HOMEOMORPHISMS OF ČECH–STONE REMAINDERS: THE ZERO-DIMENSIONAL CASE

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Abstract. We prove, using a weakening of the Proper Forcing Axiom, that any homemomorphism between Čech–Stone remainders of any two locally compact, zero-dimensional Polish spaces is induced by a homeomorphism between their cocompact subspaces.

1. Introduction

The Čech–Stone remainder (also known as corona) $\beta X \setminus X$ of a topological space X will be denoted X^*. A continuous map $\varphi : X^* \to Y^*$ is called *trivial* if there is a continuous $e : X \to Y$ such that $\varphi = e^*$, where $e^* = \beta e \setminus e$ and βe is the unique continuous extension of e to βX. It follows that two remainders X^* and Y^* are homeomorphic via a trivial map if and only if there are cocompact subspaces of X and Y which are themselves homeomorphic. In this paper we prove the following (see §2 for the definitions).

Theorem 1.1. *OCA and $MA_{\aleph_1}$ together imply that every homeomorphism between Čech–Stone remainders of locally compact, zero-dimensional, Polish spaces is trivial.*

This proves a special case of the rigidity conjecture that forcing axioms imply all homeomorphisms between Čech–Stone remainders of locally compact, noncompact Polish spaces are trivial (see [10], [9], [3]). In contrast, the Continuum Hypothesis (CH), implies that Čech–Stone remainders of locally compact, noncompact, zero-dimensional Polish spaces are homeomorphic. This is a consequence of Parovičenko's topological characterization of ω^* (see e.g., [25]). Stone duality between compact, zero-dimensional, Hausdorff spaces and Boolean algebras of their clopen sets provides a model-theoretic reformulation of this malleability phenomenon. For a locally compact, noncompact Hausdorff space X let $\mathscr{C}(X)$ denote the algebra the clopen subsets of X and let $\mathscr{K}(X)$ denote its ideal of compact-open sets. If X and Y are in addition zero-dimensional, then continuous maps from X^* to Y^* functorially correspond to Boolean algebra homomorphisms from $\mathscr{C}(Y)/\mathscr{K}(Y)$ into $\mathscr{C}(X)/\mathscr{K}(X)$. All of these algebras are elementarily equivalent and (assuming CH) saturated, and therefore isomorphic (see [6] for the details and an extension to not necessarily zero-dimensional spaces).[1]

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[1] There is a deeper metamathematical explanation of the effect of CH; see [31].

Back to rigidity, Theorem 1.1 belongs to a long line of results going back to Shelah's groundbreaking construction of an oracle-cc forcing extension of the universe in which all autohomeomorphisms of ω^* are trivial ([22]). Shelah's proof was recast in terms of forcing axioms PFA and OCA+$\mathrm{MA}_{\aleph_1}$ in [23] and [27], respectively. The latter axiom also implies that homeomorphisms between Čech–Stone remainders between countable locally compact spaces, as well as their arbitrary powers, are trivial ([9, §4]) as well as strong negations of Parovičenko's theorem ([5], [7]).

The interest in quotient rigidity results was rejuvenated by the discovery that the noncommutative analogue of 'are all automorphisms of ω^* (or of $\mathcal{P}(\omega)/\mathrm{fin}$) trivial?' was a prominent open problem in the theory of operator algebras. Motivated by their work on analytic K-homology, Brown, Douglas, and Fillmore asked whether the Calkin algebra associated with the separable, infinite-dimensional, complex Hilbert space has outer automorphisms ([2]). Like its commutative analogue, this question cannot be resolved in ZFC, with CH and OCA implying the opposite answers ([21], [11]). Other rigidity results in the setting of C*-algebras were proved for reduced products of the form $\prod_n A_n / \bigoplus_n A_n$ in case when all A_n are matrix algebras ([16], [15]), separable UHF algebras ([19]) or unital separable nuclear C*-algebras ([28], [20]).

