QUANTIFIER ELIMINATION IN C*-ALGEBRAS

CHRISTOPHER J. EAGLE, ILIJAS FARAH, EBERHARD KIRCHBERG, AND ALESSANDRO VIGNATI

Abstract. The only C*-algebras that admit elimination of quantifiers in continuous logic are $\mathbb{C}, \mathbb{C}^2$, C(Cantor space) and $M_2(\mathbb{C})$. We also prove that the theory of C*-algebras does not have model companion and show that the theory of $M_n(\mathcal{O}_{n+1})$ is not $\forall\exists$-axiomatizable for any $n \geq 2$.

Introduction

One of the key steps in using model theory in applications is to understand the *definable* objects in models of a particular theory. It is often the case that the objects which can be defined without the use of quantifiers have particularly natural descriptions, while definitions involving quantifiers are more difficult to analyze. Quantifier elimination, which is the property that every definable object can be defined without using quantifiers, is therefore a highly desirable feature for a theory to possess.

Quantifier elimination is a matter of the formal language used to study the structures of interest. It is easy to see that any theory can be extended to a theory with quantifier elimination in an expanded language by simply adding a new symbol for every object definable in the original one (for example, adding the trace to the language would give quantifier elimination for any finite-dimensional C*-algebra); the usefulness of quantifier elimination results therefore depends on using a natural language for the structures at hand. For this reason we consider C*-algebras as structures in the language for C*-algebras introduced in [FHS14]. This standard language for C*-algebras contains symbols for the natural operations in a C*-algebra; when we consider unital algebras we often add a symbol for the multiplicative identity to form the language of unital C*-algebras. These languages are sufficiently expressive that many natural classes of C*-algebras are either axiomatizable, or at least defined by the omission of certain types (many examples of this kind are given in [FHL$^+$]). Nevertheless, these languages are also sufficiently limited that quantifier-free formulas are quite simple, being continuous combinations of norms of *-polynomials with complex coefficients. We identify 2 with $\{0, 1\}$ and the Cantor space with the product space $2^{\mathbb{N}}$.

Theorem 1. *The theories of unital C*-algebras that admit quantifier elimination (in the language of unital C*-algebras) are exactly the complete theories of* $\mathbb{C}$, $\mathbb{C}^2$,

Date: June 10, 2015.
2010 *Mathematics Subject Classification.* 46L05, 03C10, 46M07 .
Key words and phrases. C*-algebras, quantifier elimination, model completeness, logic of metric structures.
IF was partially supported by NSERC.

$M_2(\mathbb{C})$ and $C(2^{\mathbb{N}})$. No noncommutative C-algebra admits quantifier elimination in the standard language of C*-algebras.*

Proof. The unital case is Theorem 2.8. The claim about not necessarily unital algebras is established in Corollary 2.11. □

We also prove that the theory of C*-algebras does not have model companion (Theorem 3.1) and give natural examples of C*-algebras whose theories are not ∀∃-axiomatizable (Corollary 3.3).

Section 1 contains preliminaries and a test for quantifier elimination. In this section we also completely answer the question of which finite-dimensional C*-algebras have quantifier elimination. In Section 2 we prove our main results, implying in particular that $M_2(\mathbb{C})$ is the only noncommutative C*-algebra whose theory admits elimination of quantifiers. In 2.2 we deal with the non-unital case. In Section 3 we show that the theory of unital C*-algebras does not have a model companion, and also obtain results related to the ∀∃-axiomatizablity of some classes of C*-algebras.

Every C*-algebra we consider is unital, unless stated specifically. The word embedding has a model theoretical sense: an embedding is a unital injective *-homomorphism. By $A^{\mathcal{U}}$ we denote an ultrapower of A associated with an ultrafilter $\mathcal{U}$. All ultrafilters are assumed to be nonprincipal ultrafilters on $\mathbb{N}$.

Acknowledgments. We are indebted to Isaac Goldbring for suggesting Theorem 3.1, and an exchange that lead to Theorem 3.2. We would also like to thank Bradd Hart for helpful remarks.

Research presented in the Appendix was supported by the Fields undergraduate summer research program in July and August 2014. This research gave the initial impetus to study that resulted in the present paper.

1. Quantifier elimination

In this section we recall the model-theoretic framework for studying C*-algebras, as well as tests for quantifier elimination. The reader interested in a more complete discussion of the model theory of C*-algebras can consult [FHS14], or the forthcoming [FHL$^+$]. For more on quantifier elimination in metric structures in general, see [BYBHU08, Section 13].

Definition 1.1. The *formulas* for C*-algebras are recursively defined as follows. In each case, $\bar{x}$ denotes a finite tuple of variables (which will later be interpreted as elements of a C*-algebra).

(1) If $P(\bar{x})$ is a *-polynomial with complex coefficients, then $\|P(\bar{x})\|$ is a formula.
(2) If $\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x})$ are formulas and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x}))$ is a formula.
(3) If $\varphi(\bar{x}, y)$ is a formula and $R \in \mathbb{R}^+$, then $\sup_{\|y\|\leq R} \varphi(\bar{x}, y)$ and $\inf_{\|y\|\leq R} \varphi(\bar{x}, y)$ are formulas.

We think of $\sup_{\|y\|\leq R}$ and $\inf_{\|y\|\leq R}$ as replacements for the first-order quantifiers ∀ and ∃, respectively. A formula constructed using only clauses (1) and (2) of the definition is therefore said to be *quantifier-free*.

The definition above is slightly different from the one in [FHS14]. In particular, we have replaced their *domains of quantification* by requiring that our suprema and infima range over closed balls of finite radius. Since the two versions are equivalent, we have chosen an approach intended to minimize the technical complexity of the definitions.

If $\varphi(\bar{x})$ is a formula, A is a C*-algebra, and $\bar{a}$ is a tuple of elements of A of the same length as the tuple $\bar{x}$, there is a natural way to evaluate φ in A with $\bar{x}$ replaced by $\bar{a}$; the result is a real number denoted $\varphi^A(\bar{a})$.

Definition 1.2. Let A be a C*-algebra, and $\bar{a} \in A^n$ be a tuple of elements from A. The *type* of $\bar{a}$ in A, denoted $\mathrm{tp}^A(\bar{a})$, is defined to be the set of all formulas $\varphi(\bar{x})$ such that $\varphi^A(\bar{a}) = 0$. Similarly, the *quantifier-free type* of $\bar{a}$, denoted $\mathrm{qftp}^A(\bar{a})$, is the set of all quantifier-free formulas $\varphi(\bar{x})$ such that $\varphi^A(\bar{a}) = 0$. If the algebra A is clear from the context we omit it from the notation.

A formula without free variables is a *sentence*. A *theory* T is a set of sentences, and a C*-algebra A is a *model* of T (written $A \models T$) if every sentence in T takes the value 0 when interpreted in A. The *theory of* A, $\mathrm{Th}(A)$, is the set of all sentences which take value 0 when interpreted in A.

