POWERS OF $\mathbb{N}^*$

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ABSTRACT. We prove that the Čech-Stone remainder of the integers, $\mathbb{N}^*$, maps onto its square if and only if there is a nontrivial map between two of its different powers, finite or infinite. We also prove that every compact space that maps onto its own square maps onto its own countable infinite product.

The structure of the Čech-Stone remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ of the integers is very sensitive to the choice of set-theoretic axioms (see [4]). For instance, the Continuum Hypothesis implies that $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^\kappa$ for all $\kappa \leq \mathfrak{c}$ ([5]), but under different axioms $\mathbb{N}^*$ does not even map onto $(\mathbb{N}^*)^2$, and moreover $(\mathbb{N}^*)^n$ does not map onto $(\mathbb{N}^*)^{n+1}$ for any $n \in \mathbb{N}$ ([2]). Obviously, if $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^2$, then $(\mathbb{N}^*)^n$ maps onto $(\mathbb{N}^*)^{n+1}$ for all $n \in \mathbb{N}$. This suggests the following question:

Question 1 (Just, [2, p. 60]). *Is it relatively consistent with the usual axioms of Set Theory that there are $m > n$ in $\mathbb{N}$ such that $(\mathbb{N}^*)^m$ maps onto $(\mathbb{N}^*)^{m+1}$, but $(\mathbb{N}^*)^n$ does not map onto $(\mathbb{N}^*)^{n+1}$?*

The following theorem implies that the answer is (rather surprisingly) negative.

Theorem 2. *The following are equivalent ($X \twoheadrightarrow Y$ stands for "X maps onto Y"):*

1. $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^2$,
2. $(\mathbb{N}^*)^n \twoheadrightarrow (\mathbb{N}^*)^{n+1}$ *for some* $n \in \mathbb{N}$,
3. $(\mathbb{N}^*)^\kappa \twoheadrightarrow (\mathbb{N}^*)^\lambda$ *for some pair of cardinals* $\kappa < \lambda$, *finite or infinite.*
4. $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^{\mathbb{N}}$.

The key behind the proof of Theorem 2 is in [1, Theorem 3] (see Theorem 5 below), where it was proved that every map from a power of $\mathbb{N}^*$ into $\mathbb{N}^*$ depends on at most one coordinate.

The clauses (1) and (4) are equivalent for all compact spaces.

Theorem 3. *A compact space X maps onto X^2 if and only if it maps onto $X^{\mathbb{N}}$.*

We also prove another surprising fact about powers of $\mathbb{N}^*$.

Theorem 4. *There is a cardinal μ such that for every pair of cardinals κ, λ the following are equivalent:*

1. $(\mathbb{N}^*)^\kappa \twoheadrightarrow (\mathbb{N}^*)^\lambda$,
2. $\kappa \geq \lambda$ *or* $\kappa < \lambda < \mu$.

We start by recounting terminology and some results of [1]. If $s \subseteq \kappa$ then $\pi^\kappa_s \colon X^\kappa \to X^s$ is the *projection* of X^κ to X^s, defined by

$$\pi^\kappa_s(\langle x_\xi : \xi < \kappa\rangle) = \langle x_\xi : \xi \in s\rangle.$$

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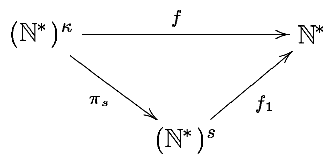
If κ is clear from the context, we will write π_s instead of π_s^κ. If $s = \{\xi\}$, then we write π_ξ instead of $\pi_{\{\xi\}}$. A map $f\colon X^\kappa \to Z$ *depends on at most one (α-th) coordinate* if there is $g\colon X \to Z$ such that $f(\langle x_\xi\rangle_{\xi<\kappa}) = g(x_\alpha)$, i.e., $f = g \circ \pi_\alpha$.

In all results below, κ is an arbitrary cardinal, and all maps are assumed to be continuous.

Theorem 5. *Assume $f\colon (\mathbb{N}^*)^\kappa \to \mathbb{N}^*$. Then $(\mathbb{N}^*)^\kappa$ can be covered by finitely many clopen sets such that f depends on at most one coordinate on each one of them. In particular, there is a finite $s \subseteq \kappa$ and a continuous map $f_1\colon (\mathbb{N}^*)^s \to \mathbb{N}^*$ such that the diagram*



commutes.

