DIMENSION PHENOMENA ASSOCIATED WITH $\beta\mathbb{N}$-SPACES

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The Čech–Stone compactification of the integers, $\beta\mathbb{N}$, and its Čech–Stone reminder, $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, are zero-dimensional spaces, thus so are all of their powers. However, some 'dimension phenomena' have been observed for these spaces. For example, van Douwen ([1]) proved that $(\beta\mathbb{N})^\kappa$ cannot be embedded into $(\beta\mathbb{N})^\lambda$, for any pair of cardinals $\kappa > \lambda$, finite or infinite. An even more amusing phenomenon associated with dimension, also first observed in [1], is the fact that in the powers of $\beta\mathbb{N}$ one can 'detect the direction of coordinate axes.' Namely, it was proved in [1, Lemma 5.1] that if $f\colon (\beta\mathbb{N})^d \to \beta\mathbb{N}$ (for a natural number d) then there is a clopen $U \subseteq \beta\mathbb{N}$ such that $f \restriction U^d$ depends on at most one coordinate. Van Douwen has proved results analogous to the above two for a more general class of $\beta\mathbb{N}$-spaces, including $\mathbb{N}^*$. Recall that a space X is a *$\beta\mathbb{N}$-space* if every countably infinite discrete $D \subseteq X$ whose closure is compact has a closure homeomorphic to $\beta\mathbb{N}$.

In the same paper, the following two conjectures were stated.

Conjecture 1 (van Douwen, [1, Conjecture 8.4]). *Let X be $\mathbb{N}^*$ or $\beta\mathbb{N}$. If f is any continuous binary operation on X, then there is a disjoint open cover $\mathcal{U}$ of X such that f depends on at most one coordinate on each one of these sets.*

Conjecture 2 (van Douwen, [1, Conjecture 8.5]). *Let X be $\mathbb{N}^*$ or $\beta\mathbb{N}$. For every autohomeomorphism F of $X^{(2)}$ there is a disjoint open cover $\mathcal{U}$ of X such that both coordinate maps $\pi_0 \circ F$ and $\pi_1 \circ F$ depend on at most one coordinate on every set in $\mathcal{U}$.*

It is easily seen that the first conjecture is stronger than the second. In [2, Theorem 4.3.2] we have verified the $\beta\mathbb{N}$-case of these two conjectures. In the present paper we extend these results to powers of arbitrary compact $\beta\mathbb{N}$-spaces and prove the following result, thus confirming and generalizing the above conjectures (a subset of X^d is a *rectangle* if it is of the form $A_0 \times A_1 \times \cdots \times A_{d-1}$ for some $A_i \subseteq X$).

Theorem 3. *Assume Z is a $\beta\mathbb{N}$-space, X is compact, κ is an arbitrary cardinal and $f\colon X^\kappa \to Z$. Then X^κ can be covered by finitely many clopen rectangles such that f depends on at most one coordinate on each one of them.*

This result confirms Conjectures 1 and 2 (see Theorem 2.1). It also has strong implications to the structure of maps between compact $\beta\mathbb{N}$-spaces. For example, it implies that a map from X^κ into Z^λ (for X, Z as in Theorem 3 and λ infinite) depends on at most λ many coordinates (Theorem 2.2). Thus a κ-th power of an infinite compact space can be embedded into a λ-th power of a $\beta\mathbb{N}$-space only if $\kappa \leq \lambda$ (this was proved by van Douwen in [1, Lemma 6.3] by using different methods). Theorem 3 has strong impact on the structure of autohomeomorphisms

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of finite powers compact $\beta\mathbb{N}$-spaces. In Theorem 3.4 we show that it also has impact on the local structure of autohomeomorphisms of the infinite powers.

One of the characterizations of $\beta\mathbb{N}$ states that it is a compactification of $\mathbb{N}$ such that every map from $\mathbb{N}$ into a compact space extends to $\beta\mathbb{N}$. In Theorem 7.5 we characterize the maps from $\mathbb{N}^d$ into a $\beta\mathbb{N}$-space that continuously extend to $(\beta\mathbb{N})^d$.

Let us consider the 'dual' statement to "$(\beta\mathbb{N})^\kappa$ does not embed into $(\beta\mathbb{N})^\lambda$ for $\kappa > \lambda$," saying that $(\mathbb{N}^*)^\kappa$ does not map onto $(\mathbb{N}^*)^\lambda$ for $\kappa < \lambda$. (This is the best that we can hope for, since $\beta\mathbb{N}$ maps onto any separable compact space.) By a classical result of Parovičenko [5], the Continuum Hypothesis implies that $\mathbb{N}^*$ maps onto every compact space of weight at most continuum, thus this 'dual' fails already for $\kappa = 1$ and $\lambda = \mathfrak{c}$. However, by a result of Just [4] in some forcing extension of the universe we may have that $(\mathbb{N}^*)^d$ does not map onto $(\mathbb{N}^*)^{d+1}$, for all natural numbers d. Relying on a work of Shelah, Just, and Velickovic, in [2, Chapter 4] we have introduced the weak Extension Principle, wEP, a set-theoretic axiom which provides an ambient for study of $\mathbb{N}^*$ and other $\beta\mathbb{N}$-spaces with properties opposite to those provided under the Continuum Hypothesis (see §4). This axiom implies, for example, that all autohomeomorphisms of a Čech–Stone remainder of every countable, locally compact space X are *trivial*, i.e., induced by a map $f\colon X \to X$ ([2, §4.7]; it was proved in [6] that all autohomeomorphisms of $\mathbb{N}^*$ can be trivial). The following conjecture was stated in an early version of [2] (recall that an ordinal is *indecomposable* if it cannot be written as a sum of two strictly smaller ordinals; we consider ordinals with their order topology).

Conjecture 4. *wEP implies that the following are equivalent.*

(1) $(\alpha^*)^m$ *maps onto* $(\gamma^*)^n$,
(2) $\alpha \geq \gamma$ *and* $m \geq n$,

for all countable indecomposable ordinals α *and* γ *and all natural numbers* m, n.

Recall that, by a result of Sierpiński, every countable locally compact space is homeomorphic to an ordinal (see e.g., [2, Proposition 4.7.1]), and that every ordinal is homeomorphic to a topological sum of a compact set and an indecomposable ordinal. (The latter is true because we can always write α as $\alpha_0 + \alpha_1$, where α_1 is indecomposable and α_0 is a successor ordinal.) Thus the above conjecture is a strong statement about the rigidity of finite powers of Čech–Stone remainders of locally compact spaces, saying that two such powers are homeomorphic if and only if they are homeomorphic for obvious reasons. Some of its special cases were verified in [2, §4.6]: when $m = n = 1$ and α, γ are arbitrary and when $\alpha = \omega$ and γ, m, n are arbitrary.

An evidence that wEP is the right axiom in this context is given in [2, §4.6], where it was proved that (roughly) (1) and (2) are consistently equivalent if and only if wEP implies that they are equivalent. In Theorem 5.3 we verify Conjecture 4 in an even stronger form, allowing m and n to be arbitrary infinite cardinals.

In §1 we introduce the notation and terminology, in §§2–5 we give applications of Theorem 3, while §§8–9 are devoted to its proof. In §2 we give its generalization to maps into Z^λ and two lemmas describing the local behavior of a map between the infinite powers. In §3 we prove that the finite powers of compact $\beta\mathbb{N}$-spaces have only the obvious autohomeomorphisms. In §4 we state the weak Extension Principle. In §5 we confirm Conjecture 4 and use it to give an analogous criterion for when the powers of countable locally compact spaces are homeomorphic, thus confirming another conjecture from an early version of [2].