Definition 1.3. A theory T has *quantifier elimination* if for every formula $\varphi(\bar{x})$ and every $\epsilon > 0$ there is a quantifier-free formula $\psi_\epsilon(\bar{x})$ such that whenever $A \models T$ and $\bar{a}$ is a tuple of the appropriate length and sorts from A, we have

$$\left|\varphi^A(\bar{a}) - \psi_\epsilon^A(\bar{a})\right| \leq \epsilon.$$

By a standard abuse of language, we say that a C*-algebra A has quantifier elimination if $\mathrm{Th}(A)$ does.

We record some general consequences of quantifier elimination for a C*-algebra. We will apply these results in the subsequent sections to show no noncommutative C*-algebra other than $M_2(\mathbb{C})$ admits elimination of quantifiers. The first of these results, Lemma 1.4, is straightforward but very useful as it gives an analytic description of a quantifier-free type of a tuple of commuting normal elements. The *joint spectrum* of commuting normal elements $a_1, \dots, a_n$, $j\sigma(\bar{a})$, is the set of all $\bar{\lambda} \in \mathbb{C}^n$ such that $\{\lambda_1 - a_1, \lambda_2 - a_2, \dots, \lambda_n - a_n\}$ generates a proper ideal. Equivalently, if these elements belong to a commutative algebra $C(X)$ then

$$j\sigma(a_1, \dots, a_n) = \{(a_1(x), \dots, a_n(x)) \colon x \in X\}.$$

Lemma 1.4. *In any C*-algebra, for two finite tuples of commuting normal elements $\bar{a}$ and $\bar{b}$ the following conditions are equivalent:*

(1) $\bar{a}$ and $\bar{b}$ have the same quantifier-free type
(2) $j\sigma(\bar{a}) = j\sigma(\bar{b})$
(3) the C-algebras generated by $\bar{a}$ and $\bar{b}$ are isomorphic via an isomorphism that sends $\bar{a}$ to $\bar{b}$.*

Consequently, if a C-algebra A has quantifier elimination, then two finite tuples of commuting normal elements in A have the same type if and only if they have the same joint spectrum.*

Proof. Let $\bar{a}$ and $\bar{b}$ be as in the hypothesis. By [EV14, Proposition 5.23] the joint spectrum $j\sigma(\bar{a})$ is quantifier-free definable from $\bar{a}$, and hence if $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$ then $j\sigma(\bar{a}) = j\sigma(\bar{b})$. Moreover, if $j\sigma(\bar{a}) = j\sigma(\bar{b})$, then $C^*(\bar{a}, 1) \cong C^*(\bar{b}, 1)$ by an isomorphism sending $\bar{a}$ to $\bar{b}$, and so $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. Finally, it follows from the definition of formulas that $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$ if and only if $\|P(\bar{a})\| = \left\|P(\bar{b})\right\|$ for all *-polynomials P with complex coefficients. In particular, if the C*-algebras generated by $\bar{a}$ and $\bar{b}$ are isomorphic via an isomorphism sending $\bar{a}$ to $\bar{b}$, then $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. □

Remark 1.5. By the Weyl-von Neumann theorem (see e.g. [Dav96]) it is true that if a and b are self-adjoint elements of the Calkin algebra such that $\sigma(a) = \sigma(b)$, then $\mathrm{tp}(a) = \mathrm{tp}(b)$. This is not true, however, for normal elements. If s is the unilateral shift in $\mathcal{B}(H)$ then its image $\pi(s)$ under the quotient map is a unitary with full spectrum and (because of the Fredholm index obstruction) it satisfies $\|\pi(s) - u^2\| \geq 1$ for all unitaries u. As pointed out in the introduction to [PW07], this failure of quantifier elimination is one of the reasons why it was difficult to construct an outer automorphism of the Calkin algebra.

We shall need the following consequence of [BYBHU08, Prop. 13.6].

Proposition 1.6. *The following are equivalent for every unital C*-algebra A.*

1. *(1) A has quantifier elimination.*
2. *(2) Whenever B is elementarily equivalent to A and separable, and N is a finitely generated unital subalgebra of B, then every embedding $\Phi\colon N \to B^{\mathcal{U}}$ can be extended to an embedding $\varphi\colon B \to B^{\mathcal{U}}$.*

Proof. Assume (1) holds and fix B, N, Φ as in (2). Let $\bar{a}$ be a tuple that generates N. By Lemma 1.4, the quantifier-free type of $\bar{a}$ determines N up to the isomorphism. By (1), the quantifier-free type of $\bar{a}$ determines its type. Since B is separable, one can enumerate a dense subset of its unit ball as b_n, for $n \in \mathbb{N}$. By the countable saturation of ultrapowers (see e.g., [FHS14]) one can recursively choose c_n, for $n \in \mathbb{N}$, in $B^{\mathcal{U}}$ such that $\mathrm{tp}^B(\bar{a}, b_1, \dots, b_n) = \mathrm{tp}^{B^{\mathcal{U}}}(\Phi(\bar{a}), c_1, \dots, c_n)$ for all n. Then the map $b_n \mapsto c_n$ is an isometry and its continuous extension to B is φ as required.

The converse implication is an easy consequence of [BYBHU08, Prop. 13.6] and we shall not need it. □

We state a more appealing weakening of the assertion (2) in which $B = A$.

($\star$) whenever F is a finitely-generated unital C*-algebra, $\iota\colon F \to A$ and $\Phi\colon F \to A^{\mathcal{U}}$ are embeddings then there is an embedding $\varphi\colon A \to A^{\mathcal{U}}$ that makes the diagram commute.

1.1. **Finite-dimensional C*-algebras.** To conclude this section we treat the case of finite-dimensional C*-algebras. When A is finite-dimensional the closed balls of A of finite radius are compact, and it follows that A is (up to isomorphism) the only model of its theory (see [BYBHU08, p. 24]). We say that a projection p is *minimal* if $pAp \cong \mathbb{C}$.

Theorem 1.7. *For a finite-dimensional C*-algebra the following are equivalent.*

(1) A is isomorphic to a subalgebra of $M_2(\mathbb{C})$.
(2) Every commutative subalgebra of A is isomorphic to $\mathbb{C}$ or to $\mathbb{C}^2$.
(3) A is isomorphic to one of $\mathbb{C}, \mathbb{C}^2$, or $M_2(\mathbb{C})$.
(4) A has quantifier elimination.
(5) A satisfies ($\star$).

Proof. The equivalence of (1), (2) and (3) is an easy consequence of the fact that every finite-dimensional C*-algebra is isomorphic to a direct sum of full matrix algebras. Clauses (4) and (5) are vacuously equivalent for finite-dimensional algebras.

