbf{T}$ in W.

Proposition 4.3. *Suppose L is a language of metric structures, M is an L-structure, φ is an L-formula, $\bar{a}$ is a tuple in M of the appropriate length, and $r \in \mathbb{R}$. The following are absolute between models of a large enough fragment of ZFC.*

(1) $\varphi(\bar{a})^M = r$.
(2) *Type* $\operatorname{tp}_M(\bar{a})$ *of an n-tuple $\bar{a}$ in M.*

(3) $\mathrm{Th}(M)$.
(4) *The set of types realized in* M*.*
(5) *the distance* $d(\mathbf{t}, \mathbf{s})$ *between complete types as defined in* §*1.2.4.*
(6) *The assertion '*$\mathbf{t}$ *is principal.'*

Proof. Suppose $V \subseteq W$ are models of a large enough fragment of ZFC and M is in V.

(1) Since M is dense in $\tilde{M}$ and the interpretations of L-formulas are uniformly continuous, the absoluteness of $\varphi(\bar{a})^M = r$ is proved by induction on the complexity of φ. Clearly (1) implies (2).

(3) The theory of M is identified with an affine functional on $\mathfrak{W}_0(L)$, has the unique continuous extension to an affine functional on $\widetilde{\mathfrak{W}_0(L)}$ of the theory of M to the completion $\widetilde{\mathfrak{W}_0(L)}$ of (the ground-model) $\mathfrak{W}_0(L)$.

(4) The set of types realized in M is $\mathbf{\Sigma}^1_1$ and therefore absolute by Lemma 2.1.

(5) is a consequence of Lemma 1.8.

(6) By (5), for a type $\mathbf{t}$ (complete or not) having a metric ε-neighbourhood which is nowhere dense in the logic topology is absolute. □

4.3. **Omitting types in the generic model** M_G**.** Recall (§3.1) that $\mathbb{P}_{\mathbf{T}}$ denotes the so-called 'infinite' forcing, i.e. $\mathbb{P}_{\mathbf{T},\Sigma}$ in the case when Σ is the set of all formulas of the language of $\mathbf{T}$.

Lemma 4.4. *If a complete theory* $\mathbf{T}$ *in a countable language has an atomic model* N *then* $\mathbb{P}_{\mathbf{T}}$ *forces that* M_G *is atomic, and therefore isometric to the completion of* N*.*

Proof. By Theorem 3.8 there are countably many dense subsets of $\mathbb{P}_{\mathbf{T}}$ such that if G meets each one of them then every nonprincipal type is omitted in M_G^n. Therefore every complete n-type for every $n \geq 1$ realized in M_G is principal and M_G is an atomic model of $\mathbf{T}$. By uniqueness of the atomic model of $\mathbf{T}$ ([2, Corollary 12.9]), M_G is isometric to the completion of N. □

Lemma 4.4 can be recast as the assertion that if $\mathbf{T}$ has an atomic model then every omissible type is forced to be omitted in the generic model. Surprisingly, the assumption that $\mathbf{T}$ has an atomic model cannot be dropped from this assertion (Corollary 6.10).

4.4. **Strong homogeneity of** $\mathbb{P}_{\mathbf{T},\Sigma}$**.** A forcing notion is *homogeneous* if for any two conditions p and q there exists an automorphism Φ such that $\Phi(p)$ is compatible with q. Since the Cohen forcing is homogeneous, $\mathbb{P}_{\mathbf{T},\Sigma}$ is equivalent to a homogeneous forcing by Lemma 4.1. This does not imply that $\mathbb{P}_{\mathbf{T},\Sigma}$ is homogeneous itself, but we shall prove that even more is true in case when $\mathbf{T}$ is complete. By S_∞ we denote the group of *all* permutations of ω. To a permutation $h \in S_\infty$ we associate an automorphism Φ_h of $\mathbb{P}_{\mathbf{T},\Sigma}$ which sends d_j to $d_{h(j)}$ for all $j \in \omega$ (writing $h[F] := (h(f_0), \dots, h(f_{n-1}))$ where n is the length of F)

$$\Phi_h((\psi, F, \varepsilon)) := (\psi, h[F], \varepsilon).$$

Lemma 4.5. *Assume* $\mathbf{T}$ *is a complete* L*-theory and* Σ *is a set of* L*-formulas satisfying* (Σ1)–(Σ3)*. For any two conditions* p_1 *and* p_2 *in* $\mathbb{P}_{\mathbf{T},\Sigma}$ *there is* $h \in S_\infty$ *such that* p_1 *and* $\Phi_h(p_2)$ *are compatible.*

Proof. Let $p_j = (\psi_j, F_j, \varepsilon_j)$ for $j < 2$. We shall write $d(j)$ and $x(j)$ for d_{F_j} and x_{F_j}, respectively. By Lemma 1.1 we may assume $\varepsilon_1 = \varepsilon_2 = 1$. Since $\mathbf{T}$ is complete we have $\mathbf{T} \models \inf_{\bar{x}(j)} \ \psi_j(\bar{x}(j)) < 1$ for $j < 2$ and therefore

$$\mathbf{T} \models \max(\inf_{\bar{x}(1)} \psi_0(\bar{x}(0)), \inf_{\bar{x}(2)} \psi_1(\bar{x}(1))) < 1.$$

Let h be such that the tuples $h[F_1]$ and F_2 have no common entries. Then

$$q = (\max(\psi_0(\bar{d}(0)), \psi_2(\bar{d}_{h(F_1)}), F_0 \cup h(F_1), \varepsilon)$$

is a condition in $\mathbb{P}_{\mathbf{T},\Sigma}$ which extends both p_0 and $\Phi_h(p_1)$. $\square$

In the following M_G^0 denotes the countable dense submodel of M_G whose universe consists of interpretations of constants d_j, for $j \in \omega$, as defined in the proof of Theorem 3.3. We introduce a convenient ad-hoc terminology. A statement $\Theta(x)$ of ZFC (possibly with parameters) is *symmetric* if for every generic filter G and every $h \in S_\infty$ we have

$$\Theta(M_G^0) \leftrightarrow \Theta(M^0_{\Phi_h(G)}).$$

For example, "M_G omits type $\mathbf{t}$" and "$M_G \models \mathbf{T}$" are both symmetric but "$d(\mathbf{d}_1, \mathbf{d}_2) < 1/2$" is not.

Corollary 4.6. *Assume* $\mathbf{T}$ *is a complete* L*-theory and* Σ *is a set of* L*-formulas satisfying* (Σ1)–(Σ3)*. If* $\Theta(\bar{y})$ *is a symmetric statement of ZFC with parameters in the ground model, then* $\mathbb{P}_{\mathbf{T},\Sigma}$ *either forces* $\Theta(M_G^0)$ *or it forces* $\neg\Theta(M_G^0)$*.*

In particular, for every ground-model type $\mathbf{t}$ *forcing notion* $\mathbb{P}_{\mathbf{T},\Sigma}$ *either forces that* M_G *realizes* $\mathbf{t}$ *or it forces that* M_G *omits* $\mathbf{t}$*.*

Proof. Fix $p \in \mathbb{P}_{\mathbf{T},\Sigma}$ which decides $\Theta(M_G^0)$. By replacing Θ with its negation if needed, we may assume that p forces $\Theta(M_G^0)$. For every $q \in \mathbb{P}_{\mathbf{T},\Sigma}$ by Lemma 4.5 there exists an $h \in S_\infty$ such that $\Phi_h(p)$ is compatible with q. But Φ_h is an automorphism of $\mathbb{P}_{\mathbf{T},\Sigma}$ that sends M_G^0 to itself and M_G to itself, and since Θ is symmetric $\Phi_h(p)$ forces $\Theta(M_G^0)$. This implies that every condition in $\mathbb{P}_{\mathbf{T},\Sigma}$ decides $\Theta(M_G^0)$ the same way that p does. $\square$

Let $\mathbb{P}_{\mathbf{T}}$ be a forcing notion of the form $\mathbb{P}_{\mathbf{T},\Sigma}$ or $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$. To every first-order property Θ of a metric structure one can associate a countable family $\mathbb{F}_\Theta$ of dense subsets of $\mathbb{P}$ such that for every $p \in \mathbb{P}$ we have $p \Vdash \Theta(M_G)$ if and only if for every $\mathbb{F}_\Theta$-generic $G \subseteq \mathbb{P}$ containing p we have $\Theta(M_G)$. Therefore it is not necessary to pass to the forcing extension in order to find a sufficiently generic model M_G.

If $\mathbf{T}$ is a complete theory in a countable language that has an atomic model then Lemma 4.4 implies that $\mathbb{P}_{\mathbf{T}}$ forces M_G is atomic. In particular, $\mathbb{P}_{\mathbf{T}}$ forces that M_G omits a ground-model type $\mathbf{t}$ if and only if it is non-principal.

The set of all nonprincipal n-types is closed in $S_n(\mathbf{T})$ because a type is principal if and only if it is an isolated point in the logic topology, by [23, Lecture 3].

If $\mathbf{T}$ does not have an atomic model then Lemma 4.7 below gives a complexity estimate for the set of types forced to be omitted in M. If a theory $\mathbf{T}$ is not complete then the theory of M_G, as well as the set of types omitted in M_G, depends on the choice of G.