A general rigidity conjecture for corona C*-algebras was stated and partially verified in [3]. The model theory of coronas proved to be a bit more complex than that of Boolean algebras. While the reduced products are countably saturated ([14]), coronas posses only a modest degree of saturation ([12], [8], [30], [13]). In return, C*-algebras provided a vantage point that resulted in the construction of nontrivial autohomeomorphisms of X^* for every noncompact, locally compact, metrizable manifold using CH ([29]).[2]

We note that Theorem 1.1 is not optimal. The first author's proof that all zero-dimensional, locally compact, Polish spaces satisfy the *weak extension principle* ([9, Theorem 4.10.1]) will appear elsewhere. Dow refuted the related *strong extension principle* ([9, Question 4.11.4]) by constructing a nontrivial continuous map from ω^* into ω^* (i.e., one that does not have a continuous extension to a map from $\beta\omega$ into $\beta\omega$) in ZFC ([4]). An alternative proof of our main result from a stronger assumption (PFA) is given by [14, Theorem 4.3].

In section 2 we introduce some of the language required to prove Theorem 1.1. Section 3 treats embeddings of $\mathscr{P}(\omega)/\mathrm{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$, and we show that under OCA +$\mathrm{MA}_{\aleph_1}$, every such embedding is trivial. Much of the proof follows the work in [27] and [26] with only minor modifications, so to avoid treading the same ground we only prove one of the ingredients going into this theorem. Section 4 completes the proof of Theorem 1.1 through an analysis of *coherent families of continuous functions.*

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[2]The only previously known case was $X = \mathbb{R}$, see [17] and [14]

2. Notation

Out terminology is standard (see [18]). The assumption of Theorem 1.1 is a consequence of the *Proper Forcing Axiom*, PFA. OCA abbreviates the *Open Coloring Axiom* ([24]; not to be confused with the eponymous OCA of [1]), and $\mathrm{MA}_{\aleph_1}$ refers to Martin's Axiom for $\aleph_1$ dense sets.

If E is a set, then $[E]^2$ will denote the set of unordered pairs from E. If $M \subseteq [E]^2$, then a set $H \subseteq E$ is called M-*homogeneous* if $[H]^2 \subseteq M$. The *Open Coloring Axiom* states: for every separable metric space E and every partition $[E]^2 = M_0 \cup M_1$ such that M_0 is open (here we identify $[E]^2$ with a symmetric subset of $E \times E$ minus the diagonal), either

(1) there is an uncountable M_0-homogeneous set, or
(2) there is a cover of E by countably-many M_1-homogeneous sets.

We fix a zero-dimensional, locally compact and noncompact Polish space X. Let $\langle K_n \mid n < \omega\rangle$ be an increasing sequence of compact-open sets in X, such that $X = \bigcup K_n$. Then $\mathscr{K}(X)$ is generated by $\langle K_n \mid n < \omega\rangle$ since

$$K \in \mathscr{K}(X) \iff \exists n\, K \subseteq K_n$$

It is easy to see that $\mathscr{C}(X)$ has size continuum, whereas $\mathscr{K}(X)$ is countable. When $A, B \in \mathscr{C}(X)$ are distinct, we write $\delta(A,B)$ for the least n such that $A \cap X_n \neq B \cap X_n$. If

$$d(A,B) = \begin{cases} 2^{-\delta(A,B)} & A \neq B \\ 0 & A = B \end{cases}$$

then d is a Polish metric on $\mathscr{C}(X)$.

Let $X_0 = K_0$ and $X_{n+1} = K_{n+1} \setminus K_n$. We will often identify $\mathscr{C}(X)$ with $\prod_n \mathscr{C}(X_n)$, and $\mathscr{P}(\omega)$ with ${}^{\omega}2$. Under these identifications, $\mathscr{K}(X)$ maps to $\bigoplus_n \mathscr{C}(X_n)$ (the set of functions in $\prod_n \mathscr{C}(X_n)$ which are nonempty on only finitely many coordinates) and fin to ${}^{<\omega}2$. If Y and Z are zero-dimensional, locally compact Polish spaces, $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(Z)/\mathscr{K}(Z)$ is a homomorphism, and $U \in \mathscr{C}(Y)$, then we write $\varphi\restriction U$ for the restriction $\varphi\restriction\mathscr{C}(U)/\mathscr{K}(U)$. When working with the quotient $\mathscr{C}(X)/\mathscr{K}(X)$ we will write $[A]$ for the equivalence class of some $A \in \mathscr{C}(X)$.