We prove that (4) implies (2). If (2) fails, then there are two projections p and q in A which are both minimal, are orthogonal, and are such that $q \neq 1 - p$. If A has quantifier elimination then every nontrivial projection has the same type as p, and in particular, is minimal. This contradicts the fact that $q + p$ is a nontrivial nonminimal projection.

We now prove that $M_2(\mathbb{C})$ has quantifier elimination, using Proposition 1.6. Every ultrapower of $M_2(\mathbb{C})$ is isomorphic to $M_2(\mathbb{C})$. If M and N are isomorphic unital subalgebras of $M_2(\mathbb{C})$, then by the equivalence of (2) and (3) and easy computation the isomorphism of M and N is implemented by a unitary in $M_2(\mathbb{C})$. Therefore the isomorphism extends to an automorphism of $M_2(\mathbb{C})$, and this completes the proof.

We omit the proofs that $\mathbb{C}$ and $\mathbb{C}^2$ have quantifier elimination, which are similar but easier (see also Lemma 1.4). □

The following will be needed in §2.

Proposition 1.8. *The only C*-algebras that have a minimal projection and satisfy ($\star$) are $\mathbb{C}$, $\mathbb{C}^2$ and $M_2(\mathbb{C})$.*

Proof. Assume A satisfies ($\star$), has a minimal projection p and it is not included in the list above. Since A is not one of $\mathbb{C}$, $\mathbb{C}^2$ or $M_2(\mathbb{C})$, by Theorem 1.7 (2) $1 - p$ is not minimal. Note that, p is minimal also when considered in the ultrapower $A^{\mathcal{U}}$. Let $\iota\colon \mathbb{C}^2 \to A$ be the unital embedding that sends $(1, 0)$ to $1 - p$ and $\Phi\colon \mathbb{C}^2 \to A^{\mathcal{U}}$ sending $(1, 0)$ to p. Suppose that $\varphi\colon A \to A^{\mathcal{U}}$ is the embedding given from ($\star$). Since $1 - p$ is not minimal, there is a positive element $a \in A$ with $a \leq 1 - p$ whose spectrum has more than 2 elements. Then $\varphi(a) \leq \varphi(1 - p) = p$ has more than 2 elements in the spectrum, a contradiction. □

Remark 1.9. If we add a predicate for the trace to the language of C*-algebras then the theory of every finite-dimensional C*-algebra (as a tracial C*-algebra) has quantifier elimination: every subalgebra of a finite-dimensional C*-algebra is a direct sum of matrix algebras, and its conjugacy class is determined by traces of its central projections

2. Noncommutative C*-algebras

For the remainder of the paper all C*-algebras are assumed to be infinite-dimensional, noncommutative, and unital, unless mentioned otherwise.

Proposition 2.1. *Let A be unital, noncommutative, and infinite-dimensional. If A satisfies ($\star$) then A is purely infinite and simple.*

The proof of Proposition 2.1 requires some definitions and lemmas.

Lemma 2.2. *Assume A is a C*-algebra (not necessarily unital) with no minimal projections. Then it contains a positive contraction of full spectrum.*

Proof. We may assume A is separable. Let (X, d) be a locally compact metric space such that $C_0(X)$ is isomorphic to a masa of A. By the continuous functional calculus we need to find $f \in C_0(X)$ whose range is a nontrivial interval. Since A has no minimal projections, X has no isolated points and is therefore uncountable.

Let us first consider the case when X has an uncountable connected component Y. Choose a point $y \in Y$ and $r > 0$ small enough to have $\sup_{z\in Y} d(z,y) \geq r$ and that $\{x \in X : d(x,y) < r\}$ is relatively compact. Define $g\colon [0,\infty) \to [0,1]$ by $g(t) = \frac{2t}{r}$ if $t \leq r/2$, $g(t) = -\frac{2t}{r} + 2$ if $r/2 < t \leq r$, and $g(t) = 0$ elsewhere. Then $f\colon X \to [0,1]$ defined by $f(x) = g(d(x,y))$ is in $C_0(X)$ and its range is equal to $[0,1]$.

If there is no such Y then every connected component of X consists of a single point and therefore X is zero-dimensional. Being locally compact and with no isolated points, X has a clopen subset homeomorphic to the Cantor set. Since the Cantor set maps continuously onto $[0,1]$, we can find f as required. □

Definition 2.3 ([Cun78])**.** For positive elements a and b in a C*-algebra A we write $a \precsim b$, and say that a is *Cuntz-subequivalent* to b, if there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\lim_n \|z_n b z_n^* - a\| = 0.$$

Lemma 2.4. *For every x in every C*-algebra one has $x^*x \precsim xx^*$. Moreover, for every n there exists z_n with $\|z_n\| \leq n$ such that $\|x^*x - z_n^* xx^* z_n\| \leq 1/n$.*

Proof. For $n \in \mathbb{N}$ let $f_n\colon [0,1] \to [0,n]$ be defined by $f_n(t) = t^{-1/2}$ if $t \geq 1/n^2$ and $f_n(t) = n$ if $t < 1/n$. Let $z_n = x f_n(x^*x)$. Clearly $\|z_n\| \leq n$. Also we have the following computation, which takes place in the commutative algebra $C^*(x^*x)$:

$$z_n^* xx^* z_n = f_n(x^*x)(x^*x)^2 f_n(x^*x) = g_n(x^*x),$$

where $g_n(t) = t$ if $t \geq 1/n^2$ and $g_n(t) = t(1-tn)$ if $t < 1/n^2$. Since $|t - g_n(t)| < 1/n$ we have that $\|x^*x - z_n^* xx^* z_n\| < 1/n$, as required. □

Let us temporarily write $a \sim b$ if

(1) a and b are positive
(2) for every n there is z_n such that $\|b - z_n^* a z_n\| \leq 1/n$ and $\|z_n\| \leq n$, and
(3) for every n there is y_n such that $\|a - y_n^* b y_n\| \leq 1/n$ and $\|y_n\| \leq n$.

This is an equivalence relation, and the type of the pair a, b is capable of detecting whether $a \sim b$. The following technical lemma is needed in the proof of Proposition 2.1.

Lemma 2.5. *If A is noncommutative, infinite-dimensional, and satisfies ($\star$), then there exist $a, b \in A$ such that a and b are orthogonal positive contractions with full spectrum; moreover, for every such pair, $a \sim b$.*

Proof. Since A is noncommutative, by [Bla06, II.6.4.14], there is x such that $\|x\| = 1$ and $x^2 = 0$. Then xx^* and x^*x are orthogonal positive elements of norm 1. Since the spectra of $a = x^*x$ and $b = xx^*$ both contain 0, they are equal (for all $x, y \in A$, $\sigma(xy)$ and $\sigma(yx)$ may only differ at $\{0\}$, see e.g., [Bla06, II.1.4.2]) .