Lemma 4.7. *Suppose $\mathbf{T}$ is a theory in a countable language L and Σ is a set of L-formulas satisfying* (Σ1)–(Σ3)*.*

1. *For every $q \in \mathbb{P}_{\mathbf{T},\Sigma}$ the set of all types forced by q to be omitted in M_G is a $\mathbf{\Pi}^1_1$-set.*
2. *The set of all types such that some $q \in \mathbb{P}_{\mathbf{T},\Sigma}$ forces to be omitted in M_G is a $\mathbf{\Pi}^1_1$-set.*
3. *If $\mathbf{T}$ is in addition complete then for every type $\mathbf{t}$ we have that $\mathbb{P}_{\mathbf{T},\Sigma}$ either forces that $\mathbf{t}$ is realized in M_G or it forces that $\mathbf{t}$ is omitted in M_G.*

Proof. Let $\mathbb{P}_0$ be a fixed countable dense subset of $\mathbb{P}_{\mathbf{T},\Sigma}$ as in Lemma 4.1. A name for a function $h\colon \omega \to \omega$ can be identified with a set of triplets (p, m, n) such that $p \Vdash \dot{h}(m) = n$. Such set Z of triplets is a name for a function if and only if for every $m \in \omega$ the following hold.

1. The set of conditions p for which there exists n satisfying $(p, m, n) \in Z$ is dense in $\mathbb{P}_0$, and
2. If p and q are compatible conditions in $\mathbb{P}_0$, $(p, m, n) \in Z$, and $(q, m, k) \in Z$, then $n = k$.

The set of such names is clearly a Borel subset of $\mathbb{P}_0 \times \omega^2$.

(1) Fix $q \in \mathbb{P}_{\mathbf{T},\Sigma}$. Lemma 1.4 implies that q does not force that $\mathbf{t}$ is omitted in M_G if and only if there exists $p \leq q$ in $\mathbb{P}_0$ and a name $\dot{h}$ for an infinite branch of the tree $T_{D,\mathbf{t}}$ (with $D := M^0_G$). Condition p forces that $\dot{h}$ is an infinite branch of $T_{D,\mathbf{t}}$ if and only if for every n the set of $r \in \mathbb{P}_0$ such that r decides the values of $\dot{h}(j)$, for $j < n$, and that ($\mathbf{t}_\omega$1) and ($\mathbf{t}_\omega$2) of §1.1.3 hold for $j < n$. The set of pairs $(p, \dot{h})$ where $\dot{h}$ is a name for a function and these two conditions hold is clearly Borel.

(2) This is a consequence of (1) because $\mathbb{P}_{\mathbf{T}}$ has a countable dense set and a countable union of $\mathbf{\Pi}^1_1$ sets is $\mathbf{\Pi}^1_1$.

(3) This is an immediate consequence of Corollary 4.6. □

5. Uniform sequences of types

Suppose $\mathbf{T}$ is a theory in language L. For $m \in \omega$ a sequence $\mathbf{t}_n$, for $n \in \omega$, of m-ary types over $\mathbf{T}$ is *uniform* if there are m-ary formulas $\varphi_i(\bar{x})$ for $i \in \omega$ such that

1. $\mathbf{t}_n(\bar{x}) = \{\varphi_i(\bar{x}) \geq 2^{-n} : i \in \omega\}$ for every n, and
2. all φ_i, for $i \in \omega$, have the same modulus of uniform continuity.

If $\mathbf{t}_n$ and φ_i are as above, then

$$\psi(\bar{x}) := \inf_{i\in\omega} \varphi_i(\bar{x})$$

is an $L_{\omega_1\omega}$ formula (see [3]) whose modulus of uniform continuity is equal to the joint modulus of uniform continuity of φ_i, for $i \in \omega$. Therefore its interpretation is uniformly continuous in every L-structure A and $\sup_{\bar{x}} \psi(\bar{x})^A = 0$ if and only if A omits all $\mathbf{t}_n$ for $n \in \omega$.

For the simplicity of notation in the following we consider a single-sorted language.

Lemma 5.1. *Suppose* $\mathbf{T}$ *is a theory in a language* L *and* $\mathbf{t}_n$*, for* $n \in \omega$*, is a uniform sequence of* m*-types over* $\mathbf{T}$*. If* $M \models \mathbf{T}$ *then*

$$Z = \{\bar{a} \in M^m : \textit{for all } n,\ \mathbf{t}_n \textit{ is not realized by } \bar{a} \textit{ in } M\}$$

is a closed subset of M^m*. In particular, if* D *is a dense subset of* M*, then all* $\mathbf{t}_n$ *are omitted in* M *if and only if none of them is realized by any* m*-tuple of elements of* D*.*

Proof. Set Z is closed as the zero set of the interpretation of the continuous infinitary formula $\inf_{i\in\omega} \varphi_i(\bar{a})$. The last sentence of the lemma follows immediately. $\square$

The syntactic characterization of omissible uniform sequences of types given below is analogous to the syntactic characterization of complete omissible types given in [2]. As Itaï Ben Yaacov and Todor Tsankov pointed out, the set X of complete types extending a type in a uniform sequence of types is metrically open (§1.2.3) and therefore by a standard argument (see [2, §12] or [23, Lecture 4]) types in X are simultaneously omissible if and only if X is meager in the logic topology (§1.2.3). We spell out details of the proof below since we will need a similar argument in the case when the theory $\mathbf{T}$ is not necessarily complete in Theorem 5.3. As in §1.3, type $\mathbf{t}$ (complete or not) is said to be *principal* if for every $\varepsilon > 0$ the set

$$B_\varepsilon(\mathbf{t}) := \{\mathbf{s} \in S_n(\mathbf{T}) : \operatorname{dist}(\mathbf{s}, K_{\mathbf{t}}) < \varepsilon\}$$

is somewhere dense in the logic topology with respect to $\mathbf{T}$.

Theorem 5.2. *Suppose* $\mathbf{T}$ *is a complete theory in a countable language* L*. If for every* $m \in \omega$ *we have a uniform sequence of types*

$$\mathbf{t}^m_n = \{\varphi^m_j(\bar{x}(m)) \geq 2^{-n} : j \in \omega\},\ \textit{for } n \in \omega,$$

then the following are equivalent.

1. *None of the types* $\mathbf{t}^m_n$*, for* $m, n \in \omega$*, is principal.*
2. *Theory* $\mathbf{T}$ *has a model that omits all* $\mathbf{t}^m_n$*, for all* $n \in \omega$ *and* $m \in \omega$*.*
3. *There are no* $\delta > 0$*,* $m \in \omega$*, and condition* $\psi(\bar{x}(m)) < \varepsilon$ *such that* $\mathbf{T} \models \inf_{\bar{x}(m)} \psi(\bar{x}(m)) < \varepsilon$ *and* $\mathbf{T} + \psi(\bar{x}(m)) < \varepsilon \models \varphi^m_j(\bar{x}(m)) \geq \delta$ *for every* $j \in \omega$*.*

Proof. (1) and (2) are equivalent by Corollary 3.9.

In order to prove that (3) implies (1), assume (1) fails and let m and n be such that $\mathbf{t}^m_n$ is principal. Since types $\mathbf{t}^l_n$ for $l \neq m$ do not appear in this proof, we shall write $\bar{x}$ in place of $\bar{x}(m)$ throughout for simplicity. Since all formulas φ^m_j, for $j < \omega$, have the same modulus of uniform continuity we can find $\delta > 0$ such that for all $\bar{x}$ and $\bar{y}$ of the appropriate sort $\max_i d(x_i, y_i) < \delta$ implies $|\varphi^m_j(\bar{x}) - \varphi^m_j(\bar{y})| \leq 2^{-n-1}$ for all j. Since $\mathbf{t}^m_n$ is principal, there is a condition $\psi(\bar{d}) < \varepsilon$ such that in every $M \models \mathbf{T}$ we have

$$\{\bar{a} : \psi(\bar{a}^M) < \varepsilon\} \subseteq \{\bar{a} : \operatorname{dist}(\bar{a}, \mathbf{t}^m_n(M)) < \delta\}.$$

Therefore $\mathbf{T} + \psi(\bar{x}) < \varepsilon \models \varphi^m_j(\bar{x}) \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}$ for all j and (3) fails.

In order to prove that (2) implies (3), assume (3) fails. Fix a condition $\psi(\bar{x}) < \varepsilon$ and $\delta > 0$ such that $\mathbf{T} \models \inf_{\bar{x}} \psi(\bar{x}) < \varepsilon$ and $\mathbf{T} + \psi(\bar{x}) < \varepsilon \models \varphi^m_j(\bar{x}) \geq \delta$ for all j. If $2^{-n} < \delta$ then clearly every model of $\mathbf{T}$ realizes $\mathbf{t}^m_n$, hence (2) fails. □

Theorem 5.2 has an analogue in the case when theory $\mathbf{T}$ is not necessarily complete.

Theorem 5.3. *Suppose $\mathbf{T}$ is a not necessarily complete theory in a countable language L. If for every $m \in \omega$ we have a uniform sequence of types*

$$\mathbf{t}^m_n = \{\varphi^m_j(\bar{x}(m)) \geq 2^{-n} : j \in \omega\}, \ \textit{for } n \in \omega,$$

then the following are equivalent.

1. *$\mathbf{T}$ has a model omitting all $\mathbf{t}^m_n$, for all m and n in ω.*
2. *There are no $\delta > 0$, finite $F \subseteq \omega$, and conditions $\psi_m(\bar{x}(m)) < \varepsilon$ for $m \in F$ such that $\mathbf{T} \models \min_{m \in F} \inf_{\bar{x}(m)} \psi_m(\bar{x}(m)) < \varepsilon$ and*
$$\mathbf{T} + \psi_m(\bar{x}(m)) < \varepsilon \models \varphi^m_j(\bar{x}(m)) \geq \delta$$
for every $j \in \omega$ and $m \in F$.

Proof. Recall that by Lemma 1.1 condition (2) is equivalent to its special case in which $\varepsilon = 1$.

