A NEW BICOMMUTANT THEOREM

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Abstract. We prove an analogue of Voiculescu's theorem: Relative bicommutant of a separable unital subalgebra A of an ultraproduct of simple unital C*-algebras is equal to A.

Ultrapowers[1] $A^{\mathcal{U}}$ of separable algebras are, being subject to well-developed model-theoretic methods, reasonably well-understood (see e.g. [12, Theorem 1.2] and §2). Since the early 1970s and the influential work of McDuff and Connes central sequence algebras $A' \cap A^{\mathcal{U}}$ play an even more important role than ultrapowers in the classification of II_1 factors and (more recently) C*-algebras. While they do not have a well-studied abstract analogue, in [12, Theorem 1] it was shown that the central sequence algebra of a strongly self-absorbing algebra ([27]) is isomorphic to its ultrapower if the Continuum Hypothesis holds. Relative commutants $B' \cap D^{\mathcal{U}}$ of separable subalgebras of ultrapowers of strongly self-absorbing C*-algebras play an increasingly important role in classification program for separable C*-algebras ([22, §3], [5]; see also [26], [29]). In the present note we make a step towards better understanding of these algebras.

C*-algebra is *primitive* if it has representation that is both faithful and irreducible. We prove an analogue of the well-known consequence of Voiculescu's theorem ([28, Corollary 1.9]) and von Neumann's bicommutant theorem ([4, §I.9.1.2]).

Theorem 1. *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of primitive* C*-*algebras and A is a separable* C*-*subalgebra. In addition assume A is a unital subalgebra if $\prod_{\mathcal{U}} B_j$ is unital. Then (with $\overline{A}^{\mathrm{WOT}}$ computed in the ultraproduct of faithful irreducible representations of B_js)*

$$A = \Big(A' \cap \prod_{\mathcal{U}} B_j\Big)' = \overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j.$$

A slightly weaker version of the following corollary to Theorem 1 (stated here with Aaron Tikuisis's kind permission) was originally proved by using very different methods ($Z(A)$ denotes the center of A).

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[1] Throughout $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$.

Corollary 2 (Farah–Tikuisis, 2015)**.** *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of simple unital* C**-algebras and A is a separable unital subalgebra. Then $Z(A' \cap \prod_{\mathcal{U}} B_j) = Z(A)$.* □

At least two open problems are concerned with bicommutants of separable subalgebras of massive operator algebras. As is well-known, central sequence algebras $M' \cap M^{\mathcal{U}}$ of II_1 factors in tracial ultrapowers behave differently from the central sequence algebras of C*-algebras. For a II_1 factor M with separable predual the central sequence algebra $M' \cap M^{\mathcal{U}}$ can be abelian or even trivial. Popa conjectured that if P is a separable subalgebra of an ultraproduct of II_1 factors then $(P' \cap \prod_{\mathcal{U}} N_i)' = P$ implies P is amenable ([25, Conjecture 2.3.1]). In the domain of C*-algebras, Pedersen asked ([24, Remark 10.11]) whether the following variant of Theorem 1 is true: If the corona $M(B)/B$ of a σ-unital C*-algebra B is simple and A is a separable unital subalgebra, is $(A' \cap M(B)/B)' = A$? (For the connection between ultraproducts and coronas see the last paragraph of §3.)

The proof of Theorem 1 uses logic of metric structures ([3], [14]) and an analysis of the interplay between C*-algebra B and its second dual B^{**}.

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1. Model theory of representations

We expand the language of C*-algebras ([14, §2.3.1]) to representations of C*-algebra. Reader's familiarity with, or at least easy access to, [14, §2] is assumed. A structure in the expanded language $\mathcal{L}_{\mathrm{rep}}$ is a C*-algebra together with its representation on a Hilbert space. As in [14], the domains of quantification on C*-algebra are D_n for $n \in \mathbb{N}$ and are interpreted as the n-balls. The domains of quantification on Hilbert space are D^H_n for $n \in \mathbb{N}$ and are also interpreted as the n-balls. On all domains the metric is $d(x,y) = \|x - y\|$ (we shall denote both the operator norm on C*-algebra and the ℓ_2-norm on Hilbert space by $\|\cdot\|$). As in [14, §2.3.1], for every $\lambda \in \mathbb{C}$ we have a unary function symbol λ to be interpreted as multiplication by λ. We also have binary function $+$ whose interpretation sends $D^H_m \times D^H_n$ to D^H_{m+n}. As the scalar product $(\cdot|\cdot)$ is definable from the norm via the polarization identity, we shall freely use it in our formulas, with the understanding that $(\xi|\eta)$ is an abbreviation for $\frac{1}{4}\sum_{j=0}^{3} i^j \|\xi + i^j\eta\|$.

Language $\mathcal{L}_{\mathrm{rep}}$ also contains a binary function symbol π whose interpretation sends $D_n \times D^H_m$ to D^H_{mn} for all m and n. It is interpreted as an action of A on H.

Every variable is associated with a sort. In particular variables x, y, z range over the C*-algebra and variables ξ, η, ζ range over the Hilbert space, all of them decorated with subscripts when needed.

We shall write $\bar{x}$ for a tuple $\bar{x} = (x_1, \dots, x_n)$ (with n either clear from the context or irrelevant). *Terms* come in two varieties. On the C*-algebra side, term is a noncommutative *-polynomial in C*-variables. On the Hilbert space side terms are linear combinations of Hilbert space variables and expressions of the form $\pi(\alpha(\bar{x}))\xi$ where $\alpha(\bar{x})$ is a term in the language of C*-algebras. *Formulas* are defined recursively. Atomic formulas are expressions of the form $\|t\|$ where t is a term.

The set of all formulas is the smallest set $\mathbb{F}$ containing all atomic formulas with the following properties.

