INDEPENDENCE OF THE EXISTENCE OF PITOWSKY SPIN MODELS

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In the last several decades the study of the foundations of Mathematics is dominated by the impact of the independence phenomena in Set Theory. In many fields of Mathematics like analysis, topology, algebra etc. basic problems were shown to be independent of the accepted axiom system for Set Theory, known as Zermello-Frankel Set Theory or ZFC. The fact that undecided problems exist in any recursively stated axiom system which has some minimal strength was well known since Gödel's famous incompleteness theorem of 1931. But the undecided statement produced by Gödel's proof was considered to be somewhat esoteric and unrelated to main stream of the subject. The new aspect that came up after Cohen independence proofs of 1963 is that mathematical problems that were considered to be central to the particular discipline were shown to be undecided. So the very notion of mathematical truth was shaken. The possibility of having a multitude of mathematical universes with different properties became a definite option for the foundation of Mathematics.

To what extent these developments are relevant to Physics? When a physical theory is stated in terms of mathematical concepts like real numbers, Hilbert spaces, manifolds etc. it implicitly adapts all the mathematical facts which are accepted by the Mathematicians to be valid for these concepts. If the mathematical "truths" may depend of the foundation of Set Theory then it is possible, at least in principle, that whether a given physical theory implies a particular physically meaningful statement may depend on the foundational framework in which the implicitly assumed Mathematics is embedded.

This may seem far fetched and it is very likely that physically consequences of a physical theory will never depend on the set theoretical foundation of the mathematical reasoning that accompanied the theory but the point of this paper is that this is still a definite possibility. Let us admit from the outset that we do not have an example of a physically meaningful statement that its truth depends on the set theoretical foundation. We shall instead demonstrate that there are independent

statements whose consequences have the same flavour as some statements about the foundation of Quantum Mechanics.

In [12], [11] I. Pitowsky used Continuum Hypothesis to construct hidden variable models for spin-1/2 and spin-1 particles in quantum mechanics. His functions are not measurable and are therefore not directly subject to Bell-type no-go theorems ([1], [2], also cf. the next paragraph). Also, under the same assumption Pitowsky constructed a function that almost violates the no-go theorem of Kochen and Specker ([7]). This theorem states that one cannot assign the values 0 and 1 to points on the unit sphere in three-dimensional space, S^2, so that the sum of values for any triple of mutually orthogonal points is 2. Pitowsky constructed a function such that for every vector $x \in S^2$ there are at most countably many exceptions (see Definition 1.2). We prove that no such function exists in some model of the Zermelo–Fraenkel set theory with the Axiom of Choice, ZFC (Theorem 1.3), confirming a conjecture of Pitowsky. While this independence result probably does not have physical interpretation, it gives some weight to the conjecture that one could decide between different set theories on the basis on their scientific consequences. See [8, §5] for a discussion of this viewpoint.

Before proceeding, we should explain why this example still falls short of having direct physical relevance. Pitowsky's function s_1 provides correct probabilities for the spin values as predicted by quantum mechanics. However, a complete model for hidden variables would require having an infinite sequence of functions s_1, s_2, s_3,... such that the values provided by s_n model the n-th repetition of the experiment. If $s_n = s_1 \circ \rho_n$ for a randomly chosen sequence of rotations, then Pitowsky's function gives the correct probabilities for values of spin on many points on the sphere ([10]). However, a Bell-type argument shows that it is not possible to correctly model the infinite sequence of values of measurements on a carefully chosen quadruple of points, and the measurability of functions s_n is not required to prove this fact ([9]).

We prove two results. The first one is that if there exists a σ-additive extension of the Lebesgue measure to the power-set of the reals then Pitowsky models do not exist. The second is that the same conclusion holds in the model of ZFC known as the 'random real model.' The proofs use results of H. Friedman and D.H. Fremlin, respectively, to show that under these assumptions every Pitowsky function has to be Borel-measurable.

The first proof has a more elegant assumption and it is easier to understand. However, its assumption, also known as 'the continuum is a real-valued measurable cardinal' is a large cardinal axiom ([5]). By

a result of Solovay this is equiconsistent with the existence of a measurable cardinal and its consistency strength is therefore a bit beyond that of ZFC. Our second result shows that the assertion 'there are no Pitowsky functions' is relatively consistent with ZFC.