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1. Definitions and notation

Our terminology and notation are either standard or follow [1]. The reader is suggested to go straight to §2, and later consult this section when needed. If f is a function and A is a subset of its domain, then by $f''A$ we denote the image of A under f. If $s \subseteq \kappa$ then $\pi^\kappa_s \colon X^\kappa \to X^s$ is the projection of X^κ to X^s,

$$\pi^\kappa_s(\langle x_\xi : \xi < \kappa\rangle) = \langle x_\xi : \xi \in s\rangle.$$

Whenever κ is clear from the context, we will write π_s instead of π^κ_s. If s is a singleton $\{\xi\}$, then we write π_ξ instead of $\pi_{\{\xi\}}$.

If $I \cap J = \emptyset$, then for $x \in X^I$ and $y \in X^J$ we define $x\hat{\ }y \in X^{I\cup J}$ by

$$(x\hat{\ }y)(\xi) = \begin{cases} x(\xi), & \text{if } \xi \in I,\\ y(\xi), & \text{if } \xi \in J.\end{cases}$$

Definition 1.1. *A space X is a* $\beta\mathbb{N}$-space *if every countably infinite discrete subset D of X whose closure is compact has a closure homeomorphic to $\beta\mathbb{N}$.*

The following was proved by van Douwen who has introduced $\beta\mathbb{N}$-spaces as a generalization of F-spaces (recall that X is an *F-space* if every cozero subset of X is C^*-embedded).

Lemma 1.2. (1) *If X is a locally compact, σ-compact space, then its Čech–Stone remainder, X^*, is a $\beta\mathbb{N}$-space.*
(2) *More generally, every F-space is a $\beta\mathbb{N}$-space.*
(3) *$\mathbb{Q}^*$ is a $\beta\mathbb{N}$-space.*

Proof. See [1, §4]. □

One of the objectives of this note is to prove that all maps between powers of compact $\beta\mathbb{N}$-spaces are of a very simple kind. Let us thus describe 'very simple' maps between powers. If X and Y are topological spaces, and consider $f_\eta \colon X \to Y$ $(\eta < \lambda)$ and a map $p\colon \lambda \to \kappa$. Then

$$f(\langle x_\xi : \xi < \kappa\rangle) = \langle f_\eta(x_{p(\eta)}) : \eta < \lambda\rangle$$

defines a map $f\colon X^\kappa \to Y^\lambda$. We say that such a map is *elementary*. Elementary maps are arguably the simplest maps between powers, since no 'mixing' of coordinates is involved. For example, rotations of $\mathcal{R}^2$ are highly non-elementary maps.

Now assume that for some $k \in \mathbb{N}$ we can split X^κ and Y^λ into k many pairwise disjoint clopen rectangles in two ways

$$X^\kappa = U_0 \cup U_1 \cup \dots \cup U_{k-1}$$
$$Y^\lambda = V_0 \cup V_1 \cup \dots \cup V_{k-1},$$

and that $g_i \colon U_i \to V_i$ is an elementary map for each $i < k$. Then we say that the map $g = \bigcup_i g_i$ is *piecewise elementary*.

The notions of elementary and piecewise elementary can be extended to the maps between products of different spaces, and most of our results—along with their proofs—apply to products of spaces, but they are stated and proved for powers only for the sake of simplicity of notation.

2. The structure of maps between finite and infinite powers

We start by an immediate consequence of Theorem 3.

Theorem 2.1. *Assume X is compact and Y is a $\beta\mathbb{N}$-space. Then every map from an arbitrary power of X into a finite power of Y is piecewise elementary.*

Proof. Fix $f \colon X^\kappa \to Y^d$. Since d is finite, by applying Theorem 3 to each of the coordinate maps $\pi_i \circ f$ and intersecting, we obtain finitely many clopen rectangles such that each $\pi_i \circ f$ is elementary on each one of these rectangles. □

The assumption that the power of Y is finite is necessary; see Example 3.3. But a relative of Theorem 2.1 is still true in the case when the power of Y is infinite.

Theorem 2.2. *Assume X is compact, Y is a $\beta\mathbb{N}$-space, and λ is an infinite cardinal. Then every $f \colon X^\kappa \to Y^\lambda$ depends on at most λ many coordinates.*

Proof. Fix $f \colon X^\kappa \to Y^\lambda$. Fix $\eta < \lambda$. By Theorem 3, we can partition X^κ into finitely many clopen pieces so that $\pi_\eta \circ f$ depends on at most one coordinate on each one of them. Therefore, $\pi_\eta \circ f$ depends on at most finitely many coordinates for every $\eta < \lambda$. Since f depends on the coordinate ξ if and only if some $\pi_\eta \circ f$ does and λ is infinite, the desired conclusion follows. □

The following consequence of Theorem 2.2 was proved by van Douwen in [1, Lemma 6.3] under a weaker assumption of Y being a so-called *weak $\beta\mathbb{N}$-space*.

Corollary 2.3. *Assume X is infinite and compact and Y is a $\beta\mathbb{N}$-space. Then $\kappa > \lambda$ implies that X^κ does not embed into Y^λ.*

Proof. Assume $f \colon X^\kappa \to Y^\lambda$ and $\kappa > \lambda$. If λ is infinite, then by Theorem 2.2 there is a $\xi < \kappa$ such that f does not depend on ξ-th coordinate; since X is infinite, f is not an embedding.

If λ is finite, apply Theorem 2.1 to obtain a partition of X into rectangles such that f depends on at most λ many coordinates on each one of them. But since X is infinite, at least one of these rectangles, call it $A_0 \times \cdots \times A_{\kappa-1}$, has all of its sides A_ξ infinite, and therefore the restriction of f to this rectangle is not an embedding. □

Note that $\beta\mathbb{N}$ maps onto $(\beta\mathbb{N})^2$ (or any other compact separable space), thus the statement dual to Corollary 2.3 is false. Similarly, by a classical result of Parovičenko ([5]) Continuum Hypothesis implies that $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^2$ (as well as onto any other compact space of weight not bigger than $2^{\aleph_0}$). We shall, however, see in Theorem 5.1 that something can be said along these lines.

Let us now prove two technical lemmas about the local behavior of continuous functions from powers of compact spaces into the powers of $\beta\mathbb{N}$-spaces.

Lemma 2.4. *If $f \colon X^\kappa \to Y^\lambda$, Y is a $\beta\mathbb{N}$-space and X is compact, then there are*

$$p^x \colon \lambda \to \kappa, \qquad x \in X^\kappa$$

such that for every $\xi < \lambda$ there is a clopen $U_{\xi,x} \ni x$ such that $\pi_\xi \circ f$ depends only on $p^x(\xi)$-th coordinate on $U_{\xi,x}$.

Proof. For every $\xi < \lambda$ we can cover X^κ with clopen sets $U_0^\xi, \dots, U_{l(\xi)-1}^\xi$ such that $(\pi_\xi \circ f)$ depends on at most one variable on each one of these sets. One of these sets contains x; let $U_{\xi,x}$ be this set. If $(\pi_\xi \circ f) \restriction U_{\xi,x}$ depends on η-th coordinate, then let $p^x(\xi) = \eta$ and if $(\pi_\xi \circ f) \restriction U_{\xi,x}$ is constant then let $p^x(\xi) = 0$. These objects clearly satisfy the requirements. □

Lemma 2.5. *Assume X is compact, Y is a $\beta\mathbb{N}$-space, $\lambda > \kappa$, and $f\colon X^\kappa \to Y^\lambda$. Then there are an $l \in \mathbb{N}$ and:*

(1) *a clopen cover $X^\kappa = U_0 \cup \dots \cup U_{l-1}$,*
(2) *$\zeta_0 \neq \eta_0, \dots, \zeta_{l-1} \neq \eta_{l-1}$, all less than λ,*
(3) *$\xi_0, \dots, \xi_{l-1} < \kappa$,*

such that for every $i < l$ both $(\pi_{\zeta_i} \circ f)$ and $(\pi_{\eta_i} \circ f)$ depend only on ξ_i-th coordinate on U_i.