(i) for every n, all continuous $f\colon [0,\infty)^n \to [0,\infty)$ and all $\varphi_1, \dots, \varphi_n$ in $\mathbb{F}$ the expression $f(\varphi_1, \dots, \varphi_n)$ belongs to $\mathbb{F}$, and

(ii) if $\varphi \in \mathbb{F}$ and x, ξ are variable symbols then each of $\sup_{\|\xi\|\leq m}\varphi$, $\inf_{\|\xi\|\leq m}\varphi$, $\sup_{\|x\|\leq m}\varphi$ and $\inf_{\|x\|\leq m}\varphi$ belongs to $\mathbb{F}$ (see [14, §2.4] or [11, Definition 2.1.1]).

Suppose $\pi\colon A \to B(H)$ is a representation of a C*-algebra A on Hilbert space H. To (A, H, π) we associate the natural metric structure $\mathcal{M}(A, H, \pi)$ in the above language.

Suppose $\varphi(\bar{x}, \bar{\xi})$ is a formula whose free variables are included among $\bar{x}$ and $\bar{\xi}$. If $\pi\colon A \to B(H)$ is a representation of a C*-algebra on Hilbert space, $\bar{a}$ are elements of A and $\bar{\xi}$ are elements of H,[2] then the *interpretation* $\varphi(\bar{a}, \bar{\xi})^{\mathcal{M}(A,H,\pi)}$ is defined by recursion on the complexity of φ in the obvious way (see [3, §3]).

Proposition 1.1. *Triples (A, H, π) such that π is a representation of A on H form an axiomatizable class.*

Proof. As in [14, Definition 3.1], we need to define a $\mathcal{L}_{\mathrm{rep}}$-theory $\mathbf{T}_{\mathrm{rep}}$ such that that the category of triples (A, H, π) where $\pi\colon A \to B(H)$ is a representation of a C*-algebra A is equivalent to the category of metric structures that are models of $\mathbf{T}_{\mathrm{rep}}$, via the map

$$(A, H, \pi) \mapsto \mathcal{M}(A, H, \pi).$$

[2]Symbols $\xi, \eta, \zeta, \dots$ denote both Hilbert space variables and vectors in Hilbert space due to the font shortage; this shall not lead to a confusion.

We use the axiomatization of C*-algebras from [14, §3.1]. In addition to the standard Hilbert space axioms, we need an axiom assuring that the interpretation of D_n^H equals the n-ball of the underlying Hilbert space:

$$\sup_{\|\xi\|\leq n} \|\xi\| \leq n$$

for all n and (writing $s \dot{-} t := \max(s-t, 0)$),

(*) $\sup_{\|\xi\|\leq n}(1 \dot{-} \|\xi\|) \inf_{\|\eta\|\leq 1} \|\xi - \eta\|$.

The standard axioms,

$$\pi(xy)\xi = \pi(x)\pi(y)\xi, \quad \pi(x+y)\xi = \pi(x)\xi + \pi(y)\xi \quad (\pi(x)\xi|\eta) = (\xi|\pi(x^*)\eta)$$

are expressible as first-order sentences.[3] The axioms described here comprise theory $\mathbf{T}_{\mathrm{rep}}$.

One needs to check that the category of models of $\mathbf{T}_{\mathrm{rep}}$ is equivalent to the category of triples (A, H, π). Every triple (A, H, π) uniquely defines a model $\mathcal{M}(A, H, \pi)$. Conversely, assume $\mathcal{M}$ is a model of $\mathbf{T}_{\mathrm{rep}}$. The algebra $A_{\mathcal{M}}$ obtained from the first component of $\mathcal{M}$ is a C*-algebra by [14, Proposition 3.2]. Also, the linear space $H_{\mathcal{M}}$ obtained from the second component of $\mathcal{M}$ is a Hilbert space and the third component gives a representation $\pi_{\mathcal{M}}$ of A on H.

To see that this provides an equivalence of categories, we need to check that $\mathcal{M}(A_{\mathcal{M}}, H_{\mathcal{M}}, \pi_{\mathcal{M}}) \cong \mathcal{M}$ for every model $\mathcal{M}$ of $\mathbf{T}_{\mathrm{rep}}$. We need to show that the domains on $\mathcal{M}$ are determined by $A_{\mathcal{M}}$ and $H_{\mathcal{M}}$. The former was proved in the second paragraph of [14, Proposition 3.2], and the latter follows by (*). □

Proposition 1.1 gives us full access to the model-theoretic toolbox, such as the Łoś' theorem (see §2) and the Löwenheim–Skolem Theorem ([14, Theorem 4.6]). From now on, we shall identify triple (A, H, π) with the associated metric structure $\mathcal{M}(A, H, \pi)$ and stop using the latter notation.

Lemma 1.2. *The following properties of a representation π of A are axiomatizable:*

1. *π is faithful*
2. *π is irreducible.*

Proof. We shall explicitly write the axioms for each of the properties of π. Fix a representation π. It is faithful if and only if it is isometric, which can be expressed as

$$\sup_{\|x\|\leq 1} \inf_{\|\xi\|\leq 1} |\|x\| - \|\pi(x)\xi\|| = 0.$$

Representation π is irreducible if and only if for all vectors ξ and η in H such that $\|\eta\| \leq 1$ and $\|\xi\| = 1$ the expression $\|\eta - \pi(a)\xi\|$ can be made

[3]Our conventions are as described in [14, p. 485]. In particular $\alpha(x,\xi) = \beta(x,\xi)$ is an abbreviation for $\sup_{\|x\|\leq n} \sup_\xi \|\alpha(x,\xi) - \beta(x,\xi)\| = 0$, for all n.

arbitrarily small when a ranges over the unit ball of A. In symbols

$$\sup_{\|\xi\|\leq 1}\sup_{\|\eta\|\leq 1}\inf_{\|x\|\leq 1}|\|\xi\| \dot{-} 1|\|\eta-\pi(x)\xi\|=0.$$

The interpretation of this sentence in (A,H,π) is 0 if and only if π is irreducible. □

Triple (D,θ,K) is an *elementary submodel* of (B,π,H), and (B,π,H) is an *elementary extension* of $(D,\theta,K$, if $D\subseteq B$, $K\subseteq H$ $\theta(d)=\pi(d)\restriction H$ for all $d\in D$, and

$$\varphi(\bar{a})^{(D,\theta,K)}=\varphi(\bar{a})^{(B,\pi,H)}$$

for all formulas φ and all $\bar{a}$ in (D,θ,K) of the appropriate sort. Axiomatizable properties, such as being irreducible or faithful, transfer between elementary submodels and elementary extensions. Therefore the Downwards Löwenheim–Skolem Theorem ([14, Theorem 4.6]) and Lemma 1.2 together imply e.g. that if φ is a pure state of a nonseparable C*-algebra B then B is an inductive limit of separable subalgebras D such that the restriction of φ to D is pure. This fact was proved in [1] and its slightly more precise version will be used in the proof of Lemma 3.2.