By Theorem 5.2 it suffices to show that $\mathbf{T}$ satisfies (2) if and only if it can be extended to a complete theory that still satisfies (2). Only the direct implication requires a proof, analogous to the corresponding proof in the first order case. Let Θ_k, for $k \in \omega$, enumerate a countable dense set of L-sentences such that each sentence occurs infinitely often. By Lemma 1.1 for a closed interval $V \subseteq [0, 1]$ and a sentence Θ the condition $\Theta \in V$ is equivalent to one of the form $\Theta' = 0$ for some Θ'.

Assume $\mathbf{T}$ satisfies (2). We shall find an increasing chain of theories $\mathbf{T} = \mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \dots$ and closed intervals $U_n \subseteq [0, 1]$ of diameter at most 2^{-n} such that for all n we have

$$\mathbf{T}_n \models \Theta_n \in U_n$$

and $\mathbf{T}_n$ still satisfies (2).

Assume that for some n both $\mathbf{T}_n$ and U_k for $k \leq n$ as required have been chosen. Fix a finite cover $\mathcal{V}$ of $[0,1]$ by closed intervals of diameter at most 2^{-n+1}. We claim that there exists $V \in \mathcal{V}$ such that $\mathbf{T}_n \cup \{\Theta_n \in V\}$ (identified with a theory by using the closed condition case of Lemma 1.1) still satisfies (2).

Assume otherwise. For every $V \in \mathcal{V}$ we can find a finite $F(V) \subseteq \omega$ such that for every $m \in F(V)$ there exist an open condition $\psi_{V,m}(\bar{x}(m)) < 1$ and $\delta(V,m) > 0$ such that the following two conditions hold:

$$\mathbf{T}_n + \Theta_{n+1} \in V \models \min_{m \in F(V)} \inf_{\bar{x}(m)} \psi_{V,m}(\bar{x}(m)) < 1$$

$$\mathbf{T}_n + \Theta_{n+1} + \psi_{V,m}(\bar{x}(m)) < 1 \models \inf_{j \in \omega} \varphi_j^m(\bar{x}(m)) \geq \delta(V,m).$$

Let

$$\Theta' := \min_{V \in \mathcal{V}} \operatorname{dist}(\Theta_{n+1}, V) \quad \text{and} \quad \delta := \min_{V \in \mathcal{V}, m \in F(V)} \delta(V,m).$$

Then $\mathbf{T}_n \models \Theta' = 0$ and $\mathbf{T}_n \models \min_{V \in \mathcal{V}, m \in F(V)} \inf_{\bar{x}(m)} \psi_{V,m}(\bar{x}(m)) < 1$ but for each $V \in \mathcal{V}$ we have

$$\mathbf{T}_n + \psi_{V,m}(\bar{x}(m)) < 1 \models \inf_{j \in \omega} \varphi_j^m(\bar{x}(m)) \geq \delta,$$

contradicting (2). This contradiction implies that we can find $V \in \mathcal{V}$ such that adding condition $\Theta_n \in V$ to $\mathbf{T}_n$ preserves condition (2).

Once all $\mathbf{T}_n$ for $n \in \omega$ are constructed, theory $\mathbf{T}_\infty := \bigcup_n \mathbf{T}_n$ decides the value of each Θ_n and is therefore complete. Since the finitary condition (2) is satisfied by every finite fragment of $\mathbf{T}_\infty$, it is satisfied by $\mathbf{T}_\infty$ and Theorem 5.2 can now be applied to $\mathbf{T}_\infty$. □

5.1. **Uniform sequences of types and forcing.** A class $\mathfrak{M}$ of models is *uniformly definable by a sequence of types* if there is a set of sequences of uniform types $\langle \mathbf{t}_n^m : n \in \omega \rangle$, for $m \in I$ such that A is in $\mathfrak{M}$ if and only if it omits all of these types. This is a special case of being "definable by uniform families of formulas" as in [16, Definition 5.7.1].

If Σ is the set of all formulas of the language of $\mathbf{T}$ then we denote $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ by $\mathbb{P}_{\mathbf{T},\mathfrak{M}}$.

Proposition 5.4. *Assume $\mathfrak{M}$ is a nonempty class of models of a not necessarily complete theory $\mathbf{T}$. If $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of types that are omitted in every model in $\mathfrak{M}$ then $\mathbb{P}_{\mathbf{T},\mathfrak{M}}$ forces that M_G omits all $\mathbf{t}_n$.*

Proof. Since $\{\mathbf{d}_j : j \in \omega\}$ is dense in M_G, Lemma 5.1 implies that $\mathbf{t}_n^m$ is realized in M_G if and only if it is realized by some $\mathbf{d}_F$ Suppose that some condition p forces that a tuple $F \subseteq F^p$ realizes $\mathbf{t}_n^m$ for some m and n in ω. But there is $M \in \mathfrak{M}$ and a tuple $\bar{a}$ in M of the appropriate sort such that $M \models \psi^p(\bar{a}) < \varepsilon^p$. Since M omits $\mathbf{t}_n^m$, we can extend p to a condition that decides that F does not satisfy some condition in $\mathbf{t}_n^m$, contradicting our assumption on p. □

Some of the most important properties of C*-algebras are uniformly definable by a sequence of types. This includes being AF, UHF ([9]), $C(X)$ for a chainable continuum ([11]), TAF, Popa algebra or quasidiagonal, having nuclear dimension $\leq n$, decomposition rank $\leq n$, for $n \in \omega$, ([16, Theorem 5.7.3]). (See however [21] and [20] for some negative examples.) These types are particularly simple and we include a straightforward technical sharpening of Proposition 5.4 with an eye to potential applications.

A uniform sequence of types $\mathbf{t}_n = \{\varphi_j(\bar{x}) \geq 2^{-n} : j \in \omega\}$, for $n \in \omega$, is *universal* if every $\varphi_j(\bar{x})$ is of the form $\inf_{\bar{y}} \psi(\bar{y}, \bar{x})$ for some quantifier-free formula $\psi(\bar{y}, \bar{x})$. (By Lemma 1.1 a condition of the form $\inf_{\bar{y}} \psi(\bar{y}, \bar{x}) \geq 2^{-n}$ is equivalent to a condition of the form $\sup_{\bar{y}} \psi'(\bar{y}, \bar{x}) = 0$, and formulas on the left-hand side of this expression are commonly recognized as universal given the somewhat controversial convention that $M \models \Theta = 0$ is interpreted as 'Θ is true in M'.) The following theorem probably follows from the results from [3] but we include it for reader's convenience.

Proposition 5.5. *Assume that $\mathfrak{M}$ is a nonempty class of models of a (not necessarily complete) theory $\mathbf{T}$ and Σ is a set of formulas in the language of $\mathbf{T}$ that satisfies* (Σ1)–(Σ3). *If $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of universal types that are omitted in every model in $\mathfrak{M}$ then $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ forces that M_G omits all $\mathbf{t}_n$.*

If $\mathbf{T}$ is $\forall\exists$-axiomatizable then $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ forces that the generic model satisfies $\mathbf{T}$.

Proof. Lemma 5.1 implies that M_D realizes $\mathbf{t}_m$ if and only if some $\mathbf{d}_F$ satisfies it. Suppose that some condition p forces that a tuple $F \subseteq F^p$ realizes $\mathbf{t}_n$ for some $n \in \omega$. There are $M \in \mathfrak{M}$ and a tuple $\bar{a}$ in M of the appropriate sort such that $M \models \psi^p(\bar{a}) < \varepsilon^p$. Since M omits $\mathbf{t}_n$, there exists a condition $\inf_{\bar{y}} \psi(\bar{y}, \bar{x}) \geq 2^{-n}$ in $\mathbf{t}_n$ such that $\inf_{\bar{y}} \psi(\bar{y}, \bar{a})^M < 2^{-n} - \varepsilon$ for some $\varepsilon > 0$ and ψ is quantifier-free. If F' is of cardinality $|\bar{y}|$, by the open case of Lemma 1.1 we have that $\psi(\bar{d}_{F'}, \bar{d}_F) < 2^{-n} - \varepsilon$ is equivalent to a condition in $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$. Then M certifies that this condition is compatible with p, and it decides that d_F does not realize $\bar{t}_n$.

Since $\mathbf{T}$ is $\forall\exists$-axiomatizable, $M_G \models \mathbf{T}$ by (4) of Theorem 3.3. □

If $\mathfrak{M}$ is a class of L-models then $M \in \mathfrak{M}$ is *existentially closed* (often abbreviated as *e.c.*) for $\mathfrak{M}$ if whenever $N \in \mathfrak{M}$ is such that M is isomorphic to a submodel of N, $\bar{a} \in M$, and $\varphi(\bar{x}, \bar{y})$ is a quantifier-free L-formula then

$$\inf_{\bar{y}} \varphi(\bar{a}, \bar{y})^M = \inf_{\bar{y}} \varphi(\bar{a}, \bar{y})^N.$$

Existentially closed C*-algebras and II_1 factors were studied in [22], [16, §6], and [15].

Corollary 5.6. *Assume $\mathbf{T}$ is an $\forall\exists$-axiomatizable theory in a countable language and $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of universal types over $\mathbf{T}$. If the class $\mathfrak{M}$ of all models of $\mathbf{T}$ that omit all $\mathbf{t}_n$, for $n \in \omega$, is nonempty then it contains a model that is existentially closed for $\mathfrak{M}$.*

Proof. Let Σ be the set of all quantifier-free formulas in the language of $\mathbf{T}$. Therefore $\mathbb{P}_{\mathbf{T},\Sigma}$-generic models are "finitely generic" in the terminology of A. Robinson (as used in [22]). By Proposition 5.5, the $\mathbb{P}_{\mathbf{T},\Sigma}$-generic model M_G is forced to omit all $\mathbf{t}_n$ and to satisfy $\mathbf{T}$, and is therefore forced to belong to $\mathfrak{M}$.