Let us prove that we may assume $\sigma(a) = [0,1]$. If $\sigma(a) \neq [0,1]$ then by continuous functional calculus we can find a nonzero projection $p \in C^*(a)$. Since by Proposition 1.8 A has no minimal projections, the algebra pAp is infinite-dimensional and by Lemma 2.2 we can find positive $a_1 \in pAp$ such that $\sigma(a_1) = [0,1]$. Let $x_1 = pa_1$. Note that $x_1 = xa_1 = a_1x$, hence $(x_1)^2 = 0$. Also, if we let $a_2 = x_1^* x_1$ then $a_2 = a_1^2$ and hence $\sigma(a_2) = [0,1]$. Therefore by replacing x with x_1 and re-evaluating a and b we may assume $\sigma(a) = [0,1]$. In particular, A contains a unital copy of $C([0,1])$.

If c and d are positive orthogonal elements with $\sigma(c) = \sigma(d) = [0,1]$ we have that $j\sigma(a,b) = j\sigma(c,d)$, so $F = C^*(a,b,1) \cong C^*(c,d,1)$. Let ι be the inclusion map and $\Phi\colon F \to A^{\mathcal{U}}$ the unital embedding that sends a to c and b to d. Let x_n, y_n witnessing that $a \sim b$ and $\varphi\colon A \to A^{\mathcal{U}}$ be the embedding that makes the diagram commute provided by $(\star)$. Then $c \sim d$ (using $\varphi(x_n)$ and $\varphi(y_n)$), in $A^{\mathcal{U}}$. Since the diagonal embedding is elementary, we have $\mathrm{tp}^A(c,d) = \mathrm{tp}^{A^{\mathcal{U}}}(c,d)$ and in particular $(\inf_{\|y\|\leq n}(\|c-yy^*\|+\|d-y^*y\|) \leq 1/n)^{A^{\mathcal{U}}}$ hence $(\inf_{\|y\|\leq n}(\|c-yy^*\|+\|d-y^*y\|) \leq 1/n)^A$. By Lemma 2.4 we also have $c \sim d$ in A. □

We now have all of the ingredients necessary to prove Proposition 2.1.

Proof of Proposition 2.1. We are assuming that A is unital, noncommutative, infinite-dimensional and satisfies $(\star)$. To show that A is simple and purely infinite it suffices, by [Rør02, Proposition 1.4.1 (i)], to check that for every two positive elements a and b we have $b \precsim a$. Since being purely infinite and simple is elementary (see [FHL+] or [GS14]) it suffices to prove that $A^{\mathcal{U}}$ is purely infinite and simple.

Before doing so we will need two preliminary claims.

Claim 2.5.1. Suppose that $f, g \in A^{\mathcal{U}}$ are positive contractions with full spectrum, and $fg = gf = g$. Then $f \sim g$.

Proof of Claim 2.5.1. Choose elements a, b and c in A such that $ab = 0$ and $bc = cb = c$, each with full spectrum - such elements can be found in $C([0,1])$ and by Lemma 2.5 A contains a unital copy of $C([0,1])$. Then again by Lemma 2.5 we have that $c \sim a \sim b$ and therefore $c \sim b$. In particular there are $z_n, y_n \in A$ witnessing $b \sim c$. Since the spectra of f and g are both $[0,1]$ and $fg = gf = g$ we have $\mathrm{qftp}(f,g) = \mathrm{qftp}(b,c)$, hence $F = C^*(f,g,1) \cong C^*(b,c,1)$. Let ι be the inclusion map and Φ the unital embedding sending f to b and g to c. Then by $(\star)$ there is $\varphi\colon A \to A^{\mathcal{U}}$ a unital embedding making the diagram commute. It is easy to see that $\varphi(z_n)$ and $\varphi(y_n)$ witness that $f \sim g$. □

Claim 2.5.2. Every positive contraction $f \in A^{\mathcal{U}}$ with full spectrum satisfies $1 \precsim f$.

Proof of Claim 2.5.2. Choose positive contractions a, b, c in $A^{\mathcal{U}}$ each with full spectrum and such that $ab = bc = ac = 0$, and let $d = 1 - a$. Then $\sigma(d) = [0,1]$ and $db = bd = b$, hence $d \sim b$ by Claim 2.5.1. Also $a \sim c$. In fact, we will only need to know that $d \precsim b$ and $a \precsim c$. Since $bc = 0$, by [Cun78, Proposition 1.1] we have that $a + d \precsim b + c$. But $a + d = 1$ and $\sigma(b + c) = [0,1]$. In particular, $b + c$ is a positive contraction with full spectrum such that $1 \precsim b + c$. The same argument used in Claim 2.5.1 and in Lemma 2.5 shows that $1 \precsim f$ for every $f \in A^{\mathcal{U}}$ as required for the claim. □

Now given positive elements a, b, we show that $b \precsim a$. First, suppose that $\sigma(a) = [0,1]$. For every positive b with $\|b\| \leq 1$ we have $b \leq 1$ and therefore $b \precsim 1$. Then Claim 2.5.2 implies that we have $b \precsim a$ in this case.

Now consider the case when $\sigma(a) \neq [0,1]$ but $\|a\| = 1$. Let $a_1 = (a - 1/2)_+$, the positive part of the self-adjoint $a - 1/2$. By Lemma 2.2 we can find $a_2 \in a_1 A a_1$ such that $\sigma(a_2) = [0,1]$, and for such a_2 we have $a_2 \leq 2a$. Fix $\epsilon > 0$. Since $1 \precsim a_2$ there is x such that $x a_2 x^* = 1 - \epsilon$, and hence

$$2xax^* \geq xa_2x^* \geq 1 - \epsilon.$$

Therefore, by continuous functional calculus, one can find z such that $zaz^* = 1-2\epsilon$. Since ϵ was arbitrary, [Rør92, Proposition 2.4] implies that $1 \precsim a$ and hence $b \precsim a$. This completes the proof. $\square$

2.1. $\mathcal{O}_2$ **and quantifier elimination.** Any C*-algebra generated by n isometries with orthogonal ranges with sum 1 is isomorphic to the Cuntz algebra $\mathcal{O}_n$ ([Cun77]). Hence $\mathcal{O}_2$ is the universal algebra defined by the relations $s^*s = t^*t = 1$ and $ss^* + tt^* = 1$. This algebra plays a pivotal role in Elliott's classification program (see [Rør02]). Notably, $\mathcal{O}_2$ has some properties implied by quantifier elimination; for example, every unital embedding of $\mathcal{O}_2$ into itself, or into any other model of its theory, is elementary (see e.g., [GS14] or [FHRT15]).