1. PITOWSKY FUNCTIONS

For $a \in S^2$ and $0 \le \theta \le \pi$ let $c(a, \theta)$ be the set of all $x \in S^2$ such that the the angle between a and x is equal to θ and let $\mu_{a,\theta}$ be the normalized Lebesgue measure on $c(a, \theta)$. We shall omit the subscript a, θ whenever it is clear from the context.

Definition 1.1. A *Pitowsky spin function* is a function F from the 2-dimensional sphere S^2 into $[-1, 1]$ satisfying the following for all $a \in S^2$:

(1) $F(a) = -F(-a)$, and
(2) For all $\theta \in [0, \pi]$ we have $\int_{c(a,\theta)} F(x)\, d\mu(x) = F(a)\cos(\theta)$.

For $a \in S^2$ denote the great circle $c(a, \pi/2)$ by $C(a)$.

Definition 1.2. A *KSP function* is a function F from S^2 into $\{0, 1\}$ such that for all $a \in S^2$ we have

(1) $F(-a) = F(a)$, and
(2) $F \upharpoonright C(a)$ is a measurable function with respect to $\mu_{a,\pi/2}$, and
(3) the set

$$\{b \in C(a) | \forall c \in C(a) \cap C(b)\, F(a) + F(b) + F(c) \ne 2\}$$

is of $\mu_{a,\pi/2}$-measure zero.

Special cases of Pitowsky spin functions, *spin-1/2 functions* and *spin-1 functions*, were constructed by Pitowsky in [12, §II] and [12, §IV], respectively, while KSP functions were constructed in [13]. All of these constructions use Continuum Hypothesis or weaker Martin's Axiom.

In the following by 'random real model' we mean a model obtained by forcing with the homogeneous measure algebra of Maharam character $(2^{\aleph_0})^+$ (i.e., 'adding $(2^{\aleph_0})^+$ random reals') over a model of ZFC.

Theorem 1.3. *Assume that the continuum is real-valued measurable or that the universe is the random real model. Then there is no Pitowsky spin function and there is no KSP function.*

Proof. Assume F is a Pitowsky spin function. By Lemma 3.2 and Lemma 3.3, F is Borel-measurable. It is a well-known consequence of the CHSH inequality ([2]) that there are no measurable spin functions.

Now assume F is a KSP function. By Lemma 3.4 and Lemma 3.5, it is Borel-measurable. From Lemma 3.6 and the fact that for a measurable F the value of $I_{a,b,c}(F)$ does not depend on the choice of a, b and c we can conclude that there are no measurable KSP functions. $\square$

J. Shipman ([14, p. 480]) announced that the conclusion of Theorem 1.3 is relatively consitent with ZFC. The proof was not included and we were not able to reconstruct his arguments neither directly nor after corresponding with him. In personal communication David Fremlin sketched a clever Fubini-type argument that provides an alternative proof of Theorem 1.3.

2. $\mathcal{F}$-MEASURABILITY

2.1. **Parameterizations.** A function $f \colon S^2 \to \mathbb{R}^2$ is a *parameterization* if it is injective and measure-class preserving. To a pair of points a and $b \neq \pm a$ in S^2 we associate three parameterizations. Let $L(a, b)$ be the half great circle starting from a and going through b. For $\theta \in [0, 2\pi]$ let $L(a, b, \theta)$ be the half great circle one gets by rotating $L(a, b)$ by the angle θ clockwise around a.

With a and b fixed, to a point $x \in S^2$ associate the following angles

$$\phi^a(x) = \angle(x, a)$$
$$\theta^{ab}(x) \text{ is } \theta \in [0, 2\pi) \text{ such that } x \in L(a, b, \theta)$$

Then the following are parameterizations with the range equal to $[0, 2\pi) \times (0, \pi)$ (plus two points).

$$p^{ab}(x) = (\theta^{ab}(x), \phi^a(x))$$
$$q^{ab}(x) = (\theta^{ab}(x), \theta^{ba}(x) \bmod \pi).$$

Note that $\theta^{ba}(x)$ is taken modulo π in order to assure the uniqueness of $q^{ab}(x)$.

2.2. $\mathcal{F}$-measurability.