Some other properties of representations (such as not being faithful) are axiomatizable but we shall concentrate on proving Theorem 1.

2. Saturation and representations

It was known to logicians since the 1960s that the two defining properties of ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ in axiomatizable categories are Łoś' Theorem ([14, Proposition 4.3]) and countable saturation ([14, Proposition 4.11]). By the former, the diagonal embedding of metric structure M into its ultrapower is elementary. More generally, if $\varphi(\bar{x})$ is a formula and $\bar{a}(j)\in M_j$ are of the appropriate sort then

$$\varphi(\bar{a})^{\prod_{\mathcal{U}} M_j}=\lim_{j\to\mathcal{U}}\varphi(\bar{a}(j))^{M_j}.$$

In order to define countable saturation, we recall the notion of a type from the logic of metric structures ([14, §4.3]). A *closed condition* (or simply a *condition*; we shall not need any other conditions) is any expression of the form $\varphi\leq r$ for formula φ and $r\geq 0$ and *type* is a set of conditions ([14, §4.3]). As every expression of the form $\varphi=r$ is equivalent to the condition $\max(\varphi,r)\leq r$ and every expression of the form $\varphi\geq r$ is equivalent to the condition $\min(0,r-\varphi)\leq 0$, we shall freely refer to such expressions as conditions. For m and n in $\mathbb{N}$ such that $m+n\geq 1$, an (m,n)*-type* is a type $\mathbf{t}$ such that all free variables occurring in conditions of $\mathbf{t}$ are among $\{x_1,\dots,x_m\}\cup\{\xi_1,\dots,\xi_n\}$.

Given a structure (A,H,π) and a subset X of $A\cup H$, we expand the language $\mathcal{L}_{\mathrm{rep}}$ by adding constants for the elements of X (as in [14, §2.4.1]). The new language is denoted $(\mathcal{L}_{\mathrm{rep}})_X$. C*-terms in $(\mathcal{L}_{\mathrm{rep}})_X$ are *-polynomials in C*-variables and constants from $X\cap A$. Hilbert space terms are linear combinations of Hilbert space variables, constants in $X \cap H$, and expressions of the form $\pi(\alpha)\xi$ where α is a C*-term in the expanded language. The interpretation of a $(\mathcal{L}_{\mathrm{rep}})_X$-formula is defined recursively in the natural way (see e.g. [11], paragraph after Definition 2.1.1).

Type *over* X is a type in $(\mathcal{L}_{\mathrm{rep}})_X$. Such type is *realized* in some elementary extension of (A, H, π) if the latter contains a tuple satisfying all conditions from the type. A type is *consistent* if it is realized in some ultrapower of (A, H, π), where the ultrafilter is taken over an arbitrary, not necessarily countable set. This is equivalent to the type being realized in some elementary extension of (A, H, π).

By Łoś' Theorem, type $\mathbf{t}$ is consistent if and only every finite subset of $\mathbf{t}$ is ε-realized in (A, H, π) for every $\varepsilon > 0$ ([14, Proposition 4.8]).

A structure (A, H, π) is said to be *countably saturated* if every consistent type over a countable (or equivalently, norm-separable) set is realized in (A, H, π). Ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ are always countably saturated ([14, Proposition 4.11]). A standard transfinite back-and-forth argument shows that a structure of density character $\aleph_1$ is countably saturated if and only if it is an ultraproduct. (*Density character* is the smallest cardinality of a dense subset.)

In the case when $A = B(H)$ we have $(B(H), H)^{\mathcal{U}} = (B(H)^{\mathcal{U}}, H^{\mathcal{U}})$, in particular $B(H)^{\mathcal{U}}$ is identified with a subalgebra of $B(H^{\mathcal{U}})$. These two algebras are equal (still assuming $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$) if and only if H is finite-dimensional. As a matter of fact, no projection $p \in B(H^{\mathcal{U}})$ with a separable, infinite-dimensional, range belongs to $B(H)^{\mathcal{U}}$ (this is proved by a standard argument, see e.g. the last two paragraphs of the proof of [13, Proposition 4.6]).

In the following, π will always be faithful and clear from the context and we shall identify A with $\pi(A)$ and suppress writing π. We shall therefore write (A, H) in place of (A, H, id).

The following two lemmas are standard (they were used by Arveson in the proof of Corollary 2 on p 344 of [2]) but we sketch the proofs for the reader's convenience.

Lemma 2.1. *Suppose A is a* C*-*algebra and φ is a functional on A. Then there are a representation $\pi\colon A \to B(K)$ and vectors ξ and η in K such that $\varphi(a) = (\pi(a)\xi|\eta)$ for all $a \in A$.*

Proof. Let $\bar{\varphi}$ be the unique extension of φ to a normal functional of the von Neumann algebra A^{**}. By Sakai's polar decomposition for normal linear functionals (see e.g. [23, Proposition 3.6.7]) there exists a normal state ψ of A^{**} and a partial isometry v such that $\varphi(a) = \psi(av)$ for all $a \in A^{**}$. Let $\pi\colon A^{**} \to B(K)$ be the GNS representation corresponding to ψ. If η is the corresponding cyclic vector and $\xi = v\eta$, then the restriction of π to A is as required. $\square$

Lemma 2.2. *Suppose A is a proper unital subalgebra of $C = \mathrm{C}^*(A,b)$. Then there exists a representation $\pi\colon C \to B(K)$ and a projection q in $\pi(A)' \cap B(K)$ such that $[q,b] \neq 0$.*