We shall need $C^*_r(\mathbb{F}_2)$, the reduced C*-algebra of $\mathbb{F}_2$, the free group on 2 generators. This algebra is exact (see [Kir93, p. 453, 1., 1-3], or [BO08, Proposition 5.1.8]) and therefore embeds into $\mathcal{O}_2$ (see [KP00]).

Our main goal in this section is to prove the following Theorem whose proof extends ideas used in the proof that stably finite exact C*-algebras are not necessarily embeddable into a stably finite nuclear C*-algebra (see the discussion preceding [Bro06, Corollary 4.2.3]).

Theorem 2.6. *If A is a separable, infinite-dimensional, noncommutative C*-algebra then A does not satisfy ($\star$). In particular, it does not have quantifier elimination.*

A formula φ is *weakly stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every C*-algebra A and every $a \in A$, $\varphi(a) < \delta$ implies that the distance from a to the zero-set of φ in A is $< \epsilon$. In the language of logic of metric structures, the zero-sets of weakly stable formulas are precisely the *definable* sets (as defined in [BYBHU08]). See [CCF$^+$14, Lemma 2.1] for details. It is shown in [BYBHU08] that every formula involving quantification over a definable set is equivalent to a standard formula.

Lemma 2.7. *Let A be a unital infinite-dimensional noncommutative C*-algebra that satisfies ($\star$). Then $\mathcal{O}_2$ embeds unitally in A.*

Proof. We write $p \sim q$ if p and q are Murray–von Neumann equivalent projections and note that $p \sim q$ in A if and only if $p \sim q$ in $A^{\mathcal{U}}$. Hence for a pair of commuting projections p and q in A the joint spectrum of p and q determines whether $p \sim q$ or not. By Proposition 2.1 we know that A is purely infinite. Hence there is $s \in A$ such that $s^*s = 1$ and $p_1 = ss^* < 1$. Moreover, $p_2 = s^2(s^*)^2$ satisfies $p_2 \lneq p_1$ and $p_2 \sim p_1$. Therefore $p_1 \sim p_1 - p_2$, and so the corner p_1Ap_1 has a unital copy of $\mathcal{O}_2$. Since $p_1 \sim 1$, this concludes the proof. $\square$

Proof of Theorem 2.6. By Lemma 2.7, $\mathcal{O}_2$ embeds into A. Let $F = C^*_r(\mathbb{F}_2)$, and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Each $M_n(\mathbb{C})$ embeds into $\mathcal{O}_2$, and hence also embeds into A. We therefore have an embedding of the ultraproduct $M = \prod_{\mathcal{U}} M_n(\mathbb{C})$ inside $\mathcal{O}_2^{\mathcal{U}} \subseteq A^{\mathcal{U}}$, denoted by $\iota_2 \colon M \to A^{\mathcal{U}}$. By [HT05, Thm. B], there are unitary representations $\pi_n \colon \mathbb{F}_2 \to M_n(\mathbb{C})$ such that for every $m \in \mathbb{N}$, all $h_1, \dots, h_m$ in $\mathbb{F}_2$ and all $c_1, \dots, c_m$ in $\mathbb{C}$ we have (with λ denoting the left-regular representation of $\mathbb{F}_2$ on $\ell_2(\mathbb{F}_2)$):

$$\lim_{n\to\infty} \| \textstyle\sum_{j=1}^m c_j \pi_n(h_j)\| = \| \sum_{j=1}^m c_j \lambda(h_j)\|.$$

Therefore the map

$$\sum_{j=1}^m c_j h_j \mapsto (\sum_{j=1}^m c_j \pi_n(h_j))/\mathcal{U}$$

is an isometry from a dense subset of F into M. The continuous extension $\psi\colon F \to M$ of this isometry is therefore a unital *-homomorphism, and we can take $\Phi = \iota_2 \circ \psi$.

We claim that Φ cannot be extended to an embedding φ of $\mathcal{O}_2$ into $A^{\mathcal{U}}$ (and in particular it cannot be extended to an embedding of A into $A^{\mathcal{U}}$), as $(\star)$ would require. Otherwise, by the nuclearity of $\mathcal{O}_2$ and the Choi–Effros lifting theorem ([CE76]) there exists a completely positive contraction $\psi'\colon \mathcal{O}_2 \to \ell_\infty(A)$ such that $\varphi = \pi \circ \psi'$, where $\pi\colon \ell_\infty(A) \to A^{\mathcal{U}}$ is the quotient map.

Since each $M_n(\mathbb{C})$ is an injective von Neumann algebra, by [Bla06, Proposition IV.2.1.4] there are coordinatewise conditional expectations $\theta_n\colon A \to M_n(\mathbb{C})$. Let $\theta\colon \ell_\infty(A) \to \prod M_n(\mathbb{C})$ be the conditional expectation induced by the expectations θ_n. Then $\theta \circ \psi'$ a completely positive contraction, hence

$$\theta \circ \psi'\colon F \to \prod M_n(\mathbb{C})$$

is a completely positive contractive lifting for Φ. We have therefore constructed an embedding of $C^*_r(\mathbb{F}_2)$ into M with a completely positive contractive lifting. C*-algebras with this property are said to be quasidiagonal (see [Bro04]). However, by a result of Rosenberg ([BO08, Corollary 7.1.18]) quasidiagonality of $C^*_r(\mathbb{F}_2)$ implies amenability of the nonamenable group $\mathbb{F}_2$. This contradiction concludes the proof. □

Theorem 2.8. *The only C*-algebras with quantifier elimination in the language of unital C*-algebras are* $\mathbb{C}$, $\mathbb{C}^2$, $M_2(\mathbb{C})$ *and* $C(2^{\mathbb{N}})$.

Proof. That the given list includes all finite-dimensional examples is Theorem 1.7, and that the list includes all noncommutative examples is Theorem 2.6. In the appendix (written with D.C. Amador, B. Hart, J. Kawach, and S. Kim), we show that if there is a commutative example not on our list then it is of the form $C(X)$ for an indecomposable continuum X. Finally, in [EGV15, Corollary 3.4] it was proved that no such example exists. □

2.2. **Non-unital C*-algebras.** In this subsection we prove that there is no non-commutative non-unital C*-algebra admitting quantifier elimination.

Recall that if A is a C*-algebra then the *unitazion* of A, $\tilde{A}$, is the smallest unital C*-algebra containing A as an ideal ([Bla06, II.1.2]). For example, $\tilde{C}_0(X) = C(X \cup \{\infty\})$ for a locally compact Hausdorff space X. Note that $\tilde{A}$ is also well-defined for unital C*-algebras A and that for example $\tilde{\mathbb{C}} \cong \mathbb{C}^2$. We identify A with a subalgebra of $\tilde{A}$, omitting a reference to the embedding.

In the non-unital case there is a natural version of property $(\star)$ in terms of non-unital embeddings. The following establishes the connection between the unital and non-unital cases.