Proof. Let $p^x\colon \lambda \to \kappa$ ($x \in X^\kappa$) be as guaranteed by Lemma 2.4. Since $\lambda > \kappa$, for every x there are $\zeta(x) \neq \eta(x) < \lambda$ and $\xi(x) < \lambda$ such that

$$p^x(\zeta(x)) = p^x(\eta(x)) = \xi(x).$$

Let $U(x) = U_{\zeta(x),x} \cap U_{\eta(x),x}$. This is a clopen neighborhood of x such that the restrictions of both $\pi_{\zeta(x)} \circ f$ and $\pi_{\eta(x)} \circ f$ to $U(x)$ depends only on $\xi(x)$-th coordinate. Let $x_0, \dots, x_{l-1}$ be such that $U(x_0), \dots, U(x_{l-1})$ form a finite subcover of $U(x)$ ($x \in X^\kappa$). Then $U_i = U(x_i)$, $\eta_i = \eta(x_i)$, $\zeta_i = \zeta(x_i)$ and $\xi_i = \xi(x_i)$ ($i < l$) are as required. □

3. Autohomeomorphisms of compact $\beta\mathbb{N}$-spaces

The following result was proved for $X = \beta\mathbb{N}$ in [2, Theorem 4.4.2]; its instance when $X = \mathbb{N}^*$ confirms van Douwen's Conjecture 2.

Theorem 3.1. *If X is a compact $\beta\mathbb{N}$-space, then all autohomeomorphisms of X^d are piecewise elementary.*

Proof. This follows immediately from Theorem 2.1. □

In some cases we can say even more. Recall that $\mathbb{H} = [0, \infty)$.

Theorem 3.2. (a) *Every continuous map $f\colon (\mathbb{H}^*)^\kappa \to \mathbb{H}^*$ is elementary.*
(b) *All autohomeomorphisms of an arbitrary power $(\mathbb{H}^*)^\kappa$ of $\mathbb{H}^*$ are elementary.*

Proof. Since $\mathbb{H}^*$ is connected, Theorem 3 implies that f depends on at most one coordinate. To prove (b), fix $f\colon (\mathbb{H}^*)^\kappa \to (\mathbb{H}^*)^\kappa$ and apply (a) to each $\pi_\eta \circ f$, for $\eta < \kappa$. □

Theorem 3.1 is in general false for infinite powers, as the following example shows.

Example 3.3. An autohomeomorphism g of $(\beta\mathbb{N})^\mathbb{N}$ which is not piecewise elementary. Let

$$(\mathcal{U}_0, \mathcal{U}_1) \mapsto (f_0(\mathcal{U}_0, \mathcal{U}_1), f_1(\mathcal{U}_0, \mathcal{U}_1))$$

be an autohomeomorphism of $(\beta\mathbb{N})^2$ which is not elementary. Write $(\beta\mathbb{N})^{\mathbb{N}} = ((\beta\mathbb{N})^2)^{\mathbb{N}}$, and define f to be $\prod_{i=0}^{\infty} f_i$, i.e.,

$$f(\mathcal{U}_0, \mathcal{U}_1) = (f_0(\mathcal{U}_0, \mathcal{U}_1), f_1(\mathcal{U}_0, \mathcal{U}_1), f_0(\mathcal{U}_2, \mathcal{U}_3), f_1(\mathcal{U}_2, \mathcal{U}_3), \dots).$$

Since every clopen subset of $(\beta\mathbb{N})^{\mathbb{N}}$ depends on at most finitely many coordinates, the function f is not elementary on any clopen set, and therefore it cannot be piecewise elementary.

Lemma 2.4 can be used to describe autohomeomorphisms of infinite powers of compact $\beta\mathbb{N}$-spaces. The following result gives some local information.

Theorem 3.4. *Let X be $\beta\mathbb{N}$ or $\mathbb{N}^*$, and let f be an autohomeomorphism of X^κ. Fix $\vec{\mathcal{U}} = \langle \mathcal{U}_\xi : \xi < \kappa\rangle \in X^\kappa$ and let $f(\vec{\mathcal{U}}) = \vec{\mathcal{V}} = \langle \mathcal{V}_\xi : \xi < \kappa\rangle$. Then*

(1) *$\vec{\mathcal{U}} \in \mathbb{N}^\kappa$ if and only if $\vec{\mathcal{V}} \in \mathbb{N}^\kappa$.*
(2) *The sets $\{\xi < \kappa : \mathcal{U}_\xi$ is a P-point$\}$ and $\{\xi < \kappa : \mathcal{V}_\xi$ is a P-point$\}$ are of the same cardinality.*

The statement (1) is most likely already known. The choice of (2) is rather arbitrary; it will be clear that the same proof gives a number of analogous statements.

Proof. Let $p^{\vec{\mathcal{U}}} : \kappa \to \kappa$ and $p^{\vec{\mathcal{V}}} : \kappa \to \kappa$ be the maps guaranteed by applying Lemma 2.4 to f and f^{-1}, respectively. Recall that $p^{\vec{\mathcal{U}}}(\xi) = \eta$ means that $\pi_\xi \circ f$ depends only on η-th coordinate in a clopen neighborhood of $\vec{\mathcal{U}}$. Since f is an autohomeomorphism, $\pi_\eta \circ f^{-1}$ depends only on ξ-th coordinate in a clopen neighborhood of $\vec{\mathcal{V}}$. Therefore, $p^{\vec{\mathcal{V}}}$ is the inverse of $p^{\vec{\mathcal{U}}}$, and $p^{\vec{\mathcal{U}}}$ is a bijection.

Since a clopen subset of $\beta\mathbb{N}$ (or $\mathbb{N}^*$) is either a singleton or homeomorphic to $\beta\mathbb{N}$ (or $\mathbb{N}^*$, respectively), this shows that $p^{\vec{\mathcal{U}}}(\xi) = \eta$ implies that $\mathcal{U}_\eta$ is being sent to $\mathcal{V}_\xi$ by an autohomeomorphism of $\beta\mathbb{N}$ (or $\mathbb{N}^*$, respectively). Therefore (1) and (2) follow. $\square$

4. The weak Extension Principle

The weak Extension Principle, wEP, was introduced in [2, Chapter 4] and proved to follow from OCA and MA in [2, Theorem 4.9.1]. This axiom provides an ambient for study of $\mathbb{N}^*$ and related spaces with properties opposite to the one provided by the Continuum Hypothesis. For example, by [5], CH implies that $\mathbb{N}^*$ maps onto a space X unless this is clearly impossible (namely, unless X is not compact or its weight is too large). On the other hand, wEP implies that $\mathbb{N}^*$ does not map onto a space X unless there is an obvious reason for this (i.e., a simply definable map; see Definition 4.1 below). The reader is invited to consult [2, §2.1] for a thorough discussion of this approach (see also the paragraph after Theorem 5.3).

In the following V and W stand for topological spaces.

Definition 4.1. *A map $f : (V^*)^\kappa \to (W^*)^\lambda$ is* trivial *if there is a (continuous) map $h : V^\kappa \to W^\lambda$ which continuously extends to*

$$(\beta h) : (\beta V)^\kappa \to (\beta W)^\lambda$$

so that $f = (\beta h) \restriction (V^*)^\kappa$.

wEP(V, W)**:** For a continuous $F : V^* \to W^*$ there is a clopen partition $V^* = U_0 \cup U_1$ such that $F''U_0$ is nowhere dense inside W^* and $F \restriction U_1$ is trivial.

Definition 4.2. *By* wEP *we denote the statement that* wEP(α,γ) *holds for all countable locally compact spaces* α,γ.

Note that we cannot strengthen wEP to "all $F\colon V^* \to W^*$ are trivial," since no constant map is trivial. Whether wEP can be strengthened to provide more information on the structure of $F \restriction U_0$ is still open (see [2, §4.11]). The definition of wEP given above is apparently weaker than the one given in [2, §4.1], since the original statement was about maps between arbitrary finite powers of V^* and W^*. However, Lemma 4.3 below implies that the two are equivalent.