In order to prove that M_G is forced to be existentially closed suppose it is isomorphic to a submodel of a model N of $\mathbf{T}$. By continuity it suffices to assure that for a countable dense set of quantifier-free L^+-formulas $\varphi(\bar{d}_F, \bar{y})$ we have

(1) $\inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y})^{M_G} = \inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y})^N$.

Fix $\varphi(\bar{d}_F, \bar{y})$ and $r \in \mathbb{Q}$. Let k be the length of $\bar{y}$. Consider the set $\mathbf{D}_{\varphi(\bar{d}_F,\bar{y}),r}$ of conditions $q \in \mathbb{P}_{\mathbf{T}}$ such that $F^q \supseteq F$ and one of the following holds.

(2) $\mathbf{T} + q \models \inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y}) \geq r$, or
(3) $\mathbf{T} + q \models \inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y}) < r$.

We claim that this set is dense in $\mathbb{P}_{\mathbf{T}}$. Suppose $p \in \mathbb{P}$ is such that $\mathbf{T} + p \not\models \inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y}) \geq r$. Fix N that satisfies p and $\inf_{\bar{y}} \varphi(\bar{d}_F, \bar{y})^N < r$. Since φ is quantifier-free, we can extend p to decide $\bar{b}$ in N such that $\varphi(\bar{d}_F, \bar{b})^N < r$. This gives a condition $q \leq p$ in $\mathbf{D}_{\varphi(\bar{d}_F,\bar{y}),r}$, and proves the density.

Let $\varphi_n(\bar{x}, \bar{y})$, for $n \in \omega$, be an enumeration of a dense set of L^+ formulas with two distinguished tuples of free variables. Choose filter G of $\mathbb{P}_{\mathbf{T}}$ that meets dense open subsets of the form $\mathbf{D}_{\varphi_n(\bar{d}_F,\bar{y}),r}$ for all finite $F \subseteq \omega$ of the appropriate size and all $r \in \mathbb{Q}_+$. Then M_G satisfies (1) for each φ_n and each F of the appropriate size. By the continuity, M_G is existentially closed. $\square$

We conclude this section with a complexity result related to Proposition 1.11 and Theorem 1 (see §7.1 for applications). A standard Borel space of L-structures $\hat{\mathcal{M}}(L)$ was defined in §1.2.1.

Proposition 5.7. *Assume* $\mathbf{T}$ *is a theory in a countable language* L*. Then every class* $\mathfrak{M}$ *of separable models of* $\mathbf{T}$ *uniformly definable by a sequence of types forms a Borel subset of the space of models of* $\mathbf{T}$*.*

Proof. This is a special case of the continuous variant of the standard fact that the set of models of an $L_{\omega_1\omega}$ sentence is Borel. Fix a set of sequences of uniform types $\langle \mathbf{t}^m_n : n \in \omega\rangle$, for $m \in \omega$, as in §5.1. By Lemma 5.1, model M omits a uniform sequence of types $\{\mathbf{t}^m_n : n \in \omega\}$ if and only if each of the elements of its countable dense set omits this sequence. Thus for $\gamma \in \hat{\mathcal{M}}(L)$ we have $M(\gamma) \in \mathfrak{M}$ if and only if (let $l(m,n)$ denote the arity of $\mathbf{t}^m_n$, let $[\omega]^{l(m,n)}$ denote the set of subsets of ω of cardinality $l(m,n)$, and let $\gamma_F := \langle \gamma_i : i \in F\rangle$)

$$(\forall m)(\forall n)(\exists k \geq 1)(\forall F \in [\omega]^{l(m,n)})(\exists j \in \omega)\varphi^m_n(\gamma_F) \geq 1/k.$$

Since all quantifiers range over countable sets, this is a Borel condition. $\square$

6. Simultaneous omission of types

We prove Theorem 2 by constructing an example of a complete theory $\mathbf{T}_4$ in a countable language L_4 and types $\mathbf{s}_n$, for $n \in \omega$, such that for every k there exists a model of $\mathbf{T}_4$ that omits all $\mathbf{s}_n$ for $n \leq k$ but no model of $\mathbf{T}_4$ simultaneously omits all $\mathbf{s}_n$.

We shall prove that in addition all types $\mathbf{s}_n$, for $n > 0$, are simultaneously omissible in a single model M_4 of $\mathbf{T}_4$. As a matter of fact, we shall first define M_4 and then take $\mathbf{T}_4$ to be its theory. Type $\mathbf{s}_0$ has a distinguished role in this construction and it is not to be confused with the type $\mathbf{s}_0$ as defined in §2.4.

Let $\operatorname{rank}(T)$ denote the rank of a well-founded tree T and let ρ_T denote the rank function on T. Hence $\operatorname{rank}(T) = \sup_{t \in T} \rho_T(t)$, with $\sup \emptyset = 0$. We write $\operatorname{rank}(T) = \infty$ if T is ill-founded. The disjoint union of a sequence T_i for $i \in \omega$ of trees is denoted by $\bigoplus_i T_i$. Thus $\operatorname{rank}(\bigoplus_i T_i) = \sup_i \operatorname{rank}(T_i)$. We write $\bigoplus_\omega T$ for $\bigoplus_i T_i$ if all T_i are equal to T.

6.1. **Some definable sets.** Given a tree T of height ω, an L_N^- structure and its theory $\mathbf{T}_T$ were defined in §2.3. As discussed in §2.3, any model N of $\mathbf{T}_T$ contains a tree T_N as a dense subset and the remaining elements of N are naturally identified with the infinite branches of T_N. The following lemma was added because of a request of the anonymous referee.

Lemma 6.1. *Suppose T is a tree of height ω, $N \models \mathbf{T}_T$, and $m \geq 1$.*

(1) *Both the set of all nodes of height $> m$ and the set of all nodes of height $\leq m$ are definable subsets of N.*
(2) *The set of all nodes of height m (called the mth* level *of N) is a definable subset of N.*
(3) *For a node a in N and $k > m = |a|$ the set*
$$\operatorname{Succ}_k(a) = \{b \in N : f_m(b) = a \text{ and } |b| = k\}$$
is a definable (from parameter a) subset of N.
(4) *For m and n in ω the set*
$$N_{m,n} := \{a \in N : |a| = m, \rho_N(a) \geq n+1\}$$
is a definable subset of N.

Proof. Since the height of s is $> m$ if and only if $d(s, f_m(s)) < 1/m$, and this is equivalent to $d(s, f_m(s)) = 0$, (1) follows by Lemma 1.13.

(2) Let
$$\varphi_m(x) := d(f_m(x), x) + \left| \frac{1}{m} - d(f_{m-1}(x), x) \right|.$$
Then $\varphi_m(a)^N = 0$ if and only if $f_m(a) = a$ and $d(f_{m-1}(a), a) \geq 1/m$. If $b \in N$ is a node and $|b| > m$ or it is a branch then $d(f_m(b), b) \geq 1/(m+1)$. If $b \in N$ is a node with $|b| < m$ then $d(f_{m-1}(b), b) > 1/m$. Therefore the mth level of N is the zero-set of φ_m. Since $\varphi_m(b)^N < 1/m$ implies $\varphi_m(b)^N = 0$, by Lemma 1.13 the mth level of N is definable.

(3) For $m < k$ let $\psi_{m,k}(x,y) := d(f_m(x),y) + \varphi_k(x)$. Then $\mathrm{Succ}_k(a)$ is the zero-set of $\psi_{m,k}(x,a)$ and $\psi_{m,k}(x,a) < 1/(k+1)$ implies $\psi_{m,k}(x,a) = 0$. Therefore $\mathrm{Succ}_k(a)$ is definable from parameter a by Lemma 1.13.

(4) Let φ_m and $\psi_{m,n}$ be as in the earlier part of this proof. Then $N_{m,n}$ is the zero-set of $\varphi_m(x)+\inf_y \psi_{m,n}(y,x)$, and it is definable by Lemma 1.13. □

6.2. **Trees $T_1, T_2, T_{1,m}$, and $S^\frown T$.** Let T_1 be the tree of all strictly decreasing sequences of natural numbers:

$$T_1 = \{s \colon n \to \omega : n \in \omega, s(i) > s(i+1) \text{ for } 0 \leq i < n-1\}$$

ordered by the extension. This is a well-founded tree of rank ω and

$$\rho_{T_1}(s) = s(|s|-1)$$

for all $s \in T_1 \setminus \{\langle\rangle\}$. Let T_2 denote the 'wider' version of the tree T_1 defined as (a function $s \colon n \to \omega \times \omega$ is identified with the pair of functions $s_0 \colon n \to \omega$ and $s_1 \colon n \to \omega$):

$$T_2 = \{s \colon n \to \omega \times \omega : n \in \omega, s_0(i) > s_0(i+1) \text{ for } 0 \leq i < n-1\}.$$

This is also a well-founded tree of rank ω and $\rho_{T_2}(s) = s_0(|s|-1)$ for all s.

Lemma 6.2. *For every node t in T_2 and every immediate successor s of t there are infinitely many immediate successors s' of t such that there is an automorphism of T_2 swapping s and s'.*

Proof. Suppose s is an immediate successor of t. Let s' be an immediate successor of t such that $s'_0 = s_0$ and $s'_1 \neq s_1$. Let Φ be an automorphism of T_2 that swaps the cones $\{r \in T_2 : s \sqsubseteq r\}$ and $\{r \in T_2 : s' \sqsubseteq r\}$ and leaves all other nodes in T_2 fixed. Since there are infinitely many choices for $s'_1(|s|)$, for every s there are infinitely many such pairs s', Φ. □

Given trees S and T, by $S^\frown T$ we denote the tree obtained by tagging ω copies of T to every node of S. Formally, we identify S and T with some trees of finite sequences from a large enough set (every tree is isomorphic to one of this form) and with $s^\frown t$ denoting the concatenation of s and t, we let (assuming that S and $\bigoplus_\omega T$ are disjoint)

$$S^\frown T = \{s^\frown t : s \in S, t \in \textstyle\bigoplus_\omega T\}$$

with the extension ordering. If T is a tree of finite sequences of elements from some set, then it has a unique root denoted $\langle\rangle$, and S can be identified with the subtree $\{s^\frown\langle\rangle : s \in S\}$ of $S^\frown T$. We record two straightforward facts about this construction.