Theorem 2.9. *Let A be a C*-algebra such that whenever F is a finitely generated C*-algebra, and $\iota\colon F \to A$ and $\Phi\colon F \to A^{\mathcal{U}}$ are injective *-homomorphisms, then there is an injective *-homomorphism $\varphi\colon A \to A^{\mathcal{U}}$ that makes the diagram commute. Then $\tilde{A}$ satisfies $(\star)$.*

The following Lemma is known to operator algebraists and model theorists will recognize it as an immediate consequence of the easy fact that the unitization of A belongs to A^{eq}. We nevertheless sketch the proof for the reader's convenience.

Lemma 2.10. *Let A be a C*-algebra. Then $\widetilde{(A^{\mathcal{U}})} \cong (\tilde{A})^{\mathcal{U}}$.*

Proof. We need to show that the natural embedding of $\widetilde{A^{\mathcal{U}}}$ into $(\tilde{A})^{\mathcal{U}}$ is surjective. As a Banach space, $\widetilde{A^{\mathcal{U}}}$ is isomorphic to $A^{\mathcal{U}} \oplus \mathbb{C}$ and $(\tilde{A})^{\mathcal{U}}$ is isomorphic to $(A \oplus \mathbb{C})^{\mathcal{U}}$, and surjectivity of the inclusion map is an easy consequence of the local compactness of $\mathbb{C}$. □

Proof of Theorem 2.9. We want to prove that $\tilde{A}$ has property $(\star)$. Fix a finitely generated unital C*-algebra F and two unital embeddings $\iota\colon F \to \tilde{A}$ and $\Phi\colon F \to (\tilde{A})^{\mathcal{U}}$. By the definition of the unitization and the fact that F is unital, there are $b_1, \dots, b_n \in A$ such that $\iota(F) = C^*(\bar{b}, 1)$ where $\bar{b} = (b_1, \dots, b_n)$. Let $G = C^*(\bar{b})$. Then $\iota(G) \subseteq A$ and, by Lemma 2.10, there is an embedding $\Phi'\colon G \to A^{\mathcal{U}}$. By hypothesis there is an embedding $\varphi'\colon A \to A^{\mathcal{U}}$ making the diagram commute; defining $\varphi\colon \tilde{A} \to (\tilde{A})^{\mathcal{U}}$ as $\varphi((a, \lambda)) = (\varphi'(a), \lambda)$ and using again Lemma 2.10, we get property $(\star)$. □

Corollary 2.11. *No noncommutative C*-algebra admits elimination of quantifiers in the standard language of C*-algebras.*

Proof. Assume A admits quantifier elimination. By Theorem 2.9, $\tilde{A}$ satisfies property $(\star)$. It follows from Theorems 1.7 and 2.6 that $\tilde{A} \cong M_2(\mathbb{C})$, which is impossible because $M_2(\mathbb{C})$ is simple. □

The theories of $\mathbb{C}$ and $C_0(2^{\mathbb{N}} \setminus \{0\})$ admit quantifier elimination in the standard language of C*-algebras. We conjecture that there are no other algebras with this property.

3. Model completeness and model companions

A theory is said to be *model complete* if every embedding between models of the theory is *elementary*, in the sense of preserving the values of all formulas. It is easy to see that quantifier elimination implies model completeness, while the converse is false. For example, the theory of every finite-dimensional C*-algebra is model complete. Model completeness is a useful tool in applications of model theory to algebra; for example, the fact that the (discrete) theory of algebraically closed fields is model complete is the key ingredient in a model-theoretic proof of Hilbert's Nullstellensatz (see [Mar02, Theorem 3.2.11]).

A theory T^* is said to be a *model companion* of theory T if: (i) every model of T is a submodel of a model of T^* and vice versa, and (ii) T^* is model complete. It is well-known that a theory can have at most one model companion. For example, the model companion of the theory of fields is the theory of algebraically closed fields. In [GS14, Proposition 5.10] it was proved that, assuming Kirchberg's Embedding Problem has a positive solution, the theory of C*-algebras does not have a model companion. Isaac Goldbring has observed that, using methods of [GS14], we can remove the dependence on Kirchberg's Embedding Problem.

Theorem 3.1. *The theory of unital C*-algebras does not have a model companion.*

Proof. Assume otherwise and let T^* be the model companion of the theory of unital C*-algebras. By Theorem 2.6, T^* does not have quantifier elimination.

On the other hand, for universally axiomatizable theories (such as the theory of unital C*-algebras), quantifier elimination for the model companion is equivalent to the original theory having amalgamation (the proof is the same as the one in the discrete setting, for which see [CK90, Prop. 3.5.19]). Unital C*-algebras have amalgamation (i.e., free products—see [Bla06, II.8.3.5]). Unital C*-algebras are only $\forall\exists$-axiomatizable (see definition below) in the language introduced above, but they are universally axiomatizable in an expanded language that has predicates for all *-polynomials (see [FHS14, p. 485]). However, the difference between the languages and axiomatizations of C*-algebras is only definitional and it does not rise to the level of concern for elementary substructures. This completes the proof. □

A regularity property of a theory weaker than model completeness is the theory being $\forall\exists$-*axiomatizable.* A sentence φ is $\forall\exists$ if it is of the form

$$\sup_{\bar{x}} \inf_{\bar{y}} \varphi(\bar{x}, \bar{y})$$

where $\varphi(\bar{x}, \bar{y})$ is quantifier-free, and a theory is $\forall\exists$-axiomatizable if it has a set of $\forall\exists$ axioms. If a theory T is model complete then it is preserved by taking inductive limits of its models. By the standard preservation theorem T is $\forall\exists$-axiomatizable (see e.g., [FHL$^+$]).

The Cuntz algebra $\mathcal{O}_2$ belongs to the important class of strongly self-absorbing C*-algebras. A C*-algebra D is *strongly self-absorbing* (s.s.a.) if $D \cong D \otimes D$ and the embedding of D into $D \otimes D$ that sends d to $d \otimes 1$ is approximately unitarily equivalent to an isomorphism between D and $D \otimes D$ ([TW07]). S.s.a. C*-algebras play an important role in the classification program of C*-algebras and exhibit interesting model-theoretic properties (see [Far14, §2.2 and §4.5] and [FHRT15]).

Theorem 3.2. *Assume A has the same universal theory as an s.s.a. algebra D. If the theory of A is model complete (or even just $\forall\exists$-axiomatizable), then A is elementarily equivalent to D.*

A standard use of saturation of ultrapowers shows that the hypothesis of Theorem 3.2 is equivalent to asserting that D embeds into an ultrapower of A and A embeds into an ultrapower of D (see e.g., [GS14]).

Proof of Theorem 3.2. The proof uses the sandwich argument of [GHS13, Proposition 3.2].