3. Embeddings of $\mathscr{P}(\omega)/\mathrm{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$

Let $e : X \to \omega$ be a continuous map. If $e^{-1}(n)$ is compact for every n, then we say e is *compact-to-one*. If e is compact-to-one, then the map $a \mapsto e^{-1}(a)$, from $\mathscr{P}(\omega)$ to $\mathscr{C}(X)$, induces a homomorphism $\varphi_e : \mathscr{P}(\omega)/\mathrm{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$. Moreover, φ_e is injective if and only if e is finite on compact sets. We call a homomorphism $\varphi : \mathscr{P}(\omega)/\mathrm{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$ *trivial* if it is of the form φ_e for some compact-to-one, continuous e.

In this section we prove

Theorem 3.1. *Assume OCA+$\mathrm{MA}_{\aleph_1}$, and suppose*

$$\varphi : \mathscr{P}(\omega)/\mathrm{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$$

is an injective homomorphism. Then φ is trivial.

Towards the proof of Theorem 3.1, we fix an injective homomorphism $\varphi : \mathscr{P}(\omega)/\mathrm{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$ and we define

$$\mathscr{I} = \{a \subseteq \omega \mid \varphi \restriction a \text{ is trivial}\}$$

Note that $\mathscr{I}$ is an ideal on ω.

A family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is called *almost disjoint* if for all distinct $a, b \in \mathscr{A}$, $a \cap b =^* \emptyset$. Such a family $\mathscr{A}$ is called *treelike* if there is some tree T on ω and a bijection $t : \omega \to {}^{<\omega}\omega$ under which each $a \in \mathscr{A}$ corresponds to a branch through T, and vice-versa. The following lemma is proven in [27].

Lemma 3.2. *Assume $MA_{\aleph_1}$. Then for every uncountable almost-disjoint family $\mathscr{A}$ of subsets of ω we may find an uncountable $\mathscr{B} \subseteq \mathscr{A}$ and partitions $b = b_0 \cup b_1$ for $b \in \mathscr{B}$ such that each family $\mathscr{B}_i = \{b_i \mid b \in \mathscr{B}\}$ is treelike.*

The following three lemmas do not directly follow from the work in [27], but their proofs are nearly the same, modulo some minor modifications. Recall that an ideal $\mathscr{J} \subseteq \mathscr{P}(\omega)$ is a *P-ideal* if for each countable sequence $A_n \in \mathscr{J}$ $(n < \omega)$ there is an $A \in \mathscr{J}$ such that for all $n < \omega$, $A_n \subseteq^* A$.

Lemma 3.3. *Assume OCA+$MA_{\aleph_1}$. If $\mathscr{I}$ is a dense P-ideal then φ is trivial.*

Lemma 3.4. *Assume $\mathfrak{b} > \aleph_1$. If $\mathscr{I}$ is not a dense P-ideal, then there is an uncountable almost disjoint family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ which is disjoint from $\mathscr{I}$.*

Lemma 3.5. *Assume OCA. Let $\mathscr{A}$ be an uncountable, tree-like, almost-disjoint family of subsets of ω. Then $\mathscr{I} \setminus \mathscr{A}$ is countable.*

Theorem 3.1 now follows from a straightforward combination of Lemmas 3.2, 3.3, 3.4 and 3.5. To illustrate the kind of modifications necessary in translating from [27], we will give a proof of Lemma 3.3.