Proof. This is a special case of [1, Theorem 3] when $X = Z = \mathbb{N}^*$. □

Corollary 6. *The following are equivalent for every cardinal λ:*

(1) $(\mathbb{N}^*)^\kappa \twoheadrightarrow (\mathbb{N}^*)^\lambda$ *for some $\kappa < \lambda$,*
(2) $(\mathbb{N}^*)^n \twoheadrightarrow (\mathbb{N}^*)^\lambda$ *for some finite $n < \lambda$.*

Proof. We will prove only the nontrivial direction; note that we may assume λ is infinite. Assume $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$ is a surjection. By Theorem 5, for each $\xi < \lambda$ there is a finite $s_\xi \subseteq \kappa$ such that $\pi_\xi \circ f$ depends only on coordinates in s_ξ. By a counting argument, there is $A \subseteq \lambda$ of size λ and a finite $s \subseteq \kappa$ such that $s_\xi = s$ for all $\xi \in A$. Hence there is a map $g\colon (\mathbb{N}^*)^s \to (\mathbb{N}^*)^A$ such that $g \circ \pi_s^\kappa = \pi_A \circ f$. Since the map on the right-hand side is surjective, so is g. This concludes the proof. □

Proof of Theorem 2. Implicatons (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. The equivalence of (4) and (1) follows from Theorem 3 proved below. Thus it only remains to prove (3) implies (1).

Assume $\kappa < \lambda$ and $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$ is a surjection. By Corollary 6, we may assume κ is finite, say $\kappa = n$. We may also assume $\lambda = n+1$. By applying Theorem 5 to $\pi_i \circ f$ for all $i < n+1$ and refining the obtained clopen coverings, we can cover $(\mathbb{N}^*)^n$ by finitely many clopen sets so that $\pi_i \circ f$ depends on at most one coordinate on each one of them for each $i < n+1$. The image of at least one of these clopen sets, call it U, includes a nonempty clopen subset, call it V, of $(\mathbb{N}^*)^{n+1}$. There are $i < j < n+1$ and $k < n$ such that both $\pi_i \circ f$ and $\pi_j \circ f$ depend at most on the k-th coordinate. Since V is clopen, its image under the projection $\vec{y} \mapsto \langle \pi_i(\vec{y}), \pi_j(\vec{y})\rangle$ is a clopen subset, call it V', of $(\mathbb{N}^*)^2$. Let $h_l\colon U \to \mathbb{N}^*$ be such that $\pi_l \circ f = h_l \circ \pi_k$ for $l \in \{i, j\}$ on U. Pick $\langle \mathcal{U}, \mathcal{V}\rangle \in V'$, and define $h\colon \mathbb{N}^* \to V$ by

$$h(\vec{x}) = \begin{cases} \langle h_i(\vec{x}), h_j(\vec{x})\rangle, & \text{if } \langle h_i(\vec{x}), h_j(\vec{x})\rangle \in V', \\ \langle \mathcal{U}, \mathcal{V}\rangle, & \text{otherwise.} \end{cases}$$

This is a continuous surjection from $\mathbb{N}^*$ onto V'; but V' is homeomorphic to $(\mathbb{N}^*)^2$, and this concludes the proof. □

Proof of Theorem 3. We prove only the nontrivial direction. Let $f_i\colon X \to X$ ($i \in \{1,2\}$) be continuous maps such that $x \mapsto (f_1(x), f_2(x))$ maps X onto X^2. Define $g_n\colon X \to X$ ($n \in \mathbb{N}$) as follows:

$$\begin{aligned} g_1 &= f_1 \\ g_2 &= f_1 \circ f_2 \\ g_3 &= f_1 \circ f_2 \circ f_2 \\ g_n &= f_1 \circ \underbrace{f_2 \circ \cdots \circ f_2}_{n-1 \text{ times}}, \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

We claim that $x \mapsto (g_n(x))_{n=1}^{\infty}$ maps X onto $X^{\mathbb{N}}$. Since X is compact, it suffices to show that the image of X under this map is dense in $X^{\mathbb{N}}$, and it turn it suffices to show that for every $n \in \mathbb{N}$ and $\vec{y} = (y_i)_{i=1}^n$ in X^n there is an $x \in X$ such that $\vec{z} = f(x)$ satisfies $z_i = y_i$ for all $i \leq n$. Equivalently, we need to prove that for every n-tuple $(y_1, \dots, y_n)$ there is an $x \in X$ such that $g_i(x) = y_i$ for $i \leq n$. We prove this by induction on n.

For $n = 2$, find $x' \in X$ such that $f_1(x') = y_2$ (possible, since f_1 is onto), and then, by using the fact that $x \mapsto (f_1(x), f_2(x))$ maps X onto X^2, find $x \in X$ such that $f_1(x) = y_1$ and $f_1(x) = x'$. Then we have $g_2(x) = f_1(f_2(x)) = f_1(x') = y_2$.