Lemma 4.3. wEP(V,W) *is equivalent to the following* (κ *is an arbitrary cardinal, while* l *is a natural number*):

wEP(V,W)**:** *For a continuous* $F\colon (V^*)^\kappa \to (W^*)^l$ *there is a clopen partition* $(V^*)^\kappa = U_0 \cup U_1$ *such that* $F''U_0$ *is nowhere dense inside* $(W^*)^l$ *and* $F \restriction U_1$ *is trivial.*

Proof. By Theorem 2.1 every $f\colon (V^*)^\kappa \to (W^*)^l$ is piecewise elementary, hence we can apply wEP(V,W) to each one of the pieces and obtain the desired decomposition. $\square$

Corollary 4.4. *Assume* wEP(V,W). *Then every open map (in particular, every homeomorphism)* $f\colon (V^*)^\kappa \to (W^*)^\lambda$ *is trivial.*

Proof. Apply wEP. If the f-image of a clopen set is nowhere dense, then this clopen set has to be empty. Therefore $\pi_\eta \circ f$ is trivial for all η, thus f is trivial itself. $\square$

Corollary 4.5. wEP *implies that all autohomeomorphisms of* $(X^*)^\kappa$ *are trivial, for every countable locally compact space* X. $\square$

Let us now prove a strengthening of [2, Theorem 4.5.1].

Theorem 4.6. *Assume* wEP(V,W) *and that* V *is homeomorphic to the topological sum* $V \oplus V$. *Then the following are equivalent*

(1) $(V^*)^\kappa$ *maps onto* W^* *for some* κ,
(2) V^* *maps onto* W^*,
(3) *there is a map* $f\colon V \to W$ *such that the* f*-preimages of compact sets are compact and* $W \setminus f''V$ *is relatively compact.*

Proof. In [2, Theorem 4.5.1] we have proved that (2) and (3) are equivalent, and (2) clearly implies (1). Hence it remains to prove that (1) implies (2).

Let $f\colon (V^*)^\kappa \to W^*$ be a surjection as in (1). By Theorem 3, there is a clopen cover $U_0,\dots,U_{l-1}$ of $(V^*)^\kappa$ such that f is elementary on each U_i. Let ξ_i $(i<l)$ be such that $f \restriction U_i$ depends on ξ_i-th coordinate (let $\xi_i = 0$ if $f \restriction U_i$ is constant). Fix an $x = \langle x_\xi : \xi < \kappa\rangle \in (V^*)^\kappa$, and let $g\colon \bigoplus_{i=0}^{l-1} V^* \to (V^*)^\kappa$ be defined by sending the i-th copy of V^* homeomorphically onto the set

$$\{x \restriction \xi_i\} \times V^* \times \{x \restriction [\xi_i+1,\kappa)\}.$$

Since V is homeomorphic to $V \oplus V$, V^* is homeomorphic to $\bigoplus_{i=0}^{l-1} V^*$, thus we can think of g as having V^* as its domain. But then $(f\circ g)\colon V^* \to W^*$ is a surjection, and therefore (2) holds. $\square$

A classical result of Mazurkiewicz and Sierpinski implies that every locally compact countable space is homeomorphic to an ordinal, taken with its order-topology (see e.g., [2, Proposition 4.7.1]). Recall that an ordinal α is *indecomposable* if

$\alpha = \alpha_0 + \alpha_1$ implies that α_1 is either 0 or α. Since $\alpha = \alpha_0 + \alpha_1$ and $\alpha_1 \neq 0$ implies that α^* and α_1^* are homeomorphic, every Čech–Stone remainder of a countable ordinal is homeomorphic to a Čech–Stone remainder of an indecomposable ordinal.

Corollary 4.7. *If α and γ are countable indecomposable ordinals, then* wEP *implies that the following are equivalent:*

(1) $(\alpha^*)^\kappa$ *maps onto* γ^* *for some* κ,
(2) α^* *maps onto* γ^*,
(3) $\alpha \geq \gamma$.

In particular, $(\mathbb{N}^*)^\kappa$ *does not map onto* $(\omega^2)^*$ *for any* κ.

Proof. Since α is homeomorphic to $\alpha \oplus \alpha$, by Theorem 4.6 the conclusion follows. □

5. Rigidity properties for powers of $\beta\mathbb{N}$-spaces

We are now ready to give the promised 'dual' to Corollary 2.3. Recall that CH implies that $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^2$, thus we need to assume an additional set-theoretic assumption like wEP to prove the following result.

Theorem 5.1. *Assume* wEP(X, Y), *where X and Y are σ-compact and locally compact, but not compact. Then $\kappa < \lambda$ implies that $(X^*)^\kappa$ does not map onto $(Y^*)^\lambda$.*

Proof. By Lemma 1.2, the Čech–Stone remainder of a σ-compact, locally compact space is a $\beta\mathbb{N}$-space. Assume $\kappa < \lambda$ and fix $f\colon (X^*)^\kappa \to (Y^*)^\lambda$. We will prove that the range of f is nowhere dense. By Lemma 2.5, there is a clopen cover $U_0, \dots, U_{k-1}$ of $(X^*)^\kappa$, $\zeta_0 \neq \eta_0, \dots, \zeta_{l-1} \neq \eta_{l-1}$ less than λ, and $\xi_0, \dots, \xi_{l-1}$ less than κ such that for every $i < l$ both $(\pi_{\zeta_i} \circ f)$ and $(\pi_{\eta_i} \circ f)$ depend only on ξ_i-th coordinate on U_i. Apply wEP to $\pi_{\eta_i} \circ f \restriction \overline{U_i}$ and to $\pi_{\zeta_i} \circ f \restriction \overline{U_i}$, for every i. This gives a decomposition of $\overline{U_i}$ into two relatively clopen sets, V_i^0 and V_i^1, such that the image of V_i^0 is nowhere dense and $f \restriction V_i^1$ is trivial. The following lemma implies that $f''V_i^1$ is nowhere dense.

Lemma 5.2. *Assume X, Y and Z are σ-compact, locally compact, but not compact. If $f : X \to Y \times Z$ is continuous and such that $(\beta f)''X^* \subseteq Y^* \times Z^*$ then $(\beta f)''X^*$ is nowhere dense in $Y^* \times Z^*$.*

Proof. The sets of the form $U^* \times V^*$, where U and V are open in Y and Z respectively and with non-compact closures, form a base for the topology of $Y^* \times Z^*$; here U^* denotes $Y^* \setminus \overline{Y \setminus U}$. The statement that $(\beta f)''X^*$ is nowhere dense in $Y^* \times Z^*$ is translated into the following statement:

(*) for every pair of open sets U and V, with non-compact closures one can find open sets $U' \subseteq U$ and $V' \subseteq V$, also with non-compact closures, such that $f''X \cap (U' \times V') = \emptyset$.

It is (*) that we shall now verify. Let $f_0\colon X \to Y$ and $f_1\colon X \to Z$ be such that

$$f(x) = \langle f_0(x), f_1(x) \rangle.$$

To prove (*), we construct $U_i \subseteq Y$ and $V_i \subseteq Z$ such that for all i:

(a) $U_i \subseteq U$ and $V_i \subseteq V$ are open, relatively compact, and nonempty.
(b) U_i is disjoint from $\bigcup_{j<i} f_0''(\overline{f_0^{-1}(U_j)} \cup \overline{f_1^{-1}(V_j)})$.
(c) V_i is disjoint from $\bigcup_{j<i} f_1''(\overline{f_0^{-1}(U_j)} \cup \overline{f_1^{-1}(V_j)})$.