Proof of Lemma 3.3. For each $a \in \mathscr{I}$, we fix $Z_a \in \mathscr{C}(X)$ and a continuous, compact-to-one map $e_a : Z_a \to a$ such that $\varphi([a]) = [Z_a]$ and for all $b \subseteq a$, $\varphi([b]) = [e_a^{-1}(b)]$. We define $f_a : \omega \to \mathscr{C}(X)$ by

$$f_a(n) = e_a^{-1}(\{n\})$$

Define a partition $[\mathscr{I}]^2 = M_0 \cup M_1$ by placing $\{a, b\} \in M_0$ if and only if there is some $n \in a \cap b$ such that $f_a(n) \neq f_b(n)$. Then M_0 is open when $\mathscr{I}$ is given the topology obtained by identifying $a \in \mathscr{I}$ with $(a, f_a) \in \mathscr{P}(\omega) \times {}^{\omega}\mathscr{C}(X)$.

Claim 3.6. There is no uncountable, M_0-homogeneous subset H of $\mathscr{I}$.

Proof. Assume H is such a set, and that $|H| = \aleph_1$. Since $\mathscr{I}$ is a P-ideal, there is a set $\bar{H} \subseteq \mathscr{I}$ such that for every $a \in H$ there is some $b \in \bar{H}$ with $a \subseteq^* b$, and moreover $\bar{H}$ is a chain of order-type ω_1 with respect to $\subseteq^*$. By OCA, there is an uncountable subset of $\bar{H}$ which is homogeneous for one of the two colors M_0 and M_1; hence, by passing to this subset, we may assume $\bar{H}$ is either M_0 or M_1 homogeneous.

Say $\bar{H}$ is M_1-homogeneous. Put $\bar{a} = \bigcup \bar{H}$, and $\bar{f} = \bigcup_{a\in\bar{H}} f_a$. Then $\bar{f} : \bar{a} \to \mathscr{C}(X)$, and for all $a \in H$ we have $a \subseteq^* \bar{a}$ and $f_{\bar{a}}\restriction(a\cap\bar{a}) =^* f_a\restriction(a\cap\bar{a})$. Choose n so that for uncountably many $a \in H$, we have $a \setminus n \subseteq \bar{a}$, and $f_{\bar{a}}\restriction a \setminus n = f_a\restriction a \setminus n$. Then if $a, b \in H$ are such, and $f_a\restriction n = f_b\restriction n$, we have $\{a, b\} \in M_1$, a contradiction.

So $\bar{H}$ is M_0-homogeneous. Define a poset $\mathbb{P}$ as follows. Put $p \in \mathbb{P}$ if and only if $p = (A_p, m_p, H_p)$ where $m_p < \omega$, $A_p \in \mathscr{C}(K_{m_p})$, and $H_p \in [\bar{H}]^{<\omega}$, and for all distinct $a, b \in H_p$, there is an $n \in a \cap b$ such that

$$\neg(f_a(n) \cap A_p = \emptyset \iff f_b(n) \cap A_p = \emptyset)$$

That is, one of $f_a(n)$, $f_b(n)$ is disjoint from A_p, and the other isn't. Put $p \le q$ if and only if $m_p \ge m_q$, $A_p \cap K_{m_q} = A_q$, and $H_p \supseteq H_q$.

First we must show that $\mathbb{P}$ is ccc. Suppose $\mathcal{X}$ is an uncountable subset of $\mathbb{P}$. We may assume without loss of generality that for some fixed m and $A \in \mathscr{C}(K_m)$, and for all $p \in \mathcal{X}$, $m_p = m$ and $A_p = A$, and moreover that H_p is the same size for all $p \in \mathcal{X}$. Let a_p be the minimal element of H_p under $\subseteq^*$, for each $p \in \mathcal{X}$. Find n_p so that for all $a \in H_p$,

$$f_{a_p}\restriction(a_p \setminus n_p) \subseteq f_a \qquad e''_{a_p} K_m \subseteq n_p$$