Since A has the same universal theory as D, D embeds into an ultrapower of A and A embeds into an ultrapower of D. We therefore have a chain

$$D \to A^{\mathcal{U}} \to (D^{\mathcal{U}})^{\mathcal{U}}.$$

Since D is s.s.a., every embedding of it into its ultrapower is elementary (e.g., [FHRT15, Theorem 2.15]). Taking ultrapower of the diagram and iterating the construction, we obtain a sequence of embeddings $B_0 \to A_0 \to B_1 \to A_1 \to \dots$ such $B_i \equiv D$, $A_i \equiv A$ and embeddings $B_i \to B_{i+1}$ are elementary for all i. The inductive limit is elementarily equivalent to D (by the elementarity) and to A (by the well-known fact that $\forall\exists$-theories are preserved under direct limits), and the conclusion follows. □

A purely infinite, simple, and nuclear C*-algebra (i.e., *Kirchberg* algebra) is said to be in *standard form* if A is unital and $[1_A] = 0$ in $K_0(A)$. This is equivalent to A having a unital copy of $\mathcal{O}_2$ (see e.g., [FHRT15, §3]).

Corollary 3.3. *If A is a Kirchberg algebra in standard form other than $\mathcal{O}_2$ then its theory is not $\forall\exists$-axiomatizable. In particular, If $n \geq 3$ then the theory of $M_{n-1}(\mathcal{O}_n)$ is not $\forall\exists$-axiomatizable.*

Proof. We first note that $\mathcal{O}_2$ is the only separable nuclear model of its theory (this is a consequence of Kirchberg's theorem that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for all separable, nuclear, unital simple C*-algebras A; see [GS14] or [FHL$^+$]). The fact that $M_n(\mathcal{O}_{n+1})$ is in standard form is well-known; see e.g., [Rør02]. □

It is shown in [GS14, Proposition 5.7] that a positive solution to the Kirchberg's Embedding Problem implies that $\mathrm{Th}(\mathcal{O}_2)$ is not model complete. We should also remark that in the case of II_1 factors the only strongly self-absorbing algebra is the hyperfinite II_1 factor R, and its theory is shown (relying on [Bro11]) not to be model-complete in [GHS13].

Having shown that many natural examples of C*-algebras do not have quantifier *elimination*, we may ask whether they have quantifier *reduction*, that is, whether it can be shown that every formula is equivalent to one with a fixed number of alternations of quantifiers. For example, in the discrete setting Sela [Sel06] showed that in the theory of nonabelian free groups every formula is equivalent to a boolean combination of $\forall\exists$ formulas.

Question 3.4. Is there a natural example of a C*-algebra which admits quantifier reduction?

Given the primarily negative nature of our results, a natural question is to determine if there is a useful expansion of the language of C*-algebras in which wider classes of algebras do have quantifier elimination. As we described in the introduction, for such an expansion to be useful we should add only a small number of symbols for natural predicates which are definable, but not quantifier-free definable, in the original language for C*-algebras. A classical example of this from discrete logic, due to Tarski, is that the theory of $\mathbb{R}$ has quantifier elimination in the language of *ordered* fields, while it does not have quantifier elimination in the language of fields without the ordering.

4. Appendix with Diego Caudillo Amador, Bradd Hart, Jamal Kawach, and Se-jin Kim

We provide a first step of the proof (completed in [EGV15, Corollary 3.4]) that the only theories of commutative C*-algebras that admit elimination of quantifiers are $\mathbb{C}$, $\mathbb{C}^2$ and $C(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ denotes the Cantor space. In [EV14] it was proved that the latter algebra has quantifier elimination. Since $2^{\mathbb{N}}$ is (up to homeomorphism) the unique zero-dimensional, compact metrizable space with no isolated points, $C(2^{\mathbb{N}})$ is the unique separable model of its theory. Therefore, if X is any compact zero-dimensional space with no isolated points then $C(X)$ is elementarily equivalent to $C(2^{\mathbb{N}})$.

Lemma 4.1. *Let $C(X)$ be an infinite-dimensional commutative C*-algebra that admits elimination of quantifiers. Then either X is connected, or $C(X)$ is elementarily equivalent to $C(2^{\mathbb{N}})$.*

Proof. By Löwenheim-Skolem we may assume that X is metrizable, and we assume also that X is not connected. We first show that X continuously surjects onto the Cantor space $2^{\mathbb{N}}$. Let $A \subseteq X$ be a nontrivial clopen set, and let p be the characteristic function of A. Then p is a nontrivial projection in $C(X)$. Assume for a contradiction that A is connected. Then p is a minimal projection, so by quantifier elimination all nontrivial projections in $C(X)$ are minimal; in particular, $X \setminus A$ is also connected. Let $f, g \in C(X)$ be such that $f[A] = [0,1]$, $f[X \setminus A] = \{0\}$, $g[A] = [0, 1/2]$, and $g[X \setminus A] = [1/2, 1]$. Note that f and g are self-adjoints with the same spectrum, so by Lemma 1.4 they have the same type. However, there is a nontrivial projection q (namely p) such that $fq = f$, while for any projection q we have $\|gq - g\| \geq 1/2$, a contradiction.

We have shown that no nontrivial clopen subset of X is connected; by repeatedly splitting each clopen set we obtain a binary tree, and hence a continuous surjection of X onto $2^{\mathbb{N}}$. On the other hand, X is a compact metrizable space, so $2^{\mathbb{N}}$ continuously surjects onto X. Now we use the sandwich method previously used in Theorem 3.2 above. We form a chain:

$$C(X) \hookrightarrow C(2^{\mathbb{N}}) \hookrightarrow C(X) \hookrightarrow C(2^{\mathbb{N}}) \hookrightarrow C(X) \hookrightarrow \cdots$$

By assumption $C(X)$ has quantifier elimination, and it is shown in [EV14] that $C(2^{\mathbb{N}})$ has quantifier elimination as well. Therefore both $C(X)$ and $C(2^{\mathbb{N}})$ are elementarily equivalent to the limit of the chain, and hence also to each other. □

It follows from Lemma 4.1 and Theorem 1.7 that the only theories of unital commutative real rank zero C*-algebras with quantifier elimination are the theories of $\mathbb{C}, \mathbb{C}^2$, and $C(2^{\mathbb{N}})$. We now turn to the other side of the dichotomy in Lemma 4.1, and consider the case where X is connected. Recall that a connected compact Hausdorff space (i.e., a *continuum*) is said to be *indecomposable* if it is not the union of two of its proper subcontinua. This property is equivalent (see e.g., [Kur68, §48, Theorem 2]) to not having a connected non dense open set.