Proof. By the Hahn–Banach separation theorem there exists a functional φ on C of norm 1 such that φ annihilates A and $\varphi(b) = \mathrm{dist}(A,b)$. Let $\pi\colon C \to B(K)$, η and ξ be as guaranteed by Lemma 2.1. Let L be the norm-closure of $\pi(A)\xi$. Since A is unital $L \neq \{0\}$. As $0 = \varphi(a) = (\pi(a)\xi|\eta)$ for all $a \in A$, η is orthogonal to L and therefore the projection p to L is nontrivial. Clearly $p \in \pi(A)' \cap B(K)$. Since $(\pi(b)\xi|\eta) = \varphi(b) \neq 0$, $\pi(b)$ does not commute with p and we therefore have $q \in \pi(A)' \cap B(K)$ such that $\|[\pi(b),q]\| > 0$. □

The proof of Theorem 1 would be much simpler if Lemma 2.2 provided an irreducible representation. This is impossible in general as the following example shows. Let A be the unitization of the algebra of compact operators $\mathcal{K}(H)$ on an infinite-dimensional Hilbert space and let b be a projection in $B(H)$ Murray-von Neumann equivalent to $1-b$. Then $C = \mathrm{C}^*(A,b)$ has (up to equivalence) three irreducible representations. Two of those representations annihilate A and send b to a scalar, and the third representation is faithful and the image of b is in the weak operator closure of the image of A.

It is well-known that for a Banach space X the second dual X^{**} can be embedded into an ultrapower of X ([19, Proposition 6.7]). In general, the second dual A^{**} of a C*-algebra A cannot be embedded into an ultrapower of A for at least two reasons. First, A^{**} is a von Neumann algebra ([4, §III.5.2]) and it therefore has real rank zero, while A may have no nontrivial projections at all. Since being projectionless is axiomatizable ([11, Theorem 2.5.1]), if A is projectionless then Łoś's Theorem implies that $A^{\mathcal{U}}$ is projectionless as well and A^{**} cannot be embeded into it. The anonymous referee pointed out an another, much subtler, obstruction. In the context of Banach spaces, the embeddability of X^{**} into $X^{\mathcal{U}}$ is equivalent to a finitary statement, the so-called *local reflexivity* of Banach spaces the C*-algebraic version of which does not hold for all C*-algebras [8, §5]. In particular, for a large class of C*-algebras the diagonal embedding of A into $A^{\mathcal{U}}$ cannot be extended even to a unital completely positive map from A^{**} into $A^{\mathcal{U}}$. The anonymous referee also pointed out that a result of J. M. G. Fell is closely related to results of the present section. It is a standard fact that a representation of a discrete group is weakly contained in another representation of the same group if and only if it can be embedded into an ultrapower of the direct sum of infinitely many copies of the latter representation. In [15, Theorem 1.2] it was essentially proved that this equivalence carries over to arbitrary C*-algebras.

All this said, Lemma 2.3 below is a poor man's C*-algebraic variant of the fact that Banach space X^{**} embeds into $X^{\mathcal{U}}$. As in [23, 3.3.6], we say that two representations π_1 and π_2 of A are said to be *equivalent* if the identity map on A extends to an isomorphism between $\pi_1(A)''$ and $\pi_2(A)''$.

Lemma 2.3. *Assume* $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$ *is an ultraproduct of faithful irreducible representations of unital* C*-*algebras and* C *is a unital separable subalgebra of* $B^{\mathcal{U}}$.

(1) *If* $C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j) = \{0\}$ *then the induced representation of* C *on* $\prod_{\mathcal{U}} H_j$ *is equivalent to the universal representation of* C.

(2) *In general, if*

$$p = \bigvee \{q : q \text{ is a projection in } C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)\}$$

then $p \in C' \cap B(\prod_{\mathcal{U}} H_j)$ *and* $c \mapsto (1-p)c$ *is equivalent to the universal representation of* $C/(C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j))$ *on* $(1-p)\prod_{\mathcal{U}} H_j$.

Proof. For a state ψ on C the $(0,1)$-type $\mathbf{t}_\psi(\xi)$ of a vector ξ implementing ψ consists of all conditions of the form $(a\xi|\xi) = \psi(a)$ for $a \in C$ and $\|\xi\| = 1$.

(1) Fix a state ψ on C. By Glimm's Lemma ([6, Lemma II.5.1]) type $\mathbf{t}_\psi$ is consistent with the theory of $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$. By the separability of C and countable saturation, there exists a unit vector $\eta \in \prod_{\mathcal{U}} H_j$ such that $\psi(c) = (c\eta|\eta)$ for all $c \in C$. Let L be the norm-closure of $C\eta$ in $\prod_{\mathcal{U}} H_j$. Then L is a reducing subspace for C and the induced representation of C on L is spatially isomorphic to the GNS representation of C corresponding to ψ. Since ψ was arbitrary, by [23, Theorem 3.8.2] this completes the proof.

(2) For every $a \in C$ we have $pa \in C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$ and therefore $pa(1-p) = 0$ Similarly $(1-p)ap = 0$, and hence $p \in C' \cap B(\prod_{\mathcal{U}} H_j)$. Let p_n, for $n \in \mathbb{N}$, be a maximal family of orthogonal projections in $C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$. It is countable by the separability of C and $p = \bigvee_n p_n$. Let ψ be a state of C that annihilates $C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$. Let $\mathbf{t}^+_\psi(\xi)$ be the type obtained from $\mathbf{t}_\psi(\xi)$ by adding to it all conditions of the form $p_n\xi = 0$ for $n \in \mathbb{N}$. By Glimm's Lemma (as stated in [6, Lemma II.5.1]) the type $\mathbf{t}^+_\psi(\xi)$ is consistent, and by the countable saturation we can find $\xi_1 \in \prod_{\mathcal{U}} H_j$ that realizes this type. Then $p\xi_1 = 0$ and therefore $\xi_1 \in (1-p)\prod_{\mathcal{U}} H_j$. Therefore every GNS representation of $C/(C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j))$ is spatially equivalent to a subrepresentation of $c \mapsto (1-p)c$, and by [23, Theorem 3.8.2] this concludes the proof. □

3. Second dual and Day's trick

The natural embedding of a C*-algebra B into its second dual B^{**} is rarely elementary. For example, having real rank zero is axiomatizable ([11, Theorem 2.5.1]) and B^{**}, being a von Neumann algebra, has real rank zero while B may have no nontrivial projections at all. However, we shall see that there is a restricted degree of elementarity between B and B^{**}, and it will suffice for our purposes.