We may assume that for some fixed n, we have $n_p = n$ for all $p \in \mathcal{X}$. Find $p, q \in \mathcal{X}$ with $f_{a_p}\restriction n = f_{a_q}\restriction n$. Since $\{a_p, a_q\} \in M_0$, there is some $k \in a_p \cap a_q$ such that $f_{a_p}(k) \neq f_{a_q}(k)$. Then $k \ge n$, and so $f_{a_p}(k) \cap K_m = f_{a_q}(k) \cap K_m = \emptyset$. At least one of $f_{a_p}(k) \setminus f_{a_q}(k)$ and $f_{a_q}(k) \setminus f_{a_p}(k)$ must be nonempty; whichever one it is, call it B. Put $A_r = A \cup B$ and $H_r = H_p \cup H_q$, and choose m_r large enough that $A_r \subseteq K_{m_r}$. Then $r = (A_r, m_r, H_r) \in \mathbb{P}$, and $r \le p, q$.

By $\mathrm{MA}_{\aleph_1}$, there is a set $A \in \mathscr{C}(X)$ and an uncountable $H^* \subseteq \bar{H}$ such that for all distinct $a, b \in H^*$,

$$\exists n \in a \cap b \quad \neg(f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

Fix $x \subseteq \omega$ such that $F(x) = A$. Then for all $a \in H^*$, $e_a^{-1}(x \cap a)\Delta(A \cap F(a))$ is compact; hence there are k_a and m_a such that

$$e_a^{-1}(x \cap a \setminus k_a) = (A \cap F(a)) \setminus K_{m_a} \qquad \text{and} \qquad e_a^{-1}(a \setminus k_a) = F(a) \setminus K_{m_a}$$

Then, for all $n \in a \setminus k_a$, $n \in x$ implies $f_a(n) \subseteq A$, and $n \notin x$ implies $f_a(n) \cap A = \emptyset$. Fix distinct $a, b \in H^*$ with $k_a = k_b = k$, and $f_a\restriction k = f_b\restriction k$. Then,

$$\forall n \in a \cap b\ (f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

This contradicts the choice of A. □

By OCA, there is a cover of $\mathscr{I}$ by countably many sets $\mathscr{I}_n$, each of which is M_1-homogeneous. Since $\mathscr{I}$ is a P-ideal, at least one of the $\mathscr{I}_n$'s must be cofinal in $\mathscr{I}$ with respect to $\subseteq^*$. Choose such an $\mathscr{I}_n$, and let $f = \bigcup\{f_a \mid a \in \mathscr{I}_n\}$. Then f is a function from some subset of ω to $\mathscr{C}(X)$. Setting $e(x) = n$ if and only if $x \in f(n)$, we get a function $e : X \to \omega$, and since $\mathscr{I}$ is dense and $\mathscr{I}_n$ cofinal in $\mathscr{I}$, $a \mapsto e^{-1}(a)$ witnesses that φ is trivial. $\square$

4. Coherent families of continuous functions

Theorem 4.1. *Assume OCA+MA$_{\aleph_1}$. Let X and Y be zero-dimensional, locally compact Polish spaces, and let $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$ be an isomorphism. Then there are compact-open $K \subseteq X$ and $L \subseteq Y$, and a homeomorphism $e : X \setminus K \to Y \setminus L$, such that for all $A \in \mathscr{C}(Y \setminus L)$, $\varphi([A]) = [e^{-1}(A)]$.*

By Stone duality, a homeomorphism $\varphi : X^* \to Y^*$ induces an isomorphism $\hat{\varphi} : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$, and any map e as in the conclusion to Theorem 4.1 will in this case be a witness to the triviality of φ. Hence Theorem 4.1 implies Theorem 1.1. Before proving Theorem 4.1 we note a corollary involving definable isomorphisms.

Corollary 4.2. *Suppose X and Y are zero-dimensional, locally compact, Polish spaces, and $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$ is an isomorphism such that the set*

$$\Gamma = \{(A, B) \in \mathscr{C}(Y) \times \mathscr{C}(X) \mid \varphi([A]) = [B]\}$$

is Borel. Then φ is trivial.