Assume U_j, V_j are chosen for $j < i$. Since Theorem 4.6 implies that $f_0^{-1}(U_j)$ and $f_1^{-1}(V_j)$ are relatively compact, $F = \bigcup_{j<i} f_0''\Big(\overline{f_0^{-1}(U_j)} \cup \overline{f_1^{-1}(V_j)}\Big)$ is a compact subset of Y. Therefore $U \setminus F$ is nonempty, and by the local compactness of Y we can pick a nonempty open and relatively compact $U_i \subseteq U \setminus F$. Similarly, we can pick $V_i \subseteq V$ which satisfies the requirements.

Assume U_i and V_i satisfy (a)–(c) above for all i. Let

$$U' = \bigcup_{i=0}^{\infty} U_{2i}, \qquad V' = \bigcup_{i=0}^{\infty} V_{2i+1}.$$

It remains to prove that $f_0(x) \in U'$ implies $f_1(x) \notin V'$ for every $x \in X$. Assume $f_0(x) \in U'$, and let i be such that $f_0(x) \in U_{2i}$. Then by (b) $f_1(x) \notin V_j$ for $j < 2i$ and by (c) $f_1(x) \notin V_j$ for $j > 2i$; in particular, $f_1(x) \notin V'$. Therefore $f''X \cap U' \times V' = \emptyset$.

This concludes the proof of (*), as well as the proof of the Lemma ... □

... and the Theorem. □

The following theorem confirms (and strengthens) Conjecture 4 originally stated in an early version of [2].

Theorem 5.3. wEP *implies that the following are equivalent*

1. $(\alpha^*)^\kappa$ *maps onto* $(\gamma^*)^\lambda$,
2. $\alpha \geq \gamma$ *and* $\kappa \geq \lambda$,

for all countable indecomposable ordinals α *and* γ *and all cardinals* κ, λ.

Proof. The direction (2) implies (1) is easy and it does not require wEP; (see [2, Theorem 4.5.1]).

Assume (2) fails. If $\kappa < \lambda$, then Theorem 5.1 implies that (1) fails. Otherwise, if $\alpha < \gamma$, then Theorem 4.6 implies that (1) fails. This completes the proof. □

The main point of Theorem 5.3 is not that it is consistent that (1) and (2) are equivalent. The main point of Theorem 5.3 is that a surjection from $(\alpha^*)^\kappa$ onto $(\gamma^*)^\lambda$ can be constructed without using additional set-theoretic assumptions only when $\alpha \geq \gamma$ and $\kappa \geq \lambda$. Even better, every such surjection has to be a slight modification of the 'obvious' surjection. Analogous remarks apply to Theorem 5.4 and Corollary 5.5. See also [2, §2.1].

The following theorem was also conjectured in an early version of [2] in a weaker form, in the case when κ and λ are finite.

Theorem 5.4. *wEP implies that the following are equivalent*

1. $(\alpha^*)^\kappa$ *is homeomorphic to* $(\gamma^*)^\lambda$,
2. $\alpha = \gamma$ *and* $\kappa = \lambda$,
3. $(\alpha^*)^\kappa$ *maps onto* $(\gamma^*)^\lambda$ *and* $(\gamma^*)^\lambda$ *maps onto* $(\alpha^*)^\kappa$,

for all countable indecomposable ordinals α *and* γ *and all cardinals* κ, λ.

Proof. It is obvious that (2) implies (1) and (1) implies (3). By Theorem 5.3, (3) implies (2). □

Since every countable locally compact space is homeomorphic to an ordinal, we have the following.

Corollary 5.5. *Assume* wEP *and let X and Y be countable locally compact spaces. Then the following are equivalent:*

(1) *$(X^*)^\kappa$ and $(Y^*)^\lambda$ are homeomorphic,*
(2) *$\kappa = \lambda$ and there are compact $K_1 \subseteq X$ and $K_2 \subseteq Y$ such that $X \setminus K_1$ and $Y \setminus K_2$ are homeomorphic.* □

6. Maps of infinite powers

We start the proof of Theorem 3 by showing that it suffices to prove it for maps whose domain is a finite power of a compact space.

Theorem 6.1. *If Z is a $\beta\mathbb{N}$-space, X is compact, and $f\colon X^\kappa \to Z$, then there is a finite $s \subseteq \kappa$ and a continuous map $f_1\colon X^s \to Z$ such that the diagram*

$$\begin{tikzcd} X^\kappa \arrow[rr, "f"] \arrow[dr, "\pi_s"'] & & Z \\ & X^s \arrow[ur, "f_1"'] & \end{tikzcd}$$

commutes.

Proof. We may assume that f is onto and therefore Z is compact. The proof proceeds by transfinite induction on the cardinal κ for fixed X and Z. The statement is vacuously true for a finite κ, thus assume that κ is an infinite cardinal and that the assertion is proved for all $\lambda < \kappa$.

Case 1. The cofinality of κ is countable. Fix a strictly increasing sequence κ_n of ordinals such that $\kappa = \sup_n \kappa_n$. We try to construct sequences x_i, y_i ($i \in \mathbb{N}$) such that for all i we have:

(1) $x_i, y_i \in X^\kappa$,
(2) $x_i \restriction \kappa_i = y_i \restriction \kappa_i$, i.e., $x_i(\xi) = y_i(\xi)$, for all $\xi < \kappa_i$, and
(3) $f(x_i) \neq f(y_i)$.

Assume that we have constructed x_i, y_i for $i < n$ (possibly $n = 0$ and these sequences are empty). If we cannot find x_n and y_n satisfying the above conditions, then define $t = \kappa_n$ and let $f_0\colon X^t \to Z$ be

$$f_0(x) = f(x'),$$

where $x' \in X^\kappa$ is arbitrary such that $x' \restriction t = x$. By our assumptions, if x' and x'' are such that $x' \restriction t = x'' \restriction t$ then $f(x') = f(x'')$. [Otherwise we could let $x_i = x'$ and $y_i = x''$.] Therefore f_0 is well-defined; it is clearly continuous and satisfies $f = f_0 \circ \pi_t$. Let λ be the cardinality of κ_n. By the inductive assumption, there is a finite $s \subseteq t$ and $f_1\colon X^s \to Z$ such that $f_1 = f_0 \circ \pi_s^{\kappa_n}$, and therefore $f = f_1 \circ \pi_s^\kappa$, as required.

Now assume that the recursive construction does not stop and there are x_i, y_i ($i \in \mathbb{N}$) satisfying (1)–(3) above. We shall need the following well-known lemma (see e.g., [1, p. 28], where its case when $Z = \beta\mathbb{N}$ was attributed to Hušek) whose proof we include for reader's convenience.

Lemma 6.2. *Assume Z is a compact $\beta\mathbb{N}$-space and a_i, b_i ($i \in \mathbb{N}$) are elements of Z such that $a_i \neq b_i$ for all i. Then there is an infinite $I \subseteq \mathbb{N}$ such that*

$$\overline{\{a_i : i \in I\}} \cap \overline{\{b_i : i \in I\}} = \emptyset.$$

Proof. For $C \subseteq \mathbb{N}$ let $A_C = \{a_i : i \in C\}$ and $B_C = \{b_i : i \in C\}$. Our proof will have three stages.

Stage 1. We obtain an infinite $C \subseteq \mathbb{N}$ such that $A_C \cap B_C = \emptyset$.

By $[\mathbb{N}]^2$ we denote the set of all unordered pairs of natural numbers, and by $\{i,j\}_<$ we denote a pair in $[\mathbb{N}]^2$ such that $i < j$. Define a partition $[\mathbb{N}]^2 = K_0 \cup K_1 \cup K_2$ by

$$\{i,j\}_< \in \begin{cases} K_0, & \text{if } a_i = b_j, \\ K_1, & \text{if } a_j = b_i, \\ K_2, & \text{if } a_i \neq b_j \text{ and } a_j \neq b_i. \end{cases}$$

If $\{i,j,k\}_<$ is a K_0-homogeneous triple, then $a_j = b_k = a_i = b_j$, a contradiction. Thus there are no K_0-homogeneous triples. One similarly proves that there are no K_1-homogeneous triples either. Thus there is an infinite $C \subseteq \mathbb{N}$ such that $\{i,j\}_< \in K_2$ (and thus $a_i \neq b_j$) for all $i,j \in C$, as required.