We shall consider the language $(\mathcal{L}_{\mathrm{rep}})_B$ obtained by adding new constants for parameters in B (see §2). Term $\alpha(x)$ in the extended language is *linear* if it is of the form

$$\alpha(x) = xa + bx$$

for some parameters a and b.

A *restricted B-linear formula* is a formula of the form

(1) $\max_{j \leq m} \|\alpha_j(x) - b_j\| + \max_{j \leq n}(r_j \dot{-} \|\beta_j(x)\|)$

where

(2) all b_j, for $1 \leq j \leq m$ are parameters in B,
(3) all r_j, for $1 \leq j \leq n$ are positive real numbers,
(4) all α_j for $1 \leq j \leq m$ are linear terms with parameters in B, and
(5) all β_j for $1 \leq j \leq n$ are linear terms with parameters in B.

Proof of the following is based on an application of the Hahn–Banach separation theorem first used by Day ([7]; see also [9, Section 2] for some uses of this method in the theory of C*-algebras).

Lemma 3.1. *Suppose B is a unital* C*-*algebra and*

$$\gamma(x) = \max_{j \leq m} \|\alpha_j(x) - b_j\| + \max_{j \leq n}(r_j \dot{-} \|\beta_j(x)\|).$$

is a restricted B-linear formula. Then the following are equivalent.

(6) $\inf_{x \in B} \gamma(x) = 0$
(7) $\inf_{x \in B^{**}} \gamma(x) = 0$.

Proof. (6) implies (7) because B is isomorphic to a unital subalgebra of B^{**} and therefore $\inf_{x \in B^{**}} \gamma(x) \leq \inf_{x \in B} \gamma(x)$.

Assume (7) holds. Let a_j and c_j, for $j \leq n$, be such that $\alpha_j(x) = a_j x + x c_j$. For each j we shall identify α_j with its interpretation, linear map from B to B. The second adjoint $\alpha_j^{**} \colon B^{**} \to B^{**}$ also satisfies $\alpha_j^{**}(x) = a_j x + x c_j$, hence $\alpha_j^{**}(x)$ is the interpretation of the term $\alpha_j(x)$ in B^{**}. The set

$$Z := \langle \alpha_j(x) : x \in B_{\leq 1} \rangle$$

is, being an image of a convex set under a linear map, a convex subset of B^m and by the Hahn–Banach theorem

$$Z_1 := B^m \cap \langle \alpha_j(x) : x \in B^{**}_{\leq 1} \rangle$$

is included in the norm-closure of Z. By (7) we have $(b_1, \dots, b_m) \in Z_1$.

Fix $\varepsilon > 0$ and let

$$X_1 := \{x \in B_{\leq 1} : \max_{j \leq m} \|\alpha_j(x) - b_j\| \leq \varepsilon\}.$$

By the above this is a convex subset of the unit ball of B and (by using the Hahn–Banach separation theorem again) the weak*-closure of X_1 in B^{**} is equal to $\{x \in B^{**}_{\leq 1} : \max_{j \leq m} \|\alpha_j(x) - b_j\| \leq \varepsilon\}$.

Let $c \in B^{**}_{\leq 1}$ be such that $\gamma(c) < \varepsilon$. Then c belongs to the weak*-closure of X_1. For each $j \leq n$ we have $\|\beta_j(c)\| > r_j - \varepsilon$. Fix a norming functional $\varphi_j \in B^*$ such that $\|\varphi_j\| = 1$ and $\varphi_j(\beta_j(c)) > r_j - \varepsilon$. Then

$$U := \{x \in B^{**} : \varphi_j(\beta_j(x)) > r_j - \varepsilon \text{ for all } j\}$$

is a weak*-open neighbourhood of c and, as c belongs to the weak*-closure of X_1, $U \cap X_1$ is a nonempty subset of $B_{\leq 1}$. Any $b \in U \cap X_1$ satisfies $\gamma(b) < \varepsilon$. As $\varepsilon > 0$ was arbitrary, this shows that (6) holds. □

In the following $A \subseteq \prod_{\mathcal{U}} B_j$ is identified with a subalgebra of $B(\prod_{\mathcal{U}} H_j)$.

Lemma 3.2. *Suppose (B_j, H_j) is an irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$.*

(1) *For every $b \in \prod_{\mathcal{U}} B_j$ we have $b \in (A' \cap B(\prod_{\mathcal{U}} H_j))'$ if and only if $b \in (A' \cap \prod_{\mathcal{U}} B_j)'$.*
(2) $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = (A' \cap \prod_{\mathcal{U}} B_j)'$.

Proof. (1) Since $\prod_{\mathcal{U}} B_j \subseteq B(\prod_{\mathcal{U}} H_j)$ we clearly have $(A' \cap B(\prod_{\mathcal{U}} H_j))' \subseteq (A' \cap \prod_{\mathcal{U}} B_j)'$. In order to prove the converse inclusion, fix $b \in \prod_{\mathcal{U}} B_j$ and suppose that there exists $q \in A' \cap B(\prod_{\mathcal{U}} H_j)$ such that $\|[q, b]\| = r > 0$. We need to find $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $[d, b] \neq 0$.

Consider the $(1, 0)$-type $\mathbf{t}(x)$ consisting of all conditions of the form

$$\|[x, b]\| \geq r \quad \text{and} \quad [x, a] = 0$$

for $a \in A$. This type is satisfied in $B(\prod_{\mathcal{U}} H_j)$ by q. Since all formulas in $\mathbf{t}(x)$ are quantifier-free, their interpretation is unchanged when passing to a different algebra. Although in general a quantifier-free type realized in a larger algebra is not necessarily consistent with the theory of the smaller algebra, Lemma 3.1 implies that in our situation this is the case.