(1) $\rho_{S^\frown T}(s) = \mathrm{rank}(T) + \rho_S(s)$ for every $s \in S$.
(2) $\mathrm{rank}(S^\frown T) = \mathrm{rank}(T) + \mathrm{rank}(S)$.

For $m \geq 1$ let

$$T_{1,m} := \{s \in T_1 : \mathrm{range}(s) \subseteq m\}.$$

We'll prove that if t is a node of rank $\geq m$ in a 'sufficiently rich' tree T then the cone in T above t contains a 'terminal' copy of $T_{1,m}$.

Lemma 6.3. *Suppose T is an $L_{\mathcal{N}}^-$ (§2.3) structure elementarily equivalent to $S^\frown T_1$ for some tree S and $s \in T$ is such that $\rho_T(s) \geq n$ for some $n \geq 1$. Then there exists $\Phi\colon T_{1,n} \to T$ such that the following hold for all t and t' in $T_{1,n}$ (by $|t|$ we denote the level of a node t):*

(3) $s \sqsubseteq \Phi(t)$,
(4) $t \sqsubseteq t'$ *if and only if* $\Phi(t) \sqsubseteq \Phi(t')$,
(5) $|\Phi(t)| = |s| + |t|$,
(6) t *is a terminal node in* $T_{1,n}$ *if and only if* $\Phi(t)$ *is a terminal node in* T.

Proof. Both $T = T_1$ and $T = S^\frown T_1$ clearly have this property. We therefore have to check that it is preserved by elementary equivalence. The kth level of T is definable (Lemma 6.1) and the distance between any two of its elements is $1/k$. Therefore for every m and all $l \geq 1$ and $n \geq 1$ each of the following is a definable subset of any $L_{\mathcal{N}}$-structure:

(7) The set of all elements x of the mth level with at least l successors on the $m+1$st level.
(8) The set of all elements x of the mth level with at least l distinct immediate successors, each of which is a terminal node.
(9) The set of all elements x of the mth level such that x has has at least l distinct immediate successors, each of which has rank $\geq n$.

If t is a non-terminal node on the mth level of $S^\frown T_1$, it belongs to sets described in both (8) and (9), for all $l \in \omega$. If in addition $\rho_{S^\frown T_1}(t) \geq n+1$ then for all $l \in \omega$ node t belongs to each of the sets described in (8) and (9). By the elementarity these implications transfer to S, and Φ as required is constructed by recursion. □

6.3. **Language L_4, models $M(k)$ and M_4, and theory $\mathbf{T}_4$.** We shall describe an expansion L_4 of $L_{\mathcal{N}}^-$ (§2.3) with the additional unary predicate symbols $P_{i,j}$ for $i > 0$ and $j \geq 0$ in ω such that each $P_{i,j}$ is $i+1$-Lipshitz.

Fix an injection

$$k\colon T_2 \to \omega$$

such that $s \sqsubset t$ implies $k(s) < k(t)$ for all s and t in T_2. We define an L_4-structure $M(k)$ whose universe is T_2 as follows. The function symbols f_k for $k \in \omega$ are interpreted as in §2.2, by $f_k(a) := b$ if b is the unique element of the j-th level below a if there is such b or $f_j(a) := a$ otherwise.

Predicates $P_{i,j}$ code a colouring of the nodes of T_2 as follows. If $|t| = i$ then we let (t^- denotes the immediate predecessor of $t \in T_2$ and $t = (t_0, t_1)$)

$$P_{i,j}(t) := \begin{cases} 0, \text{ if } |t| = i, j \geq k(t^-), \text{ and } j - k(t^-) = \max\{l : 2^l | t_1(i-1)\}, \\ 1, \text{ otherwise.} \end{cases}$$

Since $d(s,t) < 1/(i+1)$ implies $\min(|s|, |t|) > i$ and $P_{i,j}(s) = P_{i,j}(t) = 1$, the predicate $P_{i,j}$ as defined above is $i+1$-Lipshitz for all $i \geq 1$ and all j.

For every $t \in T_2$ the following hold (the condition $P_{i,j}(t) = 0$ is interpreted as 'the node t is (i,j)-coloured'):

(P1) Node t is not (i,j)-coloured for any $i \neq |t|$ and $j \geq 0$.

(P2) Node t is $(|t|,j)$-coloured for exactly one j.

(P3) If $j \geq k(t)$ and t is not a terminal node of T_2 then for every $m < \rho_{T_2}(t)$ there are infinitely many immediate $(|t|+1,j)$-coloured successors s of t such that $\rho_{T_2}(s) = m$.

(P4) No immediate successor of t is (i,j)-coloured for any $j < k(t)$.

Conditions (P1), (P2), and (P4) are clearly satisfied. Condition (P3) is satisfied because for all $m \in \omega$ and $n \in \omega$, for every $s \in T_2$ with $\rho_{T_2}(s) > m$ there exists an immediate successor t of s with $\rho_{T_2}(t) = m$ and $t_1(i-1) = n$. In particular for every $l \in \omega$ there are infinitely many immediate successors t of s with $\rho_{T_2}(t) = m$ such that 2^l divides $t_1(i-1)$ but 2^{l+1} does not.

Fix a family k^n, for $n \in \omega$, of functions such that

(k0) For all n the function $k^n \colon T_2 \to \omega$ is an injection.

(k1) For all n and $s \sqsubset t$ in T_2 we have $k^n(s) < k^n(t)$ (k^n is increasing).

(k2) Every increasing injection k from a finite subtree of T_2 into ω is extended by k^n for infinitely many n.

With $M(k^n)$ denoting $M(k)$ with $k = k^n$, let

$$M_4 := \bigoplus_n M(k^n)^\frown T_2.$$

The *bottom part* of M_4 is $\bigoplus_n M(\vec{k}^n)$. All other nodes comprise the *top part* of M_4. The interpretations of f_m and $P_{i,j}$ in the bottom part of M_4 agree with those in $M(k^n)$. Suppose t is in the top part of M_4. If $i > 0, j \geq 0$, and t belongs to the top part of M_4 then we let $P_{i,j}(t) = 1$; i.e. t is not (i,j)-coloured for any pair (i,j). Clearly the interpretation of each $P_{i,j}$ is $i+1$-Lipshitz. The functions f_m are interpreted in the usual way, by letting $f_m(t) := t$ if $|t| < m$ and $f_m(t) := s$ if s is the unique node on the mth level of M_4 extended by s.

Since its underlying tree $(\bigoplus_n T_2)^\frown T_2$ is well-founded, M_4 is a discrete metric space and therefore complete. Let

$$\mathbf{T}_4 := \mathrm{Th}(M_4).$$

We record some consequences of the definitions and properties of rank stated in (1) and (2). Since M_4 is well-founded, all of its elements are nodes.

(10) A node t of M_4 is coloured if and only if it belongs to the bottom part of M_4. Its colour (i,j) satisfies $i = |t|$ and $j \geq k^n(t^-)$ for n such that $t \in M(k^n)$.

(11) A node t of M_4 satisfies $\rho_{M_4}(t) < \omega$ if t belongs to the top part of M_4 and $\omega \leq \rho_{M_4}(t) < 2\omega$ if t belongs to the bottom part of M_4.

Suppose $N \models \mathbf{T}_4$. Analogously to §2.3,

$$T_N := \{a \in N : (\exists j \in \omega) f_j(a) = a\}$$

is a tree dense in N. Every $b \in N \setminus T_N$ can be identified with the branch $\{f_j(b) : j \in \omega\}$, and this branch is a Cauchy sequence converging to b. If $s \in N$ then we write $\rho_N(s)$ for $\rho_{T_N}(s)$ if $s \in T_N$ and $\rho_N(s) = \infty$ otherwise.

then we shall frequently refer to its underlying tree as N, and write for example $\rho_N(s)$ for the rank of a node $s \in N$. Use the terminology and notation as in §2.3 and refer to the elements of T_N and $N \setminus T_N$ as *nodes* and *branches* of N. We also write $|s|$ and $s \sqsubseteq t$ for nodes s and t of N.

The *bottom part* of N is defined to be the set of all of its coloured nodes. We prove that the bottom part of N is an initial segment of N.

(12) If $N \models T_4$, t is a coloured node of N and $s \sqsubseteq t$, then s is coloured.

(13) If $N \models T_4$ and t is a node in N of rank at least $m \geq 1$, then the cone $\{s \in N : t \sqsubseteq s\}$ contains a terminal copy of

$$T_{1,m} := \{s \in T_1 : \mathrm{range}(s) \subseteq m\}$$

as in Lemma 6.3.

The latter statement follows by Lemma 6.3 and we prove (12). Since each k^n is increasing (see (k1)), if t is an (i,j)-coloured node in M_4 then its immediate successor s is $(i-1,l)$-coloured for some $l < j$. Since the first i levels of T_N form a definable set (Lemma 6.1), for a fixed i and j this can be expressed as an L_4-sentence and therefore transfers from M_4 to N. By induction, (12) follows.