Stage 2. We find an infinite $I \subseteq C$ such that $\overline{A_I} \cap B_I = \emptyset$ and $A_I \cap \overline{B_I} = \emptyset$. To this end we employ [1, Fact on page 29], reproduced here for reader's convenience.

Claim 6.3. *If Z is a compact $\beta\mathbb{N}$-space and D, E are countably infinite disjoint subsets of Z, then there is an infinite $D_2 \subseteq D$ such that $\overline{D_2} \cap E = \emptyset$.*

Proof. First find an infinite relatively discrete $D_1 \subseteq D$. Let $E_1 = E \cap \overline{D_1}$. Since $\overline{D_1}$ is homeomorphic to $\beta\mathbb{N}$, the set E_1 is nowhere dense in $\overline{D_1}$, thus there is an infinite $D_2 \subseteq D$ such that $\overline{D_2} \cap E_1 = \overline{D_2} \cap E = \emptyset$, and D_2 is as required. □

Apply Claim 6.3 with $D = A_C$ and $E = B_C$ to get $C_1 \subseteq C$ so that $\overline{A_{C_1}} \cap B_C = \emptyset$. Now apply Claim 6.3 again with $D = B_{C_1}$ and $E = A_{C_1}$ to find $I \subseteq C_1$ such that $A_{C_1} \cap \overline{B_I} = \emptyset$. Then I is as required.

Stage 3. We prove that $\overline{A_I} \cap \overline{B_I} = \emptyset$.

The sets A_I and B_I are both relatively discrete. Moreover, by the choice of I no point of A_I is an accumulation point of B_I and no point of B_I is an accumulation point of A_I. Thus $A_I \cup B_I$ is relatively discrete, and since Z is a compact $\beta\mathbb{N}$-space, the closures of A_I and B_I are disjoint.

This concludes the proof of Lemma. □

Back to the proof of Theorem. Apply Lemma 6.2 with $a_i = f(x_i)$ and $b_i = f(y_i)$ to obtain an infinite $I \subseteq \mathbb{N}$ such that the sets $\{f(x_i) : i \in I\}$ and $\{f(y_i) : i \in I\}$ have disjoint closures.

Claim 6.4. *An $a \in X^\kappa$ is an accumulation point of $S = \{x_i : i \in I\}$ if and only if it is an accumulation point of $T = \{y_i : i \in I\}$.*

Proof. Assume a is not an accumulation point of S, and let U be a Tychonoff open neighborhood of a disjoint from $\{x_i : i \in I \setminus k\}$ for some large enough k. Let $l > k$ be such that U does not depend on coordinates $\xi > \kappa_l$. Then for every $m > l$ we have $x_m \in U$ if and only if $y_m \in U$, and therefore a is not an accumulation point of T either. The same proof shows that all accumulation points of S are accumulation points of T. □

By Claim 6.4 and the compactness of X^κ, the closures of the sets S and T have a nonempty intersection. But by the choice of I, the closures of the f-images of S and T are disjoint, a contradiction. Therefore the construction of sequences x_i, y_i stops at some finite stage, as required.

Case 2. The cofinality of κ is uncountable. Let $\lambda = \mathrm{cf}(\kappa)$, and let κ_ξ $(\xi < \lambda)$ be an increasing sequence cofinal in κ. Pick an arbitrary point $x \in X^\kappa$, and for an ordinal $\alpha < \lambda$ define $f_{\alpha,x} \colon X^{\kappa_\alpha} \to Z$ by

$$f_{\alpha,x}(y) = f(y\hat{\ }(x \restriction [\kappa_\alpha, \kappa))).$$

By the inductive assumption, for every α there is a finite $s_\alpha \subseteq \kappa_\alpha$ such that $f_{\alpha,x}$ depends only on coordinates in s. By the Pressing Down Lemma there is a stationary $S \subseteq \lambda$ and $\xi = \kappa_\eta < \lambda$ such that $s_\alpha \subseteq \xi$ for all $\alpha \in S$.

Claim 6.5. *For all $y, z \in X^\kappa$ such that $y \restriction \xi = z \restriction \xi$ we have $f(y) = f(z)$.*

Proof. Assume not. Find Tychonoff open neighborhoods $U \ni y$ and $V \ni z$ such that $f''U$ and $f''V$ are disjoint. Then for a large enough α and some open subsets U_0, V_0 of X^{κ_α} we have $U = U_0 \times X^{[\kappa_\alpha,\kappa)}$ and $V = V_0 \times X^{[\kappa_\alpha,\kappa)}$. We may assume that $\kappa_\alpha \in S$. Let $y' = (y \restriction \kappa_\alpha)\hat{\ }(x \restriction [\kappa_\alpha, \kappa)))$ and $z' = (z \restriction \kappa_\alpha)\hat{\ }(x \restriction [\kappa_\alpha, \kappa)))$. Then $f(y') = f_{\alpha,x}(y')$ and $f(z') = f_{\alpha,x}(z')$, but on the other hand $f(y') \neq f(z')$ (since $y' \in U$ and $z' \in V$). Since $y' \restriction \xi = z' \restriction \xi$, we have $y' \restriction s_\eta = z' \restriction s_\eta$ (recall that $\xi = \kappa_\eta$), and therefore $f(y') = f_{\alpha,x}(y') = f_{\alpha,x}(z') = f(z')$. But this contradicts $f(y') \neq f(z')$, and concludes the proof of claim. $\square$

Since $|\xi| < \kappa$, by the induction hypothesis there is a finite $s \subseteq \xi$ and $f_1 \colon X^s \to Z$ such that $f = f_1 \circ \pi_s$. This completes the induction and the proof of theorem. $\square$

7. An extension of van Douwen's lemma

In this section we will prove an extension of [1, Lemma 14.1], stated below as Lemma 7.1. For $d \in \mathbb{N}$, $d \geq 1$ let $[\mathbb{N}]^d$ denote the set of all d-element sets of natural numbers. We can naturally identify $[\mathbb{N}]^d$ with the set of all increasing d-tuples of natural numbers. By

$$\{m_0, m_1, \dots, m_{d-1}\}_<$$

we denote an element of $[\mathbb{N}]^d$ such that $m_0 < m_1 < \dots < m_{d-1}$. By $\langle\mathbb{N}\rangle^d$ we denote the set of all nondecreasing d-tuples of natural numbers. Finally, $\Delta^d\mathbb{N}$ denotes the diagonal, i.e., the set of all constant d-tuples $(n, n, \dots, n)$ of natural numbers.

Lemma 7.1 (van Douwen)**.** *If $d \geq 2$ and $f \colon (\beta\mathbb{N})^d \to Z$ is a continuous map such that $f \restriction [\mathbb{N}]^2$ is one-to-one, then Z is not a $\beta\mathbb{N}$-space.* $\square$

We shall prove only the case $d = 2$ now; the general case is proved in Corollary 7.6 below.

Lemma 7.2. *If $f \colon (\beta\mathbb{N})^2 \to Z$ is a continuous map such that the sets $f''[\mathbb{N}]^2$ and $f''\Delta^2\mathbb{N}$ are disjoint, then Z is not a $\beta\mathbb{N}$-space.*

(To prove the case $d = 2$ of van Douwen's lemma from Lemma 7.2, compose the map f with the continuous extension of the map $(m, n) \mapsto (m, n+1)$ to $(\beta\mathbb{N})^2$.)

Proof of Lemma 7.2. The proof relies on the following fact, also due to van Douwen ([1, Fact 14.2]).