Fix a finite subset of $\mathbf{t}(x)$ and let $F \subseteq A$ be the set of parameters occurring in this subset. Then

$$\gamma_F(x) := \inf_x \max_{a \in F} \|[x, a]\| + (r \dot{-} \|[x, b]\|)$$

is a restricted $\prod_{\mathcal{U}} B_j$-linear formula. Since A is separable, we can find projection p in $\mathrm{C}^*(A, q)' \cap B(\prod_{\mathcal{U}} H_j)$ with separable range such that $q_1 := pq$ satisfies $\|[q_1, b]\| = r$.

By the Downward Löwenheim–Skolem Theorem ([14, Theorem 4.6]) there exists a separable elementary submodel (D, K) of $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$ such that $C \subseteq D$ and the range of p is included in K. Part (2) of Lemma 1.2 and Łoś' Theorem imply that $\overline{\prod_{\mathcal{U}} B_j}^{\mathrm{WOT}} = B(\prod_{\mathcal{U}} H_j)$ and (with p_K denoting the projection to K) that $\overline{p_K D p_K}^{\mathrm{WOT}} = B(p_K \prod_{\mathcal{U}} H_j)$. We can therefore identify p_K with a minimal central projection in D^{**}. Via this identification we have $q_1 \in D^{**}$. Since $\gamma_F(q_1) = 0$, Lemma 3.1 implies $\inf_{x \in D, \|x\| \leq 1} \gamma_F(x) = 0$ and (since $|\gamma_F$ is quantifier-free) $\inf_{x \in \prod_{\mathcal{U}} B_j, \|x\| \leq 1} \gamma_F(x) = 0$.

Since F was an arbitrary finite subset of A type $\mathbf{t}(x)$ is consistent with the theory of $\prod_{\mathcal{U}} B_j$. Since A is separable, by the countable saturation there exists $d \in A' \cap \prod_{\mathcal{U}} B_j$ satisfying $\|[d, b]\| \geq r$.

(2) By the von Neumann bicommutant theorem $\overline{A}^{\mathrm{WOT}} = (A' \cap B(\prod_{\mathcal{U}} H_j))'$ and therefore (1) implies $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = (A' \cap \prod_{\mathcal{U}} B_j)'$. □

4. Proof of Theorem 1

Suppose (B_j, H_j) is a faithful irreducible representation of B_j on H_j for $j \in \mathbb{N}$ and A is a separable subalgebra of $\prod_{\mathcal{U}} B_j$. By Lemma 1.2 $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$ is an irreducible faithful representation of $\prod_{\mathcal{U}} B_j$.

By (2) of Lemma 3.2 we have $\overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j = (A' \cap \prod_{\mathcal{U}} B_j)'$. Since $A \subseteq (A' \cap \prod_{\mathcal{U}} B_j)'$, it remains to prove $(A' \cap \prod_{\mathcal{U}} B_j)' \subseteq A$. Fix $b \in \prod_{\mathcal{U}} B_j$ such that $r := \mathrm{dist}(b, A) > 0$. By (1) of Lemma 3.2 it suffices to find $d \in A' \cap B(\prod_{\mathcal{U}} H_j)$ such that $[d, b] \neq 0$. Let

$$C := \mathrm{C}^*(A, b).$$

Lemma 4.1. *With $\prod_{\mathcal{U}} B_j$, A, b, C and r as above, there exists a representation $\pi\colon C/(C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)) \to B(K)$ and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.*

Since the proof of Lemma 4.1 is on the long side, let us show how it completes the proof of Theorem 1. Lemma 2.3 implies that if

$$p = \bigvee \{q : q \text{ is a projection in } C \cap \mathcal{K}(\textstyle\prod_{\mathcal{U}} H_j)\}$$

then $p \in C' \cap B(\prod_{\mathcal{U}} H_j)$ and $c \mapsto (1-p)c$ is equivalent to the universal representation of $C/(C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j))$ on $(1-p)\prod_{\mathcal{U}} H_j$. Therefore q as in the conclusion of Lemma 4.1 can be found in $A' \cap B(\prod_{\mathcal{U}} H_j)$, implying $b \notin (A' \cap B(\prod_{\mathcal{U}} H_j))'$. By Lemma 3.2 this implies $b \notin (A' \cap \prod_{\mathcal{U}} B_j)'$, reducing the proof of Theorem 1 to the following.

Proof of Lemma 4.1. An easy special case may be worth noting. If $C \cap \mathcal{K}(\prod_{\mathcal{U}} H_j) = \{0\}$ then Lemma 2.2 implies the existence of a representation $\pi\colon C \to B(K)$ and $q \in \pi(A)' \cap B(K)$ such that $[q, \pi(b)] \neq 0$.

In the general case, let q_n, for $n \in J$, be an enumeration of a maximal orthogonal set of minimal projections in $A \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$. The index-set J is countable (and possibly finite or even empty) since A is separable. Let $p_n := \bigvee_{j \leq n} q_j$.

Suppose for a moment that there exists n such that $p_n b p_n \notin A$. Since the range of p_n is finite-dimensional, by von Neumann's Bicommutant Theorem ([4, §I.9.1.2]). and the Kadison Transitivity Theorem ([4, Theorem II.6.1.13]) we have $p_n b p_n \notin (A' \cap B(\prod_{\mathcal{U}} H_j))'$. Lemma 3.2 now implies $p_n b p_n \notin (A' \cap \prod_{\mathcal{U}} B_j)'$ and $b \notin (A' \cap \prod_{\mathcal{U}} B_j)'$.