6.4. **Types $\mathbf{s}_m$.** Let $\mathbf{s}_0(x)$ be the 1-type of an infinite branch,

$$\mathbf{s}_0(x) := \{d(x, f_n(x)) = 1/(n+1) : n \in \omega\}.$$

In order to define $\mathbf{s}_m(x)$ for $m > 0$, fix a bijection $m \mapsto (m(0), m(1))$ between $\{m \in \omega : m > 0\}$ and $\{(m(0), m(1)) \in \omega^2 : m(0) \leq m(1)\}$. The type $\mathbf{s}_m(x)$ consists of conditions described in (S1)–(S3) below. As in §6.1, if a is a node in a tree T and $k > m = |a|$ then

$$\mathrm{Succ}_k(a) := \{b \in T : f_m(b) = a \text{ and } |b| = k\}$$

and

$$\varphi_m(x) := d(f_m(x), x) + \left| \frac{1}{m} - d(f_{m-1}(x), x) \right|.$$

The condition in (S3) is an abbreviation for an L_4-formula because allowing quantification over definable sets results in conservative extension of the language (see §1.5).

(S1) $\varphi_{m(0)}(x) = 0$.

(S2) $P_{m(0),m(1)}(x) = 0$.

(S3) $\sup_{y \in \mathrm{Succ}_{m(0)+1}(x)} (1 \dot{-} P_{m(0)+1,n}(y)) = 0$ for all $n \in \omega$.

Lemma 6.4. *If $N \models \mathbf{T}_4$ and $a \in N$ then a realizes $\mathbf{s}_m$ for some $m > 0$ if and only if a is a terminal node of the bottom part of N.*

Proof. Suppose a realizes $\mathbf{s}_m$ in N and $m > 0$. Then (S1) implies $|a| = m(0)$, (S2) implies a is coloured, and (S3) implies that all immediate successors of a are uncoloured. Therefore a is a terminal node of the bottom part of N. Conversely, suppose a is a terminal node of the bottom part of N. If m is such that a belongs to the $m(0)$th level of N and it is $(m(0), m(1))$-coloured, then a realizes $\mathbf{s}_m$. $\square$

6.5. **Language L_l and models $N \restriction l$.** Fix $l < \omega$ and a separable L_4-structure N. We shall be interested in the case when $N \models \mathbf{T}_4$ or when N is one of the building blocks $M(k^n)$, for $n \in \omega$, for M_4. Let

$$N \restriction l := \{s \in N : |s| \leq l\}$$

(this set is also equal to $\{f_i(a) : i \leq l, a \in N\}$). Let L_l denote the finite reduct of L_4 consisting of symbols $P_{i,j}$ and f_i for $\max(i,j) < l$. We shall consider $N \restriction l$ as an L_l-structure. Hence it is an L_l-reduct of the substructure of N defined above. A node $s \in N \restriction l$ is *L_l-coloured* if it is (i,j)-coloured for some $P_{i,j}$ in L_l. A node $s \in N \restriction l$ is *full* if there exists $t \sqsupseteq s$ with $|t| = l$. Every L_l-coloured node is full by (11). An uncoloured node s if full if and only if $|s| + \rho_N(s) \geq l$.

Consider the following substructure of $N \restriction l$.

$$T[N \restriction l] := \{s \in N \restriction l : s \text{ is full}\}.$$

Since N is separable, by (13) and recursion the universe of $T[N \restriction l]$ is isomorphic to $\omega^{<l}$, the full ω-branching tree of height l.

Suppose $s \in T[N \restriction l]$ satisfies $|s| = i < l$ and it has an L_l-coloured immediate successor. Let

$$k'(s) := \min\{j : (\exists t) s \sqsubseteq t \text{ and } P_{i+1,j}(t) = 0\}.$$

Lemma 6.5. *Fix $n < \omega$ and consider model $M(k^n)$.*

(14) *The set $X[n,l]$ of $s \in T[M(k^n) \restriction l]$ with a coloured successor is a finite subtree of $T[M \restriction l]$.*
(15) *For every $s \in X[n,l]$ we have $k'(s) = k^n(s)$.*
(16) *For a fixed l there are only finitely many isomorphism types of models $\{M(k^m) \restriction l : m \in \omega\}$.*
(17) *For every L_l-coloured node $b \in M_4 \restriction l$ there exists an automorphism Φ of $M_4 \restriction l$ such that $\Phi(b)$ is not a terminal node of the bottom part of M_4.*

Proof. (14) follows from (k0), (k1) and (P3). (15) is an immediate consequence of (P3).

Since k^n is an injection, the set $F := \{t \in T_2 : k^n(t) < l\}$ is finite. Since $M(k^n) \restriction l$ depends only on the restriction of k^n to F, (16) follows.

(17) If b has a coloured successor take Φ to be the identity map. We may therefore assume b is an immediate successor of a terminal node of $X[n,l]$ for some n. We may identify $X[n,l]$ with a subset of T_2. Choose a subtree $Y \subseteq T_2$ which is isomorphic to $X[n,l] \cup \{b\}$ but the node corresponding to b is not a terminal node in T_2. (This is possible because T_2 includes subtrees isomorphic to $\omega^{<m}$ for an arbitrarily large $m \in \omega$.) Let $\Phi_0 \colon Y \to X[n,l] \cup \{b\}$ be this isomorphism. By (k2) there exists n' such that $k^{n'} \restriction Y = k^n \circ \Phi_0$. Then Φ_0 extends to an isomorphism $\Phi_1 \colon M(k^{n'}) \restriction l \to M(k^n) \restriction l$, and Φ_0^{-1} sends b to a non-terminal node of T_2. Finally, we extend $\Phi_1 \cup \Phi_1^{-1}$ to an automorphism of $M_4 \restriction l$ that is equal to the identity on $M(k^m) \restriction l$ for all $m \notin \{n, n'\}$. $\square$

6.6. **The Extension Lemma.** We prove that the type $\mathbf{s}_m(x)$ is generically omissible for all $m > 0$. In the following $\mathbb{P}_{\mathbf{T}}$ denotes the 'infinite' forcing as defined in §3 with $\mathbf{T} = \mathbf{T}_4$, which is $\mathbb{P}_{\mathbf{T},\Sigma}$ in case when Σ is the set of all L-formulas and $\bar{x}, y, z$ stand for assorted d_j's.

Lemma 6.6 (The Extension Lemma)**.** *Assume $\varphi(\bar{x}, y) < \varepsilon$ is a condition in $\mathbb{P}_{\mathbf{T}}$ which forces that $P_{m,n}(y) = 0$ for some m and n. Then for some new variable z this condition can be extended to a condition $\varphi'(\bar{x}, y, z) < \varepsilon'$ that in addition forces $P_{m+1,k}(z) = 0$ for some k and that z is an immediate successor of y.*

Proof. Since $\varphi(\bar{x}, y) < \varepsilon$ is a consistent condition, we can find a tuple $\bar{a}, b$ in M_4 that realizes it. Suppose b is not a terminal node of the bottom part of M_4. Fix an immediate successor c of b such that $P_{i,j}(c) = 0$ for some i and j and extend condition $\varphi(\bar{x}, y) < \varepsilon$ to a condition $\varphi'(\bar{x}, y, z) < \varepsilon'$ that decides z is an immediate successor of y and $P_{i,j}(z) = 0$. This condition is as required.

Now suppose b is a terminal node of the bottom part of M_4. We may assume that φ is in the prenex normal form since such formulas are uniformly dense in the space of all formulas by [2, Proposition 6.9]. By Lemma 1.1 we may also assume that $\varepsilon < 1$. Let $\psi_j(\bar{x}, y, \bar{t})$, for $j < n$, be a list of all atomic subformulas of $\varphi(\bar{x}, y)$. Thus $\varphi(\bar{x}, y)$ is of the form (variables in ψ_j are suppressed and dummy variables t_0 and t_{l-1} are added, if necessary, so that the string of quantifiers starts with sup and ends with inf; this is done only for readability)

$$(\dagger) \qquad \sup_{t(0)} \inf_{t(1)} \dots \inf_{t(l-1)} f(\psi_0, \dots, \psi_{n-1})$$

for some $l \in \omega$, variables $t(j)$ for $j < l$, and continuous function f. Let

$$\varepsilon' := (\varepsilon - \varphi(\bar{a}, b)^{M_4})/3.$$

Since the interpretation of $f(\psi_0, \dots, \psi_{m-1})$ is uniformly continuous and its modulus of continuity does not depend on the interpretation, we can find $\delta > 0$ such that changing the values of all variables occurring in any ψ_j by $< \delta$ affects the change of the value of $f(\psi_0, \dots, \psi_{n-1})$ by $< \varepsilon'$. Let $l > 1/\delta$ be such that all pairs i, j for which predicate $P_{i,j}$ occurs in some $\psi_j(\bar{x}, y)$ satisfy $\max(i, j) < l$. By increasing l we may also assume that all projection functions f_i occurring in some $\psi_j(\bar{x}, y)$ satisfy $i < l$ and that $\bar{a}, b$ belong to one of the first l levels of M_4.

Consider L_l and $M_4 \restriction l$ as defined in §6.5.

Claim 6.7. *For any tuple $\bar{p}, d$ in $M_4 \restriction l$ of the same sort as $\bar{x}, y$ we have*

$$|\varphi(\bar{p}, d)^{M_4 \restriction l} - \varphi(\bar{p}, d)^{M_4}| < \varepsilon'.$$

Proof. For every tuple $\bar{q}$ in M_4 such that $\bar{p}, d, \bar{q}$ is of the same sort as $\bar{x}, y, \bar{t}$ (where $\bar{t}$ are the variables occurring freely in formulas ψ_j but not in φ), we have $d(q_i, f_l(q_i)) < \delta$ for all i. For every $q \in M_4$ there exists $f_l(q) \in M_4 \restriction l$ within $< \delta$ of q. Since φ is as in $(\dagger)$, the claim follows by the choice of δ. $\square$

By (17) of Lemma 6.5 there exists an automorphism Φ of $M_4 \restriction l$ which sends b to a node which is not a terminal node of the bottom part of M_4. This is certainly not an automorphism of M_4 but the Φ-image of $\bar{a}, b$ still satisfies $\varphi(\bar{x}, y) < \varepsilon$. We can fix an immediate coloured successor c of $\Phi(b)$ and extend $\varphi(\bar{x}, y) < \varepsilon$ to a condition $\psi(\bar{x}, y, z) < \varepsilon'$ which implies z is a coloured immediate successor of c. This completes the treatment of the case when b is a terminal node of the bottom part of M_4 and the proof. $\square$

Generic model M_G was defined in Theorem 3.3.