Let $\mathcal{F}$ be a family of parameterizations. A function $F\colon S^2 \to \mathbb{R}$ is $\mathcal{F}$-*measurable* if for every $p \in \mathcal{F}$ and all s and t in $\mathbb{R}$ all sections

$$t \mapsto F(p^{-1}(s,t)) \text{ and } s \mapsto F(p^{-1}(s,t))$$

are measurable. Define

$$\mathcal{F}_P = \{p^{ab}, q^{ab} : a \in S^2, b \in S^2, a \neq \pm b\}$$

and

$$\mathcal{F}_{KSP} = \{q^{ab} : a \in S^2, b \in S^2, a \neq \pm b\}.$$

A function is P-*measurable* if it is $\mathcal{F}_P$-measurable and it is KSP-*measurable* if it is $\mathcal{F}_{KSP}$-measurable. Pitowsky spin functions are P-measurable while KSP functions are KSP-measurable.

2.3. Transitivity.

Parameterizations p and q *have the common first coordinate* if the first coordinates of $p(x)$ and $q(x)$ are equal for all x.

Lemma 2.1. *Assume p and q have the common first coordinate. Then for every $X \subseteq S^2$ for almost all s the s-section $Y_s = \{t : (s,t) \in p[X]\}$ is null if and only if the s-section $Z_s = \{t : (s,t) \in q[X]\}$ is null.*

Proof. By the assumption, $q \circ p^{-1}$ is a measure-class preserving injection from $A = p[S^2]$ onto $B = q[S^2]$. Since p and q have the common first coordinate $q \circ p^{-1}$ maps $A \cap (\{s\} \times \mathbb{R})$ onto $B \cap (\{s\} \times \mathbb{R})$ for all s. Therefore the restriction of $q \circ p^{-1}$ to $A \cap (\{s\} \times \mathbb{R})$ is measure-class preserving for almost all s. For such s set Y_s is null iff Z_s is null. $\quad\square$

For a parameterization p let p^T denote the parameterization obtained by composing p with the flip of coordinates, $(s,t) \mapsto (t,s)$. Two parameterizations p and q *have a common coordinate* if at least one of p and p^T has the common first coordinate with at least one of q and q^T. A family of parameterizations $\mathcal{F}$ is *transitive* if the graph whose vertices are elements of $\mathcal{F}$ and edges are pairs $\{p, q\}$ that have a common coordinate is connected. Note that $\mathcal{F}_P$ is transitive, but $\mathcal{F}_{KSP}$ is not. To remedy this, let $q^{ab\gamma} = (\theta^{ab}, \theta^{ba} + \gamma \mod \pi)$. Then

$$\mathcal{F}_{KSP}^+ = \{q^{ab\gamma} : a \in S^2, b \in S^2, a \neq \pm b, 0 \leq \gamma \leq \pi\}$$

is transitive and every KSP-measurable function is $\mathcal{F}_{KSP}^+$-measurable.

2.4. Approximations.

For $A \subseteq \mathbb{R}^2$ and $s \in \mathbb{R}$ write $A_s = \{t : (s,t) \in A\}$ and $A^s = \{t : (t,s) \in A\}$. For $X \subseteq \mathbb{R}$ we say that A is X-*full* if for all $s \in X$ both A_s and A^s have full measure.

From now on we assume $\mathcal{D}$ is a filter on $\mathbb{R}$ such that every $X \in \mathcal{D}$ has full outer measure. For a parameterization q and $F\colon S^2 \to \mathbb{R}$ and $X \subseteq \mathbb{R}$ we say F is q-X-*approximated* if there is a Borel function G

such that the q-image of $\Delta_{F,G} = \{x : F(x) = G(x)\}$ is X-full for some X. If F is q-X-approximated for some $X \in \mathcal{D}$ we say that it is q-$\mathcal{D}$-approximated. It is $\mathcal{F}$-$\mathcal{D}$-approximated if there is a Borel function G which q-$\mathcal{D}$-approximates F for all $q \in \mathcal{F}$.