Claim 7.3. *If s_n are disjoint finite subsets of $\mathbb{N}$ such that $|s_n| \geq n$ for all n, then the closures of the sets $\bigcup [s_n]^2$ and $\Delta^2 \bigcup s_n$ in $(\beta\mathbb{N})^2$ are not disjoint.* $\square$

We shall consider three cases.

CASE 1. The closure of the set $X = f''\Delta^2\mathbb{N}$ in Z is countably infinite. Then this is a countable compact subset of Z. Therefore Z has a nontrivial convergent sequence and it is not a $\beta\mathbb{N}$-space.

CASE 2. The closure of the set $X = f''\Delta^2\mathbb{N}$ in Z is uncountable. Then we can find $\mathcal{U} \in \mathbb{N}^*$ such that

(**) $f(\langle\mathcal{U},\mathcal{U}\rangle) \notin f''\langle\mathbb{N}\rangle^2$.

Let $U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset \dots$ be clopen neighborhoods of $f(\mathcal{U},\mathcal{U})$ whose intersection is disjoint from $f''\langle\mathbb{N}\rangle^2$. By the continuity there are sets $A_n \in \mathcal{U}$ such that $f''A_n^2 \subseteq U_n$ for every n. Since each A_n is infinite, we can pick pairwise disjoint $s_n \subseteq A_n$ such that $|s_n| = n$ for all n. By Claim 7.3, the closures of the sets $X = \bigcup_n [s_n]^2$ and $Y = \bigcup_n \Delta^2 s_n$ have a nonempty intersection, and therefore the closures of their f-images intersect as well. On the other hand, the set $T = f''X \cup f''Y$ is relatively discrete in Z. This is because $T \cap \bigcap_n U_n = \emptyset$, while $T \setminus \overline{U}_n$ is finite for all n. But the closures of $f''X$ and $f''Y$ have a nonempty intersection, therefore Z is not a $\beta\mathbb{N}$-space.

CASE 3. The closure of the set $X = f''\Delta^2\mathbb{N}$ in Z is finite. Assume Z is a $\beta\mathbb{N}$-space. By going to an infinite subset of $\mathbb{N}$, we can assume that for some fixed $\bar{z} \in Z$ we have $f(m,m) = \bar{z}$ for all m. Note that by Claim 7.3, closures of the sets $[A]^2$ and $\Delta^2 A$ have a nonempty intersection for every infinite $A \subseteq \mathbb{N}$. Therefore we can derive a contradiction by finding an infinite A such that the closure of $f''[A]^2$ does not contain $\bar{z}$.

By applying Ramsey's theorem, we find an infinite $A \subseteq \mathbb{N}$ such that either (i) $f(l,m) = f(l,n)$ for all $l < m < n$ in A or (ii) $f(l,m) \neq f(l,n)$ for all $l < m < n$ in A. Applying it again, we shrink A so that either (iii) $f(l,n) = f(m,n)$ for all $l < m < n$ in A or (iv) $f(l,n) \neq f(m,n)$ for all $l < m < n$ in A.

If (i) and (iii) apply, f is constant on $[A]^2$, and the closure of $f''[A]^2$ is as required.

If (i) and (iv) apply, let $z_m = f(m,n)$ (by (i) z_m indeed does not depend on n). Since Z is a $\beta\mathbb{N}$-space, $\bar{z}$ has an open neighborhood U such that $C = \{m : z_m \notin U\}$ is infinite. But this implies that the closure of $f''[C]^2$ avoids $\bar{z}$, as required.

If (ii) and (iii) apply, define $z_n = f(m,n)$ for $m < n$ in A. Like above, we find an infinite subsequence of $\{z_n\}$ which does not accumulate to $\bar{z}$.

Now assume (ii) and (iv) apply. Define $f_1 \colon \mathbb{N}^2 \to Z$ by

$$f_1(m,n) = f(m,n+1).$$

Then f_1 continuously extends to $f_2 \colon (\beta\mathbb{N})^2 \to Z$. Since $f_2''\Delta^2\mathbb{N}$ is infinite, either Case 1 or Case 2 applies to prove that Z is not a $\beta\mathbb{N}$-space.

This completes the proof of Lemma 7.2. □

The rest of this section lists some applications of Lemma 7.2 that will not be used in the proof of Theorem 3.

Recall that $\beta\mathbb{N}$ is characterized by the fact that every map from $\mathbb{N}$ into a compact space continuously extends to $\beta\mathbb{N}$. We can now state and prove a characterization of when a map from $\mathbb{N}^d$ into a $\beta\mathbb{N}$-space continuously extends to $(\beta\mathbb{N})^d$.

The proof of this result, as well as that of Theorem 3, rely on Theorem 7.4 below, proved in the case $d = 2$ in [2, Theorem 4.2.1] and for an arbitrary natural number d in [3, Theorem 3].

Theorem 7.4. *Assume $f \colon X^d \to X$. Then exactly one of the following applies:*

(A) *X^d can be can be covered by finitely many rectangles such that f depends on at most one coordinate on each one of them.*
(B) *There is a disjoint partition $d = s\dot\cup t$ and sequences $\vec{x}_i \in X^s$, $\vec{y}_i \in X^t$ such that for all i and all $j < k$ we have $f(\vec{x}_i{}^\frown \vec{y}_i) \neq f(\vec{x}_j{}^\frown \vec{y}_k)$.* □

Theorem 7.5. *Let Z be a $\beta\mathbb{N}$-space, and let $f : \mathbb{N}^d \to Z$. Then the following are equivalent:*

(1) *f continuously extends to $(\beta\mathbb{N})^d$.*
(2) *$\mathbb{N}^d$ can be can be covered by finitely many rectangles such that f depends on at most one coordinate on each one of them.*
(3) *There is no disjoint partition $d = s\dot\cup t$ for which there are sequences $x_i \in \mathbb{N}^s$, $y_i \in \mathbb{N}^t$, such that*

$$f(x_i{}^\frown y_i) \neq f(x_i{}^\frown y_j) \text{ and } f(x_i{}^\frown y_j) \neq f(x_j{}^\frown y_j)$$

for all $i < j$.

Moreover, these are equivalent even when d is an arbitrary infinite cardinal.

Proof. Assume (3), and let $g : \mathbb{N}^2 \to Z$ be

$$g(i, j) = f(\vec{x}_i{}^\frown \vec{y}_j).$$

Then $g''\Delta^2\mathbb{N}$ and $g''[\mathbb{N}]^2$ are disjoint, thus Lemma 7.2 implies that (1) fails, and (1) implies (3).

Since (3) implies (2) by Theorem 7.4, it remains to prove that (2) implies (1). Assume (2), that $\mathbb{N}^d$ can be covered by finitely many rectangles such that f depends on at most one coordinate on each one of them. These rectangles correspond to disjoint clopen subsets of $(\beta\mathbb{N})^d$. Note that, since every function from $\mathbb{N}$ into Z continuously extends to $\beta\mathbb{N}$, every function $f : \mathbb{N}^d \to Z$ which depends on at most one coordinate continuously extends to $\beta\mathbb{N}$ as well. Since the sets are clopen, the union of these functions is continuous and therefore (1) follows.

The 'moreover' part follows by an application of Theorem 6.1. □

The same proof gives an analogous characterization of when a map $f : D^d \to Z$ can be extended to $(\beta D)^d$, where D is an arbitrary discrete space and Z is a $\beta\mathbb{N}$-space.

As an application of Theorem 7.5, we can now generalize Lemma 7.2 to higher powers, thus giving a full generalization of van Douwen's Lemma 7.1.