Lemma 6.8. *There is a countable family* $\mathbf{F}$ *of dense subsets of* $\mathbb{P}_{\mathbf{T},\Sigma}$ *such that if a filter* G *is* $\mathbf{F}$*-generic then* M_G *is a model of* $\mathbf{T}_4$ *and every coloured node of* M_G *has a coloured immediate successor.*

Proof. By Theorem 3.3 there is a countable family $\mathbb{F}_{\mathbf{T},\Sigma}$ of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ such that if a filter G is $\mathbb{F}_{\mathbf{T},\Sigma}$-generic then $M_G \models \mathbf{T}_4$. Lemma 6.6 implies that the set

$$\begin{aligned}\mathbf{D}_{i,j,k} := \{p \in \mathbb{F}_{\mathbf{T},\Sigma} : \mathbf{T}_4 + p \models d_k \text{ is not } (i,j)\text{-coloured or}\\ \mathbf{T}_4 + p \models \text{`}d_j \text{ has an } (i+1,l)\text{-coloured successor for some } l\text{'}\}\end{aligned}$$

is dense in $\mathbb{F}_{\mathbf{T},\Sigma}$ for all i, j, and k.

Let $\mathbf{F}' := \mathbb{F}_{\mathbf{T},\Sigma} \cup \{\mathbf{D}_{i,j,k} : i, j, k \in \omega\}$ and suppose G is $\mathbf{F}'$-generic. Then $M_G \models \mathbf{T}_4$, hence node of M_G is an isolated point and is therefore an interpretation of some constant d_j. By the elementarity M_G has a coloured node, and every coloured node of M_G has a coloured successor. This concludes the proof. $\square$

We now have proofs of some quotable weakennings of Theorem 2 and Theorem 3.

Corollary 6.9. *With* $\mathbf{T} := \mathbf{T}_4$*, forcing* $\mathbb{P}_{\mathbf{T}}$ *forces that all* $\mathbf{s}_m$ *for* $m \geq 1$ *are omitted in* M_G *and that the nodes of* M_G *form an ill-founded tree. In particular,* M_G *realizes* $\mathbf{s}_0$*.*

Proof. Lemma 6.8 implies that M_G is as required. $\square$

Corollary 6.10. *There exists a complete theory* $\mathbf{T}$ *in a countable language and a type* $\mathbf{t}(x)$ *which is omissible in a model of* $\mathbf{T}$ *but* $\mathbb{P}_{\mathbf{T}}$ *forces that* M_G *realizes* $\mathbf{t}$*.*

Proof. With $\mathbf{T} := \mathbf{T}_4$, type $\mathbf{s}_0$ of an infinite branch defined above is omissible in model M_4 of $\mathbf{T}_4$, but by Corollary 6.9 it is realized in M_G. $\square$

Corollary 6.11. *With* $\mathbf{T}_4$ *and* $\mathbf{s}_n$*, for* $n \in \omega$*, as defined above the following holds. For every* n *there exists a model of* $\mathbf{T}_4$ *that omits* $\mathbf{s}_n$*, but no model of* $\mathbf{T}_4$ *omits all* $\mathbf{s}_n$ *simultaneously.*

Proof. With $M_4, \mathbf{T}_4$ and $\mathbf{s}_m$ as above, M_4 omits type $\mathbf{s}_0$ by the construction. On the other hand, Corollary 6.9 implies that the generic model omits all $\mathbf{s}_m$

for $m \geq 1$. As the filter G needs to meet only countably many dense sets, we can construct such model by recursion.

Assume N is an L_4-model elementarily equivalent to M_4 which omits $\mathbf{s}_m$ for all sufficiently large m. Since the 'bottom part' of N (consisting of all coloured nodes) intersects the mth level by the elementarity, it is ill-founded. Therefore N has an infinite branch and it realizes $\mathbf{s}_0$. $\square$

6.7. **Models** $M_4(j)$**.** Fix $j \geq 1$. We shall construct $M_4(j) \models \mathbf{T}$ that omits $\mathbf{s}_m$ for all m such that $m(0) < j$. Together with Corollary 6.11 this will complete the proof of Theorem 2

Fix $n \geq 1$ and let $M(k^n, j)$ be the submodel of $M(k^n)$ (see §6.3) obtained by removing every s with no extension to the jth level. Therefore $|s| < j$ implies $\rho_{M(k^n,j)}(s) \geq j - |s|$ for $s \in M(k^n, j)$. Since the universe of $M(k^n)$ is T_2, its elements are pairs $s = (s_0, s_1)$ and $\rho_{M(k^n)}(s) = s(|s|-1)$. Therefore

$$M(k^n, j) = \{s \in M(k^n) : s_0(|s| - 1) \geq j - |s|\}.$$

The L_4-structure

$$M_4(j) := \bigoplus_n M(k^n, j)^\frown T_2$$

clearly omits $\mathbf{s}_0$ and $\mathbf{s}_m$ for all m such that $m(0) < j$.

Lemma 6.12. *For all j and l the models $M_4(j) \restriction l$ and $M_4 \restriction l$ are isomorphic.*

Proof. This is similar to the proof of Lemma 6.5. Fix $n \in \omega$. As in the discussion preceding Lemma 6.5, let $T[M(k^n, j) \restriction l]$ be the tree of all full nodes in $M(k^n, j) \restriction l$. Then $T[M(k^n, j) \restriction l]$ is a subtree of $T[M(k^n) \restriction l]$. For $s \in T[M(k^n, j) \restriction l]$ such that $|s| = i < l$ and s has an L_l-coloured immediate successor let

$$k''(s) := \min\{j : (\exists t) s \sqsubseteq t \text{ and } P_{i+1,j}(t) = 0\}.$$

The following facts have proofs analogous to the proofs of the corresponding parts of Lemma 6.5, and $X[n, l]$ is as defined there. The isomorphism type of $M(k^n, j) \restriction l$ is uniquely determined by

$$Y[n, j, l] := \{s \in T[M(k^n, j) \restriction l] : k''(s) \text{ is defined}\}.$$

By (k0)–(k2) (the choice of functions k^n) for every n there are infinitely many n' such that $Y[n, j, l]$ is isomorphic to $X[n', l]$, and vice versa. We can therefore find a permutation Φ of ω such that $M(k^n, j) \restriction l$ is isomorphic to $M(k^{\Phi(n)}) \restriction l$ for all n. Putting these isomorphisms together we obtain a required isomorphism between $M_4 \restriction l$ and $M_4(j) \restriction l$. $\square$

Proof of Theorem 2. We shall prove that $\mathbf{T}_4$ and types $\mathbf{s}_n$, for $n \in \omega$ as defined above are as required. Being theory of a model M_4, theory $\mathbf{T}_4$ is clearly complete. For every j model $M_4(j)$ is a model of $\mathbf{T}_4$ by Lemma 6.12 and Lemma 1.12. By its definition and the discussion following (S1)–(S3), model $M_4(j)$ omits all $\mathbf{s}_m$ for $m(0) < j$. Therefore for every k there exists j large enough such that model $M_4(j)$ of $\mathbf{T}_4$ omits all $\mathbf{s}_m$ for $m < k$.

By Corollary 6.11 no model of $\mathbf{T}_4$ simultaneously omits all $\mathbf{s}_m$ for $m \in \omega$, and this concludes the proof. $\square$

Proof of Theorem 3. We shall define a complete theory $\mathbf{T}_5$ in a countable language L_5 and types $\mathbf{s}$ and $\mathbf{t}$ omissible in models of $\mathbf{T}_5$ such that no model of $\mathbf{T}_5$ simultaneously omits both of them.

Let trees T_1 and T_2, language L_4, model M_4, and theory $\mathbf{T}_4$ be as in the proof of Theorem 2. We define an expansion of L_4, a two-sorted language L_5 with sorts X and Y. The sort Y corresponds to L_4, so that if N is an L_5-structure then Y^N is an L_4-structure. The sort X is equipped with the following:

(18) a discrete $\{0,1\}$-metric d,
(19) a unary function g from X into X, and
(20) a unary function h from X into Y.

Both g and h are interpreted as 1-Lipshitz functions (necessarily, since X is discrete). This describes language L_5 and we proceed to describe an L_5-model M_5. The model Y^{M_5} is isomorphic to M_4. The set X^{M_5} is a tree isomorphic to the underlying tree of M_4 and we interpret h as the tree isomorphism function from X^{M_5} onto Y^{M_5}. Finally, the interpretation of g sends $\langle\rangle$ to itself and every other node of X^{M_5} to its immediate predecessor.

This describes M_5; let $\mathbf{T}_5 := \mathrm{Th}(M_5)$.

By Lemma 6.1, predicates in sort Y defined by

$$\mathrm{P}_{>m}(s) : \text{the height of } s \text{ is } > m,$$
$$\mathrm{P}_{\leq m}(s) : \text{the height of } s \text{ is } \leq n,$$

are definable.

Lemma 6.13. *The following hold for every $N \models \mathbf{T}_5$ and a and b in X^N.*

(21) *If $a \neq b$ and $h(a)$ and $h(b)$ are nodes of Y^N then $h(a) \neq h(b)$.*
(22) *If $h(b)$ is a node of height $m+1$ then $h(g(b)) = f_m(h(b))$.*
(23) *Every node of Y^N is in the range of h.*
(24) *The set of immediate successors of $h(a)$ is equal to*

$$\{h(c) : c \in X^N \text{ and } g(c) = a\}.$$

Proof. We prove each of (21)–(24) for nodes of height $\leq m$.