Proof of Corollary 4.2. The fact that φ is an isomorphism between $\mathscr{C}(Y)/\mathscr{K}(Y)$ and $\mathscr{C}(X)/\mathscr{K}(X)$ can be written as a $\mathbf{\Pi}^1_2$ statement using Γ; hence by Schoenfield absoluteness, if $V^{\mathbb{P}}$ is a forcing extension satisfying OCA+MA$_{\aleph_1}$ (see [24]), then in $V^{\mathbb{P}}$ the map $\bar{\varphi} : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$, defined from the reinterpretation of Γ in $V^{\mathbb{P}}$, is also an isomorphism. By Theorem 4.1, then, we have in $V^{\mathbb{P}}$ that

$$\exists e \in C(X, Y)\ \forall A \in \mathscr{C}(Y)\ \bar{\varphi}([A]) = [e^{-1}(A)]$$

where $C(X, Y)$ denotes the space of continuous maps from X to Y. This can be written as a $\mathbf{\Sigma}^1_2$ statement and so by Schoenfield absoluteness again, it must be true in V with φ replacing $\bar{\varphi}$. $\square$

Before the proof of Theorem 4.1 we set down some more notation. Fix X, Y and φ as in the statement of the theorem. Let L_n be an increasing sequence of compact subsets of Y, with union Y, and let $Y_{n+1} = L_{n+1} \setminus L_n$ and $Y_0 = L_0$. Let $\mathscr{B}$ be a countable base for Y consisting of compact-open sets, such that

- for all $U \in \mathscr{B}$, the set of $V \in \mathscr{B}$ with $V \supseteq U$ is finite and linearly ordered by $\subseteq$, and
- for all $U \in \mathscr{B}$ and all $n < \omega$, either $U \subseteq Y_n$ or $U \cap Y_n = \emptyset$.

It follows that for all $U, V \in \mathscr{B}$, either $U \cap V = \emptyset$, $U \subseteq V$, or $V \subseteq U$. Let $\mathbb{P}$ be the poset of all partitions of Y into elements of $\mathscr{B}$, ordered by refinement;

$$P \prec Q \iff \forall U \in P\ \exists V \in Q \quad U \subseteq V$$

We also use $\prec^*$ to denote *eventual refinement*;

$$P \prec^* Q \iff \forall^\infty U \in P\, \exists V \in Q \quad U \subseteq V$$

When $P \prec^* Q$ we let $\Gamma(P,Q)$ be the least n such that every $U \in P$ disjoint from L_n is contained in some element of Q.

For a given $P \in \mathbb{P}$, let $s_P : Y \to P$ be the unique function satisfying $x \in s_P(x)$ for all $x \in Y$; similarly, when $P, Q \in \mathbb{P}$ and $P \prec Q$ we let $s_{PQ} : P \to Q$ be the unique function satisfying $U \subseteq s_{PQ}(U)$ for all $U \in P$. These maps induce embeddings $\sigma_P : \mathscr{P}(P)/\operatorname{fin} \to \mathscr{C}(Y)/\mathscr{K}(Y)$ and $\sigma_{PQ} : \mathscr{P}(Q)/\operatorname{fin} \to \mathscr{P}(P)/\operatorname{fin}$ in the usual way.

Finally, we need to prove a uniqueness result for maps $e : Z \to \omega$ inducing the same map $\mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(Z)/\mathscr{K}(Z)$.

Lemma 4.3. *Suppose $Z \in \mathscr{C}(X)$ and $e, f : Z \to \omega$ are continuous, compact-to-one maps, such that $e^{-1}(a)\Delta f^{-1}(a)$ is compact for every $a \subseteq \omega$. Then $\{x \in Z \mid e(x) \neq f(x)\}$ is compact.*