We may therefore assume $p_n b p_n \in A$ for all n. Let $p = \bigvee_n p_n$. Lemma 2.3 (2) implies $p \in A' \cap B(\prod_{\mathcal{U}} H_j)$, and we may therefore assume $[b, p] = 0$. Since $C = \mathrm{C}^*(A, b)$ this implies $p \in C' \cap B(\prod_{\mathcal{U}} H_j)$. Since $p_n b p_n \in A$ for all n we have $A \cap \mathcal{K}(\prod_{\mathcal{U}} H_j) pCp \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$. If $c \in C$ then for every n we have $p_n c(1-p) = 0$ and similarly $(1-p)cp_n = 0$. Since the sequence p_n, for $n \in \mathbb{N}$, is an approximate unit for $A \cap \mathcal{K}(\prod_{\mathcal{U}} H_j)$, the latter is an ideal of C. Let $\theta\colon C \to C/(A \cap \mathcal{K})$ be the quotient map. We claim that $\mathrm{dist}(\theta(b), \theta(A)) = \mathrm{dist}(b, A) > 0$. Fix $a \in A$. We need to show that $\|\theta(a - b)\| \geq r$.

Consider the $(0,1)$-type $\mathbf{t}(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \quad \|(a-b)\xi\| \geq r, \quad p_n\xi = 0,$$

for $n \in J$. To see that this type is consistent fix a finite $F \subseteq J$. Let $q := \sum_{n\in F} p_n$ and

$$a' := (1-q)a(1-q) + \sum_{n\in F} p_n b p_n.$$

As both summands belong to A, $a' \in A$ and therefore $\|a'-b\| \geq r$. Fix $\varepsilon > 0$. If $\xi \in \prod_{\mathcal{U}} H_j$ is a vector of norm ≤ 1 such that $\|(a'-b)\xi\| > r - \varepsilon$ then $\xi' = (1-q)\xi$ has the same property since $(a'-b)q = 0$. Since $\varepsilon > 0$ was arbitrary, $\mathbf{t}(\xi)$ is consistent. By the countable saturation there exists a unit vector $\xi \in \prod_{\mathcal{U}} H_j$ which realizes $\mathbf{t}(\xi)$. Since $p_n\xi = 0$ for all n we have $p\xi = 0$ and therefore $\|\theta(a-b)\| \geq \|(1-p)(a-b)(1-p)\| \geq r$. Since $a \in A$ was arbitrary, we conclude that $\operatorname{dist}(\theta(b), \theta(A)) = r$.

Suppose for a moment that $(1-p)C(1-p) \cap \mathcal{K}(\prod_{\mathcal{U}} H_j) = \{0\}$. By (2) of Lemma 2.3 the representation

$$C \ni c \mapsto (1-p)c \in B((1-p)\textstyle\prod_{\mathcal{U}} H_j)$$

is equivalent to the universal representation of C. Hence by Lemma 2.2 we can find $d \in (1-p)(A' \cap B(\prod_{\mathcal{U}} H_j))$ that does not commute with b, and by the above this concludes the proof in this case.

We may therefore assume that $(1-p)C(1-p) \cap \mathcal{K}(\prod_{\mathcal{U}} H_j) \neq \{0\}$. By the spectral theorem for self-adjoint compact operators and continuous functional calculus, there exists a nonzero projection $q \in (1-p)C(1-p)$ of finite rank. Fix $c \in C$ such that $(1-p)c(1-p) = q$.

By Lemma 3.2 it suffices to find $q \in A' \cap (1-p)B(\prod_{\mathcal{U}} H_j)(1-p)$ such that $[q, c] \neq 0$. Suppose otherwise, that $c \in (A' \cap \prod_{\mathcal{U}} B_j)'$. (2) of Lemma 3.2 implies that $c \in \overline{A}^{\mathrm{WOT}}$. By the Kaplansky Density Theorem ([4, Theorem I.9.1.3]) there is a net of positive contractions in A converging to c in the weak operator topology. By the continuous functional calculus and the Kadison Transitivity Theorem ([4, Theorem II.6.1.13]) we may choose this net among the members of

$$Z := \{a \in A_+ : \|a\| = 1, qaq = q\}.$$

Consider the $(0,1)$-type $\mathbf{t}_1(\xi)$ consisting of all conditions of the form

$$\|\xi\| = 1, \quad a\xi = \xi, \quad q\xi = 0, \quad p_n\xi = 0$$

for $n \in \mathbb{N}$ and $a \in Z$.

We claim that $\mathbf{t}_1(\xi)$ is consistent. Fix $\varepsilon > 0$ and $a_1, a_2, \dots, a_n$ in Z. Let

$$a := a_1 a_2 \dots a_{n-1} a_n a_{n-1} \dots a_2 a_1.$$

Then $a \in Z$ and $q \leq a$. By the choice of p the operator $(1-p)(a-s)$ is not compact for any $s < 1$. Therefore there exists a unit vector ξ_0 in $(1-p-q)\prod_{\mathcal{U}} H_j$ such that $\|\xi_0 - a\xi_0\|$ is arbitrarily small. By the countable saturation there exists a unit vector $\xi_1 \in (1-(p+q))\prod_{\mathcal{U}} H_j$ such that

$a\xi_1 = \xi_1$. As each a_j is a positive contraction, we have $a_j\xi_1 = \xi_1$ for $1 \leq j \leq n$. Since $a_1, \dots, a_n$ was an arbitrary subset of Z, this shows that $\mathbf{t}_1(\xi)$ is consistent. Since Z is separable, by the countable saturation there exists $\xi \in \prod_{\mathcal{U}} H_j$ realizing $\mathbf{t}_1(\xi)$. Then ξ is a unit vector in $(1-(p+q))\prod_{\mathcal{U}} H_j$ such that $a\xi = \xi$ for all $a \in Z$. As $c\xi = 0$, this contradicts c being in the weak operator topology closure of Z.

Therefore there exists $q \in A' \cap (1-p)B(\prod_{\mathcal{U}} H_j)(1-p)$ such that $[q,c] \neq 0$. Since $c \in C = \mathrm{C}^*(A,b)$ we have $[q,b] \neq 0$, and this concludes the proof. $\square$

5. Concluding remarks

In the following infinitary form of the Kadison Transitivity Theorem p_K denotes projection to a closed subspace K of $\prod_{\mathcal{U}} H_j$.