(21) The following sentence is in $\mathrm{Th}(M_5)$ for every $m \geq 1$:

$$\sup_{x \in X, y \in X} \min(\mathrm{P}_{>m}(h(x)), \mathrm{P}_{>m}(h(y)), d(x,y), \frac{1}{m} \dot{-} d(h(x), h(y))).$$

Thus in every model N of $\mathbf{T}_5$, for all $m \geq 1$ and a and b in X^N such that $\max(|h(a)|, |h(b)|) \leq m$ we have $h(a) = h(b)$ or $d(h(a), h(b)) \geq 1/m$. This implies (21).

(22) For all $m \in \omega$ and $x \in X^N$ one of (i) $|h(x)| > m+1$, (ii) $|h(x)| \leq m$, or (iii) $h(g(x)) = f_m(h(x))$ applies. Therefore $\mathrm{Th}(M_5)$ contains the sentence

$$\sup_{x \in X} \min(\mathrm{P}_{>m+1}(h(x)), \mathrm{P}_{\leq m}(h(x)), d(h(g(x)), f_m(h(x)))$$

and if $N \models \mathbf{T}_5$ and $b \in X^N$ satisfies $|h(b)| = m+1$ then $h(g(b)) = f_m(h(b)$.

(23) For every $m \geq 1$ the following sentence is in $\mathrm{Th}(M_5)$:

$$\sup_{x \in Y} \min(\mathrm{P}_{>\,m}(x), \inf_{y \in X} d(h(y), x)).$$

Since the set of nodes of height $\leq m$ is discrete, (23) holds in every model of $\mathbf{T}_5$.

(24) For every $m \geq 1$ the following sentence is in $\mathrm{Th}(M_5)$:

$$\sup_{x \in X, y \in Y} \min(\mathrm{P}_{>\,m}(h(x)), \mathrm{P}_{\leq\,m\text{-}1}(h(x)), \mathrm{P}_{>\,m\,+\,1}(y), \mathrm{P}_{\leq\,m}(y),$$
$$\frac{1}{m} \dot{-} d(f_m(y), h(x)), \inf_{z \in X} \max(d(h(z), y), d(g(z), x))).$$

Again, since the set of nodes of height $\leq m+1$ is discrete, (24) holds in every model of $\mathbf{T}_5$. $\square$

We proceed to define types $\mathbf{s}$ and $\mathbf{t}$. Type $\mathbf{s}(x)$ is the type of an infinite branch in Y^N; i.e. it is $\mathbf{s}_0(x)$ as defined in the proof of Theorem 2. Type $\mathbf{t}(x)$, for x of sort X, asserts the following (g^k is the kth iterate of g for $k \geq 1$).

(25) $\inf_{y \in X} d(x, g^k(y)) = 0$ for all $k \geq 1$.

(26) $\inf_{y \in X} \max(d(x, g(y)), P_{m,n}(h(y))) = 1$ for all m and n.

Suppose that $N \models \mathbf{T}_5$ and $a \in N$ realizes $\mathbf{t}$. By (25) for every $k \geq 1$ there exists $y \in X^N$ such that $g^k(y) = a$ and therefore $\rho_N(a) \geq \omega$. On the other hand, (26) implies that for all y such that $g(y) = a$, the node $h(y)$ is not coloured.

If $N \models \mathbf{T}_5$ is such that not all nodes of Y^N are coloured, then $\mathbf{t}$ is realized in N by any $a \in X^N$ such that $h(a)$ belongs to the top part of Y^N and $\rho_N(a) \geq \omega$ or $h(a)$ is a terminal node of the bottom part of Y^N.

Each of the types $\mathbf{s}$ and $\mathbf{t}$ is omissible in a model of $\mathbf{T}_5$. Type $\mathbf{s}$ is clearly omitted in M_5. Type $\mathbf{t}$ is omitted in $\mathbb{P}_{\mathbf{T}_5}$-generic model. Proof of the latter fact follows the proofs of Lemma 6.6, Lemma 6.8 and Corollary 6.9 almost verbatim. Alternatively, one could prove that the L_4-part of the $\mathbb{P}_{\mathbf{T}_5}$-generic model is $\mathbb{P}_{\mathbf{T}}$-generic and use these facts directly.

It remains to check that $\mathbf{s}$ and $\mathbf{t}$ are not simultaneously omissible in a model of $\mathbf{T}_5$. If $N \models \mathbf{T}_5$ omits $\mathbf{s}$ then X^N omits $\mathbf{s}_0$. Therefore all elements of X^N are nodes. In particular the bottom part of X^N is well-founded, and any of its terminal nodes realizes $\mathbf{t}$ in N. $\square$

7. Concluding remarks

We don't know whether (6) of Theorem 1 (i.e. Theorem 2.8) is sharp.

Question 7.1. *What are the possible complexities of the set of types omissible in a model of a complete theory* $\mathbf{T}$ *in a countable language? In particular, can this set be* $\mathbf{\Sigma}^1_2$*-complete?*

According to [8] and [10, Definition 4.12], a type $\mathbf{t}(\bar{x})$ is metrically principal over a theory $\mathbf{T}$ (we consider the case when L is the fragment consisting of all finitary sentences) if and only if for every $\delta > 0$ the type $\mathbf{t}^\delta(\bar{x})$, asserting that every finite subset of $\mathbf{t}$ is realizable by an n-tuple within δ of $\bar{x}$, is principal.

For example, type $\mathbf{t}$ defined in the proof of Proposition 2.5 is metrically principal over $\mathbf{T}^S$ if the tree S has height ω. This is because $\mathbf{t}_{1/n}$ is realized by any node of S that is not an end-node. This gives an example of an omissible metrically principal type. A simple argument shows that a *complete* metrically principal type cannot be omissible.

7.1. C***-algebras.** The original motivation for this study came from the model-theoretic study of C*-algebras. Many important properties of C*-algebras are axiomatizable (see [16, Theorem 2.5.1]), and numerous non-axiomatizable properties are uniformly definable by a sequence of types (see [16, Definition 5.7.1] and the discussion afterwards). The answers to some of the most prominent open problems in the theory of C*-algebras depend on whether C*-algebras with these properties (in particular, nuclear—also known as amenable—C*-algebras) can be constructed in a novel way. Therefore Theorem 5.2, Theorem 5.3, Proposition 5.4, Proposition 5.5 and Corolary 5.6 open possibilities for constructing C*-algebras with prescribed first-order properties in these classes. Also, some of the deepest recent results on classification of C*-algebras have equivalent formulation in the language of (metric) first-order logic (see [13], the introduction to [30], and [14] or [16]).

We have a machine for construction of C*-algebras with properties prescribed by a given theory. These are generic algebras obtained by Henkin construction as described in §3 and §5. Although they are assembled from finite pieces corresponding to conditions of $\mathbb{P}_\mathbf{T}$ or one of its variations, they are not obviously obtained from matrix algebras and abelian algebras by applying basic operations of taking inductive limits, crossed products by $\mathbb{Z}$, stable isomorphisms, quotients, extensions, hereditary subalgebras, or KK-equivalence (cf. the bootstrap class problem, [7, IV.3.1.16 and V.1.5.4]). At present no method for assuring that the C*-algebras obtained by using the Henkin construction do not belong to the (large or small) bootstrap class is known. Results of [16] (combined with §3 and §5) reduce several prominent open problems on classification of C*-algebras to problems about the existence of theories with certain properties.

In [27] Kechris had defined a Borel space of C*-algebras and proved that the nuclear C*-algebras form a Borel subset. We give a generalization of this result. Recall that a Borel structure on the space of models was defined in §1.2.1. Although this space is different from one used by Kechris, these representations of space of separable C*-algebras are equivalent ([18]).

Corollary 7.2. *The following sets of* C**-algebras are Borel subsets of the standard Borel space of* C**-algebras: UHF, AF, nuclear, nuclear dimension* $\leq n$ *for* $n \leq \aleph_0$*, decomposition rank* $\leq n$ *for* $n \leq \aleph_0$*, tracially AF, simple.*

Proof. Since each of the sets of C*-algebras listed above is uniformly definable by a sequence of types by [9] and [16], the conclusion follows by Proposition 5.7. □

The class of all cardinals κ for which the set of types over any set of cardinality κ in a model of theory $\mathbf{T}$ has cardinality at most κ is an important invariant of a first-order theory (see [32]; see also [2, §14], [17, §5] for the metric version). To a complete theory $\mathbf{T}$ in a countable language L in the logic of metric structures one can associate descriptive complexities of distinguished sets of (not necessarily complete) types. Our results suggest that these invariants provide nontrivial information about the theory $\mathbf{T}$. In particular, it is plausible that in the case when $\mathbf{T}$ is natural theory (e.g. theory of a C*-algebra) the set of types omissible in a model of $\mathbf{T}$ is Borel. Each theory used in our counterexamples interprets the Baire space as presented in §2.2. Is this a necessary condition for pathological behaviour of metric theories?

Problem 7.3. Find a model-theoretic characterization of complete theories $\mathbf{T}$ in a separable language with each the following properties.

(1) The set of complete types omissible in a model of $\mathbf{T}$ is Borel.
(2) If two types are omissible in models of $\mathbf{T}$, then they are simultaneously omissible in a model of $\mathbf{T}$.
(3) If types $\mathbf{t}_n$, for $n \in \omega$, are such that for any $k \in \omega$ the types $\mathbf{t}_n$, for $n < k$, are simultaneously omissible in a model of $\mathbf{T}$, then all of these types are simultaneously omissible in a model of $\mathbf{T}$.

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