Proof. Suppose not; then for some infinite set $I \subseteq \omega$ and all $n \in I$, there is a point $x_n \in Z \cap X_n$ such that $e(x_n) \neq f(x_n)$. Since e and f are compact-to-one, we may assume also that $m \neq n$ implies $e(x_m) \neq e(x_n)$ and $f(x_m) \neq f(x_n)$. Now define a coloring $F : [I]^2 \to 3$ by

$$F(\{m < n\}) = \begin{cases} 0 & e(x_m) \neq f(x_n) \wedge f(x_m) \neq e(x_n) \\ 1 & e(x_m) = f(x_n) \wedge f(x_m) \neq e(x_n) \\ 2 & e(x_m) \neq f(x_n) \wedge f(x_m) = e(x_n) \end{cases}$$

By Ramsey's theorem, there is an infinite set $a \subseteq I$ which is homogeneous for this coloring. Suppose first that a is 1-homogeneous, and let $m < n < k$ be members of a. Then

$$e(x_m) = f(x_n) \quad \text{and} \quad e(x_m) = f(x_k) \quad \text{and} \quad e(x_n) = f(x_k)$$

which implies $e(x_n) = f(x_n)$, a contradiction. Similarly, a cannot be 2-homogeneous.

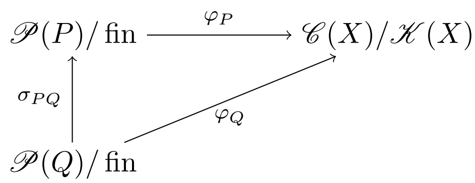
Now suppose a is 0-homogeneous. Let $a = a_0 \cup a_1$ be a partition of a into two infinite sets, and put $W_i = \{x_n \mid n \in a_i\}$ and $W = \{x_n \mid n \in a\} = W_0 \cup W_1$. From the homogeneity of a, it follows that $e''W \cap f''W = \emptyset$, and hence (as e and f are injective on W)

$$W \cap e^{-1}((e''W_0) \cup (f''W_1)) = W_0 \quad \text{and} \quad W \cap f^{-1}((e''W_0) \cup (f''W_1)) = W_1$$

So, if $b = e''W_0 \cup f''W_1$, we have $W \subseteq e^{-1}(b)\Delta f^{-1}(b)$. But W is not compact, so this is a contradiction. $\square$

Proof of Theorem 4.1. For each $P \in \mathbb{P}$, let $\varphi_P = \varphi \circ \sigma_P$. Then φ_P is an embedding of $\mathscr{P}(P)/\operatorname{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$. By Theorem 4.1, there is a continuous map $e_P : X \to P$ such that $a \mapsto e_P^{-1}(a)$ lifts φ_P. Note that if

$P, Q \in \mathbb{P}$ and $P \prec^* Q$, then the following diagram commutes:



So by Lemma 4.3, the set $\{x \in X \mid s_{PQ}(e_P(x)) \neq e_Q(x)\}$ is compact.

Now let $[\mathbb{P}]^2 = M_0 \cup M_1$ be the partition defined by

$$\{P, Q\} \in M_0 \iff \exists x \in X \quad s_{P,P\vee Q}(e_P(x)) \neq s_{Q,P\vee Q}(e_Q(x))$$

Here $P \vee Q$ is the finest partition coarser than both P and Q. If we define $f_P : \mathscr{B} \to \mathscr{C}(X)$ by

$$f_P(U) = \{x \in X \mid e_P(x) \subseteq U\}$$

then we have

$$\{P, Q\} \in M_0 \iff \exists U \in \mathscr{B} \quad f_P(U) \neq f_Q(U)$$

and it follows that M_0 is open in the topology on $\mathbb{P}$ obtained by identifying P with f_P.

Claim 4.4. There is no uncountable, M_0-homogeneous subset of $\mathbb{P}$.