Theorem 4.2. *Assume X is a continuum such that $C(X)$ has elimination of quantifiers. Then X is indecomposable.*

Proof. A *peak function* is $f \in C(X)$ such that $\sigma(f) = [0,1]$ and the set $\{x : f(x) > 4/5\}$ is connected. A *volcano function* is $f \in C(X)$ such that $\sigma(f) = [0,1]$ and $f = g + h$ for some g and h that satisfy $\sigma(g) = \sigma(h) = [0,1]$ and $gh = 0$. Proposition 2.2 implies that every every C*-algebra with no minimal projections contains a volcano function f_1. We shall construct a peak function f_2 and show that f_1 and f_2 have different types. By Lemma 1.4 this will conclude the proof.

We will show that f_1 and f_2 do not have the same type. Consider the formula

$$\varphi(x) = \inf_{y,z} \max(\|x - yy^* - zz^*\|, |1 - \|y\||, |1 - \|z\||, \|yy^*zz^*\|).$$

Taking $y = g^{1/2}$ and $z = h^{1/2}$ we see that $\varphi(f_1) = 0$,

Now we construct a peak function using our assumptions on X. Let U be a connected open subset which is not dense in X, and fix $z \in X$ such that $\mathrm{dist}(z, U) > r > 0$ for some r. With $F = X \setminus U$, the function $h_0(x) = \mathrm{dist}(x, F)$ has the closure of U as its support. We normalize and let $h = \|h_0\|^{-1} h_0$. Function $g(x) = r^{-1} \max(0, r - d(x, z))$ satisfies $\sigma(g) = [0,1]$ and its support is disjoint from U. Therefore $f_2 = \frac{1}{5}h + \frac{4}{5}(1 - g)$ is a peak function.

Assume $\varphi(f_2) < 1/10$. Then there are $a = yy^*$ and $b = zz^*$ such that

$$\max(\|f_2 - a - b\|, |1 - \|a\||, |1 - \|b\||, \|ab\|) < 1/10.$$

Then there are s and t in X such that $a(s) > 9/10$ and $b(t) > 9/10$. Since $|f_2(x) - a(x) - b(x)| < 1/10$ for all $x \in X$, we have $f_2(s) > 4/5$ and $f_2(t) > 4/5$ and $a(t) < 1/5$ and $b(s) < 1/5$. Since $\{x \in X : f_2(x) > 4/5\}$ is connected, there is $x \in X$ such that $a(x) = b(x)$ and $f_2(x) > 4/5$. Therefore $a(x) = b(x) > 7/20$ and $a(x)b(x) > 1/10$. This violates our assumptions and completes the proof. □

References

[Bla06] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.

[BO08] N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[Bro04] N. Brown, *On quasidiagonal C^*-algebras*, Operator algebras and applications, Adv. Stud. Pure Math., vol. 38, Math. Soc. Japan, Tokyo, 2004, pp. 19–64.

[Bro06] N.P. Brown, *Invariant means and finite representation theory of C^*-algebras*, Mem. Amer. Math. Soc. **184** (2006), no. 865, viii+105.

[Bro11] N. Brown, *Topological dynamical systems associated to II_1 factors*, Adv. Math. **227** (2011), no. 4, 1665–1699, With an appendix by Narutaka Ozawa.

[BYBHU08] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, eds.), Lecture Notes series of the London Mathematical Society, no. 350, Cambridge University Press, 2008, pp. 315–427.

[CCF$^+$14] K. Carlson, E. Cheung, I. Farah, A. Gerhardt-Bourke, B. Hart, L. Mezuman, N. Sequeira, and A. Sherman, *Omitting types and AF algebras*, Arch. Math. Logic **53** (2014), 157–169.

[CE76] Man-Duen Choi and Edward G Effros, *The completely positive lifting problem for C^*-algebras*, Annals of Mathematics (1976), 585–609.

[CK90] C.C. Chang and H.J. Keisler, *Model theory*, 3 ed., North Holland, 1990.

[Cun77] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185.

[Cun78] ———, *Dimension functions on simple C^*-algebras*, Mathematische Annalen **233** (1978), no. 2, 145–153.

[Dav96] K.R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[EGV15] C.J. Eagle, I. Goldbring, and A. Vignati, *The pseudoarc is a co-existentially closed continuum*, arXiv preprint 1503.03443 (2015).

[EV14] C.J. Eagle and A. Vignati, *Saturation and elementary equivalence of C^*-algebras*, to appear in Journal of Functional Analysis (2014), arXiv preprint 1406.4875.

[Far14] I. Farah, *Logic and operator algebras*, Proceedings of the Seoul ICM, 2014, arXiv preprint 1404.4978.

[FHL$^+$] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, *Model theory for nuclear C^*-algebras*, in preparation.

[FHRT15] I. Farah, B. Hart, M. Rørdam, and A.P. Tikuisis, *Relative commutants of strongly self-absorbing C^*-algebras*, arXiv preprint 1502.05228.

[FHS14] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model theory*, Israel J. Math. **201** (2014), 477–505.

[GHS13] I. Goldbring, B. Hart, and T. Sinclair, *The theory of tracial von Neumann algebras does not have a model companion*, J. Symbolic Logic **78** (2013), no. 3, 1000–1004.

[GS14] I. Goldbring and T. Sinclair, *On Kirchberg's embedding problem*, arXiv preprint 1404.1861 (2014).

[HT05] U. Haagerup and S. Thorbjørnsen, *A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group*, Annals of Mathematics **162** (2005), no. 2, 711–775.

[Kir93] E. Kirchberg, *On nonsemisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. **112** (1993), no. 3, 449–489.

[KP00] E. Kirchberg and N.C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. reine angew. Math. **525** (2000), 17–53.

[Kur68] K. Kuratowski, *Topology, vol. II*, Academic Press New York, 1968.

[Mar02] David Marker, *Model theory: An introduction*, Graduate Texts in Mathematics, vol. 217, Springer, New York, 2002.

[PW07] N.C. Phillips and N. Weaver, *The Calkin algebra has outer automorphisms*, Duke Math. Journal **139** (2007), 185–202.

[Rør92] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Functional Analysis **107** (1992), no. 2, 255–269.

[Rør02] ———, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, Springer, 2002, pp. 1–145.

[Sel06] Z. Sela, *Diophantine geometry over groups V_2. Quantifier elimination II*, Geom. Funct. Anal. **16** (2006), 707–730.

[TW07] A.S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, Trans. Amer. Math. Soc. **359** (2007), 3999–4029.

(C. J. Eagle) Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, Canada, M5S 2E4
E-mail address: cjeagle@math.toronto.edu
URL: http://www.math.toronto.edu/cjeagle/

(I. Farah) Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 34, Belgrade, Serbia
E-mail address: ifarah@mathstat.yorku.ca
URL: http://www.math.yorku.ca/∼ifarah

(E. Kirchberg) Humboldt Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, D-10099 Berlin, Germany
E-mail address: kirchbrg@mathematik.hu-berlin.de

(A. Vignati) Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada, M3J 1P3,
E-mail address: ale.vignati@gmail.com