Proposition 5.1. *Assume $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$ is an ultraproduct of faithful and irreducible representations of unital* C^**-algebras. Also assume K is a separable closed subspace of $\prod_{\mathcal{U}} H_j$ and $T \in B(K)$.*

1. *There exists $b \in \prod_{\mathcal{U}} B_j$ such that $\|b\| = \|T\|$ and $p_K b p_K = T$.*
2. *If T is self-adjoint (or positive, or unitary in $B(K)$) then b can be chosen to be self-adjoint (or positive, or unitary in $B(\prod_{\mathcal{U}} H_j)$).*

Proof. (1) is a consequence of the Kadison transitivity theorem and countable saturation of the structure $(\prod_{\mathcal{U}} B_j, \prod_{\mathcal{U}} H_j)$. Let p_n, for $n \in \mathbb{N}$, be an increasing sequence of finite-dimensional projections converging to p_K in the strong operator topology and let a_n, for $n \in \mathbb{N}$, be a dense subset of A. We need to check that the type $\mathbf{t}(x)$ consisting of all conditions of the form

$$\|p_n(x-T)p_n\| = 0, \qquad \|x\| = \|T\|$$

for $n \in \mathbb{N}$ is consistent. Since the representation of $\prod_{\mathcal{U}} B_j$ on $\prod_{\mathcal{U}} H_j$ is irreducible by Lemma 1.2, every finite subset of $\mathbf{t}(x)$ is consistent by the Kadison Transitivity Theorem. We can therefore find $b \in \prod_{\mathcal{U}} B_j$ that satisfies $\mathbf{t}(x)$ and therefore $p_K b p_K = T$ and $\|b\| = \|T\|$.

(2) If T is self-adjoint, add the condition $x = x^*$ to $\mathbf{t}(x)$. By [23, Theorem 2.7.5] the corresponding type is consistent, and the assertion again follows by countable saturation. The case when T is a unitary also uses [23, Theorem 2.7.5]. $\square$

An important consequence of full Voiculescu's theorem is that any two unital representations $\pi_j \colon A \to B(H)$ of a separable unital C^*-algebra A on H such that $\ker(\pi_1) = \ker(\pi_2)$ and $\pi_1(A) \cap \mathcal{K}(H) = \pi_2(A) \cap \mathcal{K}(H) = \{0\}$ are approximately unitarily equivalent ([28, Corollary 1.4]). The analogous statement is in general false for the ultraproducts. Let $B_n = M_n(\mathbb{C})$ for $n \in \mathbb{N}$ and let $A = \mathbb{C}^2$. Group $K_0(\prod_{\mathcal{U}} M_n(\mathbb{C}))$ is isomorphic to $\mathbb{Z}^{\mathbb{N}}$ with the natural ordering and the identity function id as the order-unit. Every unital representation of A corresponds to an element of this group that lies between 0 and id, and there are $2^{\aleph_0}$ inequivalent representations. Also,

$K_0(\prod_{\mathcal{U}} M_n(\mathbb{C}))$ is isomorphic to the ultraproduct $\prod_{\mathcal{U}} \mathbb{Z}$ and $2^{\aleph_0}$ of these extensions remain inequivalent even after passing to the ultraproduct.

We return to G. K. Pedersen's question ([24, Remark 10.11]), whether a bicommutant theorem $(A' \cap M(B)/B)' = A$ is true for a separable unital subalgebra A of a corona $M(B)/B$ of a σ-unital C*-algebra B? A simple and unital C*-algebra C is *purely infinite* if for every nonzero $a \in C$ there are x and y such that $xay = 1$.

Question 5.2. *Suppose C is a unital, simple, purely infinite, and separable and A is a unital subalgebra of C. Is $(A' \cap C^{**})' \cap C = A$?*

Let us prove that a positive answer to Question 5.2 would imply a positive answer to Pedersen's question. If A is a separable and unital subalgebra of $M(B)/B$ and $b \in M(B)/B \setminus A$, then there exists a separable elementary submodel C of $M(B)/B$ containing b. By [21], $M(B)/B$ is simple if and only if it is purely infinite, Since being simple and purely infinite is axiomatizable ([11, Theorem 2.5.1]), C is simple and purely infinite. If $(A' \cap C^{**})' \cap C = A$ then Proposition 5.3 below implies that there exists $d \in A' \cap M(B)/B$ such that $[d, b] \neq 0$.

Proposition 5.3. *Suppose B is a* C**-algebra, A is a separable subalgebra of B, $b \in B$ and $r \geq 0$. If B is an ultraproduct or a corona of a σ-unital, non-unital* C**-algebra then*

$$\sup_{d \in (A' \cap B)_+, \|d\| \leq 1} \|[d, b]\| = \sup_{d \in (A' \cap B^{**})_+, \|d\| \leq 1} \|[d, b]\|.$$

Proof. The only property of B used in the proof of Proposition 5.3 (given at the end of this section) is that of being countably degree-1 saturated ([10, Theorem 1]). Since $B \subseteq B^{**}$, it suffices to prove '$\geq$' in the above inequality. Suppose $b \in B$ and $d \in (A' \cap B^{**})_+$ are such that $\|d\| = 1$ and $r \dot{-} \|[b, d]\|$. Consider the type $\mathbf{t}(x)$ consisting of conditions $\|x\| = 1$, $x \geq 0$, $\|xb - bx\| \geq r$, and $\|[x, a]\| = 0$ for a in a countable dense subset of A, This is a countable degree-1 type. If $\phi_j = 0$, for $j < n$, is a finite subset of $\mathbf{t}(x)$ then $\gamma(x) := \max_{j<n} \phi_j(x)$ is a restricted B-linear formula and Lemma 2.3 implies that it is approximately satisfied in B. By the countable degree-1 saturation of B ([10, Theorem 1]) we can find a realization d' of $\mathbf{t}(x)$ in B. Clearly $d' \in (A' \cap B)_+$, $\|d'\| = 1$, and $\|[d', b]\| \geq r$, completing the proof. □

Some information on a special case of Pedersen's conjecture can also be found in [20].

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