Lemma 2.2. *Assume F is q-$\mathcal{D}$-approximated for all $q \in \mathcal{F}$ and $\mathcal{F}$ is transitive. Then F is $\mathcal{F}$-approximated.*

Proof. For each $q \in \mathcal{F}$ fix $X_q \in \mathcal{D}$ and a Borel G_q which q-X_q-approximates F. Fix p and q in $\mathcal{F}$ that have common first coordinate. Let $A = \{x : G_p(x) \neq G_q(x)\}$. By Lemma 2.1, $q \circ p^{-1}(X)$ has almost all of its s-sections null. Therefore the measurable set $q[A]$ is covered by the union of two sets each of which has all of its s-sections null for every $s \in X_q \cap X_p$. Since $X_q \cap X_p$ has full outer measure and $q[A]$ is measurable, $q[A]$ is null. Hence A is null as well.   □

3. FREMLIN'S AND FRIEDMAN'S FUBINI-TYPE THEOREMS

Theorem 3.1 is an immediate consequence of results of D.H. Fremlin and H. Friedman ([3, Theorem 6K(b)], [4], see also [14]), or rather their proofs, and a negligible strengthening of the latter result. While there is a well-developed machinery for showing that 'continuum is real-valued measurable' implies many statements true in random real model (see [3]) we do not have a uniform proof of these statements.

Theorem 3.1. *Assume that the continuum is real-valued measurable or that the universe is the random real model. Then there exists a filter $\mathcal{D}$ of subsets of $\mathbb{R}$ such that*
  (1) *Every set in $\mathcal{D}$ has full outer measure, and*
  (2) *For every $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that all of its sections f_s and f^s are measurable there exists a Borel function $g \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\{(s,t) : g(s,t) = f(s,t)\}$ is X-full for some $X \in \mathcal{D}$.*

Proof. Fix f. If the continuum is real-valued measurable then let $\mathcal{D} = \{X : X$ is a Borel set of full measure$\}$. By [3, Theorem 6K(b)]. there exists a measurable function $h \colon \mathbb{R}^2 \to \mathbb{R}$ which $\mathbb{R}$-approximates f. By changing h on a null set we can assume it is Borel-measurable and X-approximates f for some Borel set of full measure.

Now assume we are in a model obtained by adding $(2^{\aleph_0})^+$ random reals to a model of ZFC. For simplicity of notation we assume that the Continuum Hypothesis holds in the latter model. Let $\mathcal{R}$ denote the set of random reals. It was proved by Friedman ([4, Lemma 15 and Theorem 2]) that some Borel function g X-approximates f for a set X of full outer measure. A closer look at the proof of Lemma 15 reveals

that $X \subseteq \mathcal{R}$ is the set of all random reals added over an intermediate model in which CH holds. Such set always has full outer measure. Therefore $\mathcal{D} = \{X \subseteq \mathcal{R} : |\mathcal{R} \setminus X| \leq \aleph_1\}$ is a filter as required. $\qquad \square$

3.1. **Pitowsky spin functions.** Throughout this section we assume conclusion of Theorem 3.1 holds for a filter $\mathcal{D}$ of full outer measure subsets of $\mathbb{R}$. The following is immediate from Theorem 3.1 and Lemma 2.2.

Lemma 3.2. *Every bounded P-measurable function is P-$\mathcal{D}$-approximated.* $\qquad \square$

Lemma 3.3. *If a Pitowsky spin function F is P-$\mathcal{D}$-approximated by a Borel function G then it is Borel.*

Proof. For every $a \in S^2$ there is $X \in \mathcal{D}$ such that for all θ in $X \cap [0, \pi/2]$ we have that

$$F(a) = \frac{1}{\cos(\theta)} \int_{c(a,\theta)} F \, d\mu = \frac{1}{\cos(\theta)} \int_{c(a,\theta)} G \, d\mu.$$

Since the function $\theta \mapsto \int_{c(a,\theta)} G \, d\mu$ is measurable and constant on a set of full outer measure, it is almost everywhere equal to $F(a)$. Therefore $F(a) = \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\cos(\theta)} \int_{c(a,\theta)} G \, d\mu$ for all a, and it only remains to check that the function on the right hand side of the equality is Borel. Recall that $P(S^2)$ denotes the compact metric space of probability Borel measures on S^2 (see [6, §17]). Function $(a, \theta) \mapsto c(a, \theta)$ from $S^2 \times [0, \pi)$ to $K(S^2)$ (the space of compact subsets of S^2) is continuous, and by the Portmanteau Theorem ([6, 17.20]) so is the function $(a, \theta) \mapsto \frac{1}{\sin(\theta)} \mu_{a,\theta}$ from $S^2 \times [0, \pi)$ into $P(S^2)$. Finally, the map $\mu \mapsto \int G \, d\mu$ is a Borel map from $P(S^2)$ into $\mathbb{R}$ by [6, 17.24], and the conclusion follows. $\qquad \square$

3.2. **KSP functions.** Throughout this section we assume the conclusion of Theorem 3.1 holds for a filter $\mathcal{D}$ of full outer measure subsets of $\mathbb{R}$. Since $\mathcal{F}_{KSP}$ is transitive, a proof identical to the proof of Lemma 3.2 gives the following lemma.