Corollary 7.6. *If $f : (\beta\mathbb{N})^d \to Z$ is a continuous map such that the sets $f''[\mathbb{N}]^d$ and $f''\Delta^d\mathbb{N}$ are disjoint, then Z is not a $\beta\mathbb{N}$-space.*

Proof. If Z is a $\beta\mathbb{N}$-space, then by Theorem 7.5 we can partition $\mathbb{N}^d$ into finitely many rectangles such that f depends on at most one coordinate on each one of them. One of these rectangles, say R, has an infinite intersection with the diagonal; let $\{n_i : i \in \mathbb{N}\}$ be the increasing enumeration of the set of n such that the d-tuple $(n, n, \dots, n)$ is in R. Let $j < d$ be such that $f \restriction R$ depends only on j-th coordinate. Then

$$f(n_0, n_1, \dots, n_{d-1}) = f(n_j, n_j, \dots, n_j),$$

therefore $f''[\mathbb{N}]^d$ and $f''\Delta^d\mathbb{N}$ are not disjoint. □

8. Clopen decompositions

In Theorem 8.2 below we show that in Theorem 3 it suffices to obtain a decomposition of X^d into closed (instead of clopen) rectangles.

Lemma 8.1. *Assume $f\colon X^d \to Z$ and that P, R are rectangles in X^d such that $f \restriction P$ and $f \restriction R$ both depend only on x_n. Then $f \restriction (P \cup R)$ also depends only on x_n.*

The condition that $P \cap R \neq \emptyset$ is easily seen to be necessary—otherwise both $f \restriction P$ and $f \restriction P$ can be constant, with $f \restriction (P \cup R)$ not constant. The condition that P and R are rectangles is also easily seen to be necessary.

Proof. For convenience of notation, we may assume $n = 0$. Let $P = \prod_{i=0}^{d-1} A_i$ and $R = \prod_{i=0}^{d-1} B_i$. By the assumption, there are $g_1\colon A_0 \to Z$ and $g_2\colon B_0 \to Z$ such that $f \restriction P = g_1 \circ \pi_0$ and $f \restriction R = g_2 \circ \pi_0$. We claim that $g_1 \cup g_2$ is a function. Pick $a \in \operatorname{dom}(g_1) \cap \operatorname{dom}(g_2) = A_0 \cap B_0$. Since $P \cap R \neq \emptyset$, we can pick $\vec{x} \in \prod_{i=1}^{d-1} A_i$. Then $a^\frown\vec{x} \in P \cap R$, and $g_1(a) = f(a^\frown\vec{x}) = g_2(a)$, thus $g = g_1 \cup g_2$ is indeed a function. Therefore $f \restriction (P \cup R) = g \circ \pi_0$, and this concludes the proof. □

Theorem 8.2. *Let X and Z be arbitrary topological spaces. Assume $f\colon X^d \to Z$ is such that X^d can be covered by finitely many* **closed** *rectangles such that f is elementary on each one of them. Then X^d can be covered by finitely many* **clopen** *rectangles such that f is elementary on each one of them.*

Proof. By possibly refining the given covering of X^d by rectangles, we may assume that for every $i < d$ there are $l(i) \in \mathbb{N}$ and a covering

$$X = A^i_0 \cup A^i_1 \cup \cdots \cup A^i_{l(i)-1}$$

by closed sets such that for every $h \in \prod_{i=0}^{d-1} l(i)$ the restriction of f to $\prod_{i=0}^{d-1} A^i_{h(i)}$ is elementary. For an $h \in \prod_{i=1}^{d-1} l(i)$ let

$$C_h = \prod_{i=1}^{d-1} A^i_{h(i)}.$$

Claim 8.3. *If $A^0_i \cap A^0_j \neq \emptyset$, then the restriction of f to $(A^0_i \cup A^0_j) \times C_h$ depends on at most one coordinate for every h.*

Proof. Fix an h. If neither $f \restriction (A^0_i \times C_h)$ nor $f \restriction (A^0_j \times C_h)$ depends on x_n for any $n > 0$, then they both depend only on x_0 and the conclusion follows by Lemma 8.1.

Now assume $f \restriction (A^0_i \times C_h)$ "really depends" on x_n for some $n > 0$, meaning that there are $\langle x_0, x_1, \dots, x_{d-1}\rangle$ and $\langle y_0, y_1, \dots, y_{d-1}\rangle$ in $A^0_i \times C_h$ such that $x_l = y_l$ for all $l \neq n$ but $f(x_0, x_1, \dots, x_{d-1}) \neq f(y_0, y_1, \dots, y_{d-1})$. Pick $\bar{x}_0 \in A^0_i \cap A^0_j$. Then $f \restriction (A^0_i \times C_h)$ does not depend on x_0, thus

$$\begin{aligned} f(\bar{x}_0, x_1, \dots, x_{d-1}) &= f(x_0, x_1, \dots, x_{d-1}) \\ &\neq f(y_0, y_1, \dots, y_{d-1}) = f(\bar{x}_0, y_1, \dots, y_{d-1}). \end{aligned}$$

Since $\langle \bar{x}_0, x_1, \dots, x_{d-1}\rangle$ and $\langle \bar{x}_0, y_1, \dots, y_{d-1}\rangle$ both belong to $A^0_j \times C_h$, this implies that $f \restriction (A^0_j \times C_h)$ depends on x_n, thus by our assumption it depends only on x_n. By Lemma 8.1, the Claim follows.

The case when $f \restriction (A^0_j \times C_h)$ "really depends" on some x_n for $n > 0$ is treated analogously. This concludes the proof. □

By using Claim 8.3 repeatedly we may go on joining A_i^0's until we end up with pairwise disjoint cover $F_0^0, \dots, F_{m(0)}^0$ of X, such that the restriction of f to every $F_i^0 \times C_h$ depends on at most one coordinate. Since each A_i^0 is closed, each F_j^0 is clopen.

Repeat the above construction and replace $A_0^1, \dots, A_{d-1}^1$ with a clopen partition $F_0^1, \dots, F_{m(1)}^1$ of X such that f depends on at most one coordinate on each $F_{h(0)}^0 \times F_n^1 \times \prod_{i=2}^{d-1} A_{h(i)}^i$ for $n < m(1)$. By repeating this construction for $2, 3, \dots, d-1$, we finally obtain integers $m(j)$ $(j < d)$ and a sequence of clopen partitions F_i^j $(j < d$, $i < m(j))$ of X such that f depends on at most one coordinate on each clopen rectangle $\prod_{j=0}^{d-1} F_{h(j)}^j$. This concludes the proof. $\square$

9. The proof of Theorem 3

By Theorem 6.1, we may assume that $\kappa = d$ is finite.

Assume that the possibility (A) of Theorem 7.4 applies, i.e., that X can be covered by finitely many rectangles, $R_0, \dots, R_{l-1}$ such that the restriction of f to each one of them depends on at most one coordinate. The closures of these rectangles are still rectangles, and since f is continuous, it depends on at most one coordinate on each one of these closures. By Theorem 8.2, we can assume that the rectangles are clopen.

Therefore it suffices to prove that the alternative (B) of Theorem 7.4 leads to a contradiction. Let $d = s \cup t$, $\vec{x}_i \in X^s$ and $\vec{y}_i \in X^t$ $(i \in \mathbb{N})$ be as given by (B) of Theorem 7.4. Define a map $h_0 \colon [\mathbb{N}]^2 \to X^d$ by

$$h_0(\{m, n\}) = \vec{x}_m {}^\frown \vec{y}_n.$$

Since $\vec{x}_m \in X^s$, $\vec{y}_m \in X^t$ for all m, and X^d is compact, h_0 can be continuously extended to a map $H \colon (\beta\mathbb{N})^2 \to X^s \times X^t = X^d$. Then $g = f \circ H$ is a continuous map from $(\beta\mathbb{N})^d$ into X, and it has the property that $g(m,n) \neq g(l,l)$ for all $m < n$ and l. Therefore $g''[\mathbb{N}]^2$ and $g''\Delta^2\mathbb{N}$ are disjoint and Z is not a $\beta\mathbb{N}$-space by Lemma 7.1.

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