Proof. Suppose H is such, and has size $\aleph_1$. Using $MA_{\aleph_1}$ with a simple modification of Hechler forcing, we see that there is some $\bar{P} \in \mathbb{P}$ such that $P \succ^* \bar{P}$ for all $P \in H$. By thinning out H and refining a finite subset of $\bar{P}$, we may assume that $P \succ \bar{P}$ for all $P \in H$, and moreover that there is an $\bar{n}$ such that for all $P \in H$,

$$\left\{x \in X \mid s_{\bar{P},P}(e_{\bar{P}}(x)) \neq e_P(x)\right\} \subseteq K_{\bar{n}}$$

Now fix $P, Q \in H$ such that $e_P \restriction K_{\bar{n}} = e_Q \restriction K_{\bar{n}}$. Then $s_{P,P\vee Q} \circ e_P = s_{Q,P\vee Q} \circ e_Q$, contradicting the fact that $\{P, Q\} \in M_0$. □

By OCA, there is a countable cover of $\mathbb{P}$ by M_1-homogeneous sets; since $\mathbb{P}$ is countably directed under $\succ^*$, it follows that one of them, say $\mathbb{Q}$, is cofinal in $\mathbb{P}$. It follows moreover that for some n, we have

$$\forall P \in \mathbb{P}\ \exists Q \in \mathbb{Q} \quad \Gamma(Q, P) \leq n$$

That is, $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$ defined by

$$P \prec^n Q \iff \forall U \in P\ \ (U \cap L_n = \emptyset \implies \exists V \in Q\ U \subseteq V)$$

Claim 4.5. There is a compact set $K \subseteq X$ and a unique continuous map $e : X \setminus K \to Y$ satisfying

$$\forall x \in X \setminus K \quad e(x) \in \bigcap_{P \in \mathcal{Q}} e_P(x)$$

Proof. Fix $x \in X$. If $P, Q \in \mathbb{Q}$, then by M_1-homogeneity of $\mathbb{Q}$ we have

$$s_{P,P\vee Q}(e_P(x)) = s_{Q,P\vee Q}(e_Q(x))$$

Then, the unique member of $P\vee Q$ containing $e_P(x)$ is the same as the unique member of $P \vee Q$ containing $e_Q(x)$. It follows that $e_P(x) \cap e_Q(x) \neq \emptyset$, and so either $e_P(x) \subseteq e_Q(x)$ or vice-versa. Then the collection $\{e_P(x) \mid P \in \mathbb{Q}\}$ is a chain, and hence by compactness has nonempty intersection.

Now let

$$K = \{x \in X \mid \forall P \in \mathbb{Q}\; e_P(x) \subseteq L_n\} \subseteq \bigcap_{P\in\mathbb{Q}} e_P^{-1}(P \cap \mathscr{C}(L_n))$$

Then K is contained in a compact set. If $x \in X \setminus K$ and $P \in \mathbb{Q}$, then $e_P(x)$ is disjoint from L_n. Then for any $x \in X \setminus K$ and $\epsilon > 0$, there is some $P \in \mathbb{Q}$ such that $e_P(x)$ has diameter less than ϵ (since $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$). Thus e, as defined above, is unique.

To see that e is continuous, note that for any open $U \subseteq X$,

$$x \in e^{-1}(U) \iff \exists P \in \mathbb{Q} \quad e_P(x) \subseteq U$$

□

Claim 4.6. The map $U \mapsto e^{-1}(U)$ lifts φ.

Proof. Fix $P \in \mathbb{Q}$, and let $U \in P$. Then clearly, for all $x \in X\setminus K$, $e_P(x) = U$ if and only if $e(x) \in U$. Since there are only finitely many $U \in P$ such that one of $e_P^{-1}(\{U\})$ or $e^{-1}(U)$ meets K, it follows that

$$\forall^\infty U \in P\; e_P^{-1}(\{U\}) = e^{-1}(U)$$

Then $U \mapsto e^{-1}(U)$ lifts φ_P.

Now fix $A \in \mathscr{C}(Y)$. Then there is some $P \in \mathbb{P}$ such that A can be written as a union of a subset of P. Find $Q \in \mathbb{Q}$ with $Q \prec^* P$; then, up to a compact set, A can be written as a union of some subset a of Q. Hence,

$$\varphi[A] = \varphi_Q[a] = [e^{-1}(A)]$$

□

□

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