Lemma 3.4. *Every bounded KSP-measurable function is KS-$\mathcal{D}$-approximated.* $\qquad \square$

Lemma 3.5. *If a KSP function F is KS-$\mathcal{D}$-approximated by a Borel function G then it is Borel.*

Proof. We have $F(x) = 1 - 2 \int_{c(x,\pi/2)} F \, d\mu$ for every $x \in S^2$. Fix $a \in S^2$. Then the set of $x \in c(a, \pi/2)$ such that $\int_{c(x,\pi/2)} G \, d\mu = \int_{c(x,\pi/2)} F \, d\mu$ has full outer measure in $c(a, \pi/2)$. The function of $x \in c(a, \pi/2)$ on the right-hand side is measurable (by [6, 17.24]) and agrees with F on

a set of full outer measure in $c(a, \pi/2)$. Since F is Borel, these two functions agree almost everywhere and

$$F(a) = 1 - 2 \int_{c(a,\pi/2)} (1 - 2 \int_{c(x,\pi/2)} G(y)\, d\mu_{x,\pi/2}(y))\, d\mu_{a,\pi/2}(x)$$

and it only remains to check that the function of a on the right hand side is Borel-measurable. Like in the proof of Lemma 3.3, this follows from $a \mapsto \mu_{a,\pi/2}$ being Borel and applying [6, 17.24] twice.   □

Fix two members of S^2 a, b which are orthogonal. We represent each rotation α in SO(3) as a product $\alpha_3^{a,b}(r,s,t) \cdot \alpha_2^{a,b}(r,s) \cdot \alpha_1^{a,b}(r)$ where $\alpha_1^{a,b}(r)$ is a rotation around a by an angle r, $\alpha_2^{a,b}(r,s)$ is a rotation around $\alpha_1^{a,b}(r)(b)$ by an angle s and α_3 is a rotation around $\alpha_2^{a,b}(r,s)(\alpha_1^{a,b}(r)(b))$ by an angle t. (We are going to omit the superscript a, b if they are understood from the context.)

For any function F on S^2 and $a, b, c \in S^2$ which are mutually orthogonal define

$$I_{a,b,c}(F) = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} F(\alpha_3^{a,b}(r,s,t) \cdot \alpha_2^{a,b}(r,s) \cdot \alpha_1^{a,b}(r)(c))\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}r$$

where the measure on on the interval $[0, 2\pi]$ is normalized to be 1 and $I_{a,b,c}(F)$ is defined only if all the relevant integrals are defined. Note that if G is a measurable function on S^2 then $I_{a,b,c}(G) = \int_{SO(3)} G(\alpha(c))\, \mathrm{d}\alpha$ where we take the right invariant Haar measure on SO(3). In this case $I_{a,b,c}(G)$ does not depend on the choice of a, b, c.

Lemma 3.6. *Assume F is a KSP function on S^2. Then for every mutually orthogonal triple a, b, c $I_{a,b,c}(F)$ is defined and is equal to $(5 - F(a))/8$*

Proof. By F being a KSP function it is easily seen that for every $d \in S^2$ we have $\int_{c(d,\pi/2)} F\, d\mu = 1 - \frac{F(d)}{2}$. Fix $r, s \in [0, 2\pi]$. The inner integral in the definition of $I_{a,b,c}(F)$ is the integral of F over $C(\alpha_2(r,s)(a))$. Hence it is equal to $1 - \frac{F(\alpha_2(r,s)(a))}{2}$. The second integral (according to s) is the integral of the function $H(x) = 1 - \frac{F(x))}{2}$ on the great circle $C(\alpha_1(r)(b))$ so it is equal to $(3 + F(\alpha_1(r)(b))/4)$. The last integral is the integral of the function $K(x) = (3 + F(x))/4$ on the great circle $C(a)$ so it is equal to $(5 - F(a))/8$.   □

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