Assume the claim is true for n and prove it for $n+1$. By the induction hypothesis, find $x' \in X$ such that $g_i(x') = y_{i+1}$, for $1 \leq i \leq n$. Now pick x such that $f_1(x) = y_1$ and $f_2(x) = x'$. Then for $2 \leq i \leq n+1$ we have $g_i(x) = g_{i-1}(f_2(x)) = g_{i-1}(x') = y_i$, and this completes the proof. □

By simple cardinal arithmetic, Theorem 3 implies that a compactum of size strictly less than the continuum does not map onto its own square. The compactness assumption cannot be omitted from Theorem since $X = \mathbb{N}$ maps onto $\mathbb{N}^2$ but not onto $\mathbb{N}^{\mathbb{N}}$. J. van Mill points out that the version of this result in which X^2 and $X^{\mathbb{N}}$ are required to be homeomorphic to X is false, by [3].

S. Solecki [7] has pointed out that the above construction of the maps g_n appears in [6]. In this paper Sierpiński proves that if $x \mapsto (f_1(x), f_2(x))$ maps $[0,1]$ onto $[0,1]^2$, then $x \mapsto \langle g_n(x) : n \in \mathbb{N}\rangle$ maps $[0,1]$ onto $[0,1]^{\mathbb{N}}$. Although [6] does not use any assumptions beyond the compactness of $[0,1]$, Theorem 3 was not stated in this paper.

Definition 7. Let μ be the smallest cardinal such that $\mathbb{N}^*$ does not map onto $(\mathbb{N}^*)^{\mu}$.

Note that μ is at least two, and since the weight of $\mathbb{N}^*$ is equal to $\mathfrak{c}$, μ is at most $\mathfrak{c}^+$. (Recall that *weight* of a topological space is the minimal size of its base, and that the weight of a continuous image of a compact space does not exceed the weight of the original space.) As pointed out above, μ can assume either one of these two values: by [5], Continuum Hypothesis implies that $\mu = \mathfrak{c}^+$ while in [2] Just has obtained a forcing extension in which $\mu = 2$. By Theorem 2, $\mu > 2$ implies $\mu \geq \aleph_1$.

Proof of Theorem 4. Let μ be as in Definition 7. Then $\kappa < \lambda < \mu$ implies $(\mathbb{N}^*)^{\kappa} \twoheadrightarrow \mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^{\lambda}$, hence (2) implies (1). To prove (1) implies (2), let us assume $f\colon (\mathbb{N}^*)^{\kappa} \to (\mathbb{N}^*)^{\lambda}$ is a surjection and $\kappa < \lambda$. By Corollary 6, we may assume κ is finite. By Theorem 2, $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^{\kappa}$, hence by composing we have $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^{\lambda}$. □

Question 8. *Which of the following are relatively consistent with the usual axioms of Set Theory:*

(a) *μ is neither 2 nor $\mathfrak{c}^+$?*
(b) *$\mu = \aleph_1$?*
(c) *μ is a limit cardinal?*

The following variation on Theorem 3 shows that μ has to be regular.

Theorem 9. *Let X be an arbitrary topological space, and assume κ is a singular cardinal such that X maps onto X^λ for all $\lambda < \kappa$. Then X maps onto X^κ.*

Proof. Let κ_α ($\alpha < \mathrm{cf}(\kappa)$) be a strictly increasing sequence of cardinals with supremum equal to κ. For $\eta < \kappa$ let $f_\eta \colon X \to X$ be continuous maps such that $x \mapsto (f_\eta(x))_{\eta \in [\kappa_\alpha, \kappa_{\alpha+1})}$ maps X onto $X^{[\kappa_\alpha, \kappa_{\alpha+1})}$. Define $g_\eta \colon X \to X$ ($\mathrm{cf}(\kappa) \leq \eta < \kappa$) by:

$$g_\eta = f_\eta \circ f_\alpha \text{ if } \eta \in [\kappa_\alpha, \kappa_{\alpha+1}).$$

We claim that $x \mapsto (g_\eta(x))_{\eta=\kappa_0}^{\kappa}$ maps X onto $X^{[\kappa_0, \kappa)}$. Since the latter power is homeomorphic to X^κ, this will conclude the proof. Fix $\vec{y} = (y_\eta)_{\eta=\kappa_0}^{\kappa}$. Let x_α ($\alpha < \mathrm{cf}(\kappa)$) be such that $f_\eta(x_\alpha) = y_\eta$ for $\eta \in [\kappa_\alpha, \kappa_{\alpha+1})$. Now let $x \in X$ be such that $f_\alpha(x) = x_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$. Then if $\eta \in [\kappa_\alpha, \kappa_{\alpha+1})$ we have $g_\eta(x) = f_\eta(f_\alpha(x)) = f_\eta(x_\alpha) = y_\eta$, as required. □

Question 10. *Is it true that if X is a compact space and κ is the supremum of all λ such that X maps onto X^λ, then X maps onto X^